Locally convex quasi $C^*$-normed algebras

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Abstract

If $A_0[\| \cdot \|_0]$ is a $C^*$-normed algebra and $\tau$ a locally convex topology on $A_0$ making its multiplication separately continuous, then $\tilde{A}_0[\tau]$ (completion of $A_0[\tau]$) is a locally convex quasi $*$-algebra over $A_0$, but it is not necessarily a locally convex quasi $*$-algebra over the $C^*$-algebra $\tilde{A}_0[\| \cdot \|_0]$ (completion of $A_0[\| \cdot \|_0]$). In this article, stimulated by physical examples, we introduce the notion of a locally convex quasi $C^*$-normed algebra, aiming at the investigation of $\tilde{A}_0[\tau]$; in particular, we study its structure, $*$-representation theory and functional calculus.

1. Introduction

In the present paper we continue the study introduced in [7] and carried over in [13] and [8]. At this stage, it concerns the investigation of the structure of the completion of a $C^*$-normed algebra $A_0[\| \cdot \|_0]$, under a locally convex topology $\tau$ “compatible” to $\| \cdot \|_0$, that makes the multiplication of $A_0$ separately continuous. The case when $A_0[\| \cdot \|_0]$ is a $C^*$-algebra and $\tau$ makes the multiplication jointly continuous was considered in [7, 13], while the analogue case corresponding to separately continuous multiplication was discussed in [8], where the so-called locally convex quasi $C^*$-algebras were introduced. In this work, prompted by examples that one meets in physics, we introduce the notion of locally convex quasi $C^*$-normed algebras, which is wider than that of locally convex quasi $C^*$-algebras, starting with a $C^*$-normed algebra $A_0[\| \cdot \|_0]$ and a locally convex topology $\tau$, “compatible” to $\| \cdot \|_0$, making the multiplication of $A_0$ separately continuous. For example, let $M_0$ be a $C^*$-normed algebra of operators on a Hilbert space $H$, endowed with the operator norm $\| \cdot \|_0$, $D$ a dense subspace of $H$ such that $M_0 D \subset D$ and $\tau_s$ the strong*-topology on $M_0$ defined by $D$. Then, the $C^*$-algebra $\tilde{M}_0[\| \cdot \|_0]$ does not leave $D$ invariant, in general, and so the multiplication $ax$ of $a \in \tilde{M}_0[\tau_s]$ and $x \in \tilde{M}_0[\| \cdot \|_0]$ is not necessarily well-defined, therefore $\tilde{M}_0[\tau_s]$ is not a locally convex quasi $C^*$-algebra.
over the $C^*$-algebra $\mathcal{M}_0[\|\cdot\|_0]$. Hence, it is meaningful to study not only locally convex quasi $C^*$-algebras, but also locally convex quasi $C^*$-normed algebras.

For locally convex quasi “$C^*$-normed algebras” we obtain analogous results to those in [8] for locally convex quasi “$C^*$-algebras” despite of the lack of completion and of weakening the condition $(T_3)$ of [8].

In Section 3 we consider a $C^*$-algebra $A_0[\|\cdot\|_0]$ with a “regular” locally convex topology $\tau$ and show that every unital pseudo-complete symmetric locally convex $*$-algebra $A[\tau]$ such that $A_0[\|\cdot\|_0] \subset A[\tau] \subset \mathcal{A}_0[\tau]$ is a $GB^*$-algebra over the unit ball $\mathcal{U}(A_0)$ of $A_0[\|\cdot\|_0]$. The latter algebras have been defined by G.R. Allan [2] and P.G. Dixon [12] and play an essential role in the unbounded $*$-representation theory. In Section 4 we define the notion of locally convex quasi $C^*$-normed algebras and study their general theory, while in Section 5 we investigate the structure of commutative locally convex quasi $C^*$-normed algebras. In the final Section 6 we present locally convex quasi $C^*$-normed algebras of operators and then we study questions on the $*$-representation theory of locally convex quasi $C^*$-normed algebras and functional calculus for the “commutatively quasi-positive” elements of $\mathcal{A}_0[\tau]$.

Topological quasi $*$-algebras were introduced in 1981 by G. Lassner [15, 16], for facing solutions of certain problems in quantum statistics and quantum dynamics. But only later (see [17, p. 90]) the initial definition was reformulated in the right way, having thus included many more interesting examples. Quasi $*$-algebras came in light in 1988 (see [19], as well as [20, 9, 10]), serving as important examples of partial $*$-algebras initiated by J.-P. Antoine and W. Karwowski in [4, 5]. A lot of works have been done on this topic, which can be found in the treatise [3], where the reader will also find a relevant rich literature. Partial $*$-algebras and quasi $*$-algebras keep a very prominent place in the study of unbounded operators, where the latter are the foundation stones for mathematical physics and quantum field theory (see, for instance, [3, 14, 6, 20]).

Our motivation for such studies comes, on the one hand, from the preceding discussion and the promising contribution of the powerful tool that the $C^*$-property offers to such studies and, on the other hand, from the physical examples of locally convex quasi $C^*$-normed algebras in “dynamics of the BCS-Bogolubov model” [16] that will be shortly discussed in Section 7.

2. Preliminaries

Throughout the whole paper we consider complex algebras and we suppose that all topological spaces are Hausdorff. If an algebra $\mathcal{A}$ has an identity element, this will be denoted by $1$, and an algebra $\mathcal{A}$ with identity $1$ will be called unital.

Let $A_0[\|\cdot\|_0]$ be a $C^*$-normed algebra. The symbol $\|\cdot\|_0$ of the $C^*$-norm will also
denote the corresponding topology. Let \( \tau \) be a topology on \( A_0 \) such that \( A_0[\tau] \) is a locally convex \( * \)-algebra. The topologies \( \tau, \| \|_0 \) on \( A_0 \) are called compatible, whenever for any Cauchy net \( \{x_\alpha \} \) in \( A_0[\| \|_0] \) such that \( x_\alpha \to 0 \) in \( \tau, x_\alpha \to 0 \) in \( \| \|_0 \) \cite{8}. The completion of \( A_0 \) with respect to \( \tau \) will be denoted by \( \hat{A}_0[\tau] \). In the sequel, we shall call a directed family of seminorms that defines a locally convex topology \( \tau \), a defining family of seminorms.

A partial \( * \)-algebra is a vector space \( A \) equipped with a vector space involution \( * : A \to A : x \mapsto x^* \) and a partial multiplication defined on a set \( \Gamma \subset A \times A \) such that:

(i) \( (x, y) \in \Gamma \) implies \( (y^*, x^*) \in \Gamma \);

(ii) \( (x, y_1), (x, y_2) \in \Gamma \) and \( \lambda, \mu \in \mathbb{C} \) imply \( (x, \lambda y_1 + \mu y_2) \in \Gamma \);

(iii) for every \( (x, y) \in \Gamma \), a product \( xy \in A \) is defined, such that \( xy \) depends linearly on \( x \) and \( y \) and satisfies the equality \( (xy)^* = y^*x^* \).

Given a pair \( (x, y) \in \Gamma \), we say that \( x \) is a left multiplier of \( y \) and \( y \) is a right multiplier of \( x \).

Quasi \( * \)-algebras are essential examples of partial \( * \)-algebras. If \( A \) is a vector space and \( A_0 \) a subspace of \( A \), which is also a \( * \)-algebra, then \( A \) is said to be a quasi \( * \)-algebra over \( A_0 \) whenever:

(i)\(^\prime\) The multiplication of \( A_0 \) is extended on \( A \) as follows: The correspondences

\[ A \times A_0 \to A : (a, x) \mapsto ax \] (left multiplication of \( x \) by \( a \)) and

\[ A_0 \times A \to A : (x, a) \mapsto xa \] (right multiplication of \( x \) by \( a \))

are always defined and are bilinear;

(ii)\(^\prime\) \( x_1(x_2a) = (x_1x_2)a, (ax_1)x_2 = a(x_1x_2) \) and \( x_1(ax_2) = (x_1a)x_2 \), for all \( x_1, x_2 \in A_0 \) and \( a \in A \);

(iii)\(^\prime\) the involution \( * \) of \( A_0 \) is extended on \( A \), denoted also by \( * \), such that \( (ax)^* = x^*a^* \) and \( (xa)^* = a^*x^* \), for all \( x \in A_0 \) and \( a \in A \).

For further information cf. \cite{3}. If \( A_0[\tau] \) is a locally convex \( * \)-algebra, with separately continuous multiplication, its completion \( \hat{A}_0[\tau] \) is a quasi \( * \)-algebra over \( A_0 \) with respect to the operations:

- \( ax := \lim_{a} x_\alpha a \) (left multiplication), \( x \in A_0, a \in \hat{A}_0[\tau] \),

- \( xa := \lim_{a} x x_\alpha \) (right multiplication), \( x \in A_0, a \in \hat{A}_0[\tau] \),

where \( \{x_\alpha \}_{a \in \Sigma} \) is a net in \( A_0 \) such that \( a = \tau \)-limit \( x_\alpha \).

- An involution on \( \hat{A}_0[\tau] \) like in (iii)\(^\prime\) is the continuous extension of the involution on \( A_0 \).

A \( * \)-invariant subspace \( A \) of \( \hat{A}_0[\tau] \) containing \( A_0 \) is called a quasi \( * \)-subalgebra of \( \hat{A}_0[\tau] \) if \( ax, xa \) belong to \( A \) for any \( x \in A_0, a \in A \). One easily shows that \( A \) is a quasi \( * \)-algebra over \( A_0 \). Moreover, \( A[\tau] \) is a locally convex space that contains \( A_0 \) as a dense subspace and for every fixed \( x \in A_0 \), the maps \( A[\tau] \to A[\tau] \) with \( a \mapsto ax \) and \( a \mapsto xa \) are continuous. An algebra of this kind is called locally convex quasi \( * \)-algebra over \( A_0 \).
We denote by $L^1(D, \mathcal{H})$ the set of all (closable) linear operators $X$ such that $D(X) = D$, $D(X^*) \supseteq D$. The set $L^1(D, \mathcal{H})$ is a partial $*$-algebra with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication $\lambda X$, the involution $X \mapsto X^\dagger = X^*|D$ and the \textit{(weak)} partial multiplication $X_1 \circ X_2 = X_1^\dagger X_2$, defined whenever $X_2$ is a weak right multiplier of $X_1$ (we shall write $X_2 \in R^w(X_1)$ or $X_1 \in L^w(X_2)$), that is, if $X_2D \subseteq D(X_1^\dagger)$ and $X_1^\dagger D \subseteq D(X_2^*)$. $L^1(D, \mathcal{H})$ is neither associative nor semiassociative.

\textbf{Definition 2.1.} Let $D$ be a dense subspace of a Hilbert space $\mathcal{H}$. A $*$-\textit{representation} $\pi$ of $\mathcal{A}[\tau]$ is a linear map from $\mathcal{A}$ into $L^1(D, \mathcal{H})$ (see beginning of Section 4) with the following properties:

(i) $\pi$ is a $*$-representation of $\mathcal{A}_0$;
(ii) $\pi(a)^\dagger = \pi(a^*)$, $\forall a \in \mathcal{A}$;
(iii) $\pi(ax) = \pi(a) \circ \pi(x)$ and $\pi(xa) = \pi(x) \circ \pi(a)$, $\forall a \in \mathcal{A}$ and $x \in \mathcal{A}_0$, where $\circ$ is the (weak) partial multiplication of $\mathcal{L}^1(D, \mathcal{H})$ (ibid.). Having a $*$-representation $\pi$ as before, we write $D(\pi)$ in the place of $D$ and $\mathcal{H}_\pi$ in the place of $\mathcal{H}$. By a $(\tau, \tau_s^*)$-\textit{continuous $*$-representation} $\pi$ of $\mathcal{A}[\tau]$, we clearly mean continuity of $\pi$, when $L^1(D(\pi), \mathcal{H}_\pi)$ carries the locally convex topology $\tau_s^*$ (see Section 4).

In what follows, we shall need the concept of a $GB^*$-algebra introduced by G.R. Allan [2] (see also [12]), which we remind here. Let $\mathcal{A}[\tau]$ be a locally convex $*$-algebra with identity $1$ and let $B^*$ denote the collection of all closed, bounded, absolutely convex subsets $B$ of $\mathcal{A}[\tau]$ with the properties: $1 \in B$, $B^* = B$ and $B^2 \subseteq B$. For each $B \in B^*$, the linear span $\mathcal{A}[B]$ of $B$ is a normed $*$-algebra under the Minkowski functional $\| \cdot \|_B$ of $B$. When $\mathcal{A}[B]$ is complete for each $B \in B^*$, then $\mathcal{A}[\tau]$ is called \textit{pseudo-complete}. Every unital sequentially complete locally convex $*$-algebra is pseudo-complete [1, Proposition (2.6)]. A unital locally convex $*$-algebra $\mathcal{A}[\tau]$ is called \textit{symmetric} (resp. algebraically symmetric) if for every $x \in \mathcal{A}$ the element $1 + x^*x$ has an Allan-bounded inverse in $\mathcal{A}$ [2, pp. 91,93] (resp. if $(1+x^*x)$ has inverse in $\mathcal{A}$). A unital symmetric pseudo-complete locally convex $*$-algebra $\mathcal{A}[\tau]$, such that $B^*$ has a greatest member, say $B_0$, is said to be a $GB^*$-\textit{algebra over} $B_0$. In this case, $\mathcal{A}[B_0]$ is a $C^*$-algebra.

\section{$C^*$-normed algebras with regular locally convex topology}

Let $\mathcal{A}_0[\| \cdot \|_0]$ be a $C^*$-normed algebra and $\widetilde{\mathcal{A}}_0[\| \cdot \|_0]$ the $C^*$-algebra completion of $\mathcal{A}_0[\| \cdot \|_0]$. Consider a locally convex topology $\tau$ on $\mathcal{A}_0$ with the following properties:

(T1) $\mathcal{A}_0[\tau]$ is a locally convex $*$-algebra with separately continuous multiplication.
(T2) $\tau \preceq \| \cdot \|_0$, with $\tau$ and $\| \cdot \|_0$ being compatible.
Then, compatibility of $\tau$, $\| \cdot \|_0$ implies that:

- $A_0[\| \cdot \|_0] \hookrightarrow \widetilde{A}_0[\| \cdot \|_0] \hookrightarrow \widetilde{A}_0[\tau]$;

- $\widetilde{A}_0[\tau]$ is a locally convex quasi $*$-algebra over the $C^*$-normed algebra $A_0[\| \cdot \|_0]$, but it is not necessarily a locally convex quasi $*$-algebra over the $C^*$-algebra $\widetilde{A}_0[\| \cdot \|_0]$, since $\widetilde{A}_0[\| \cdot \|_0]$ is not a locally convex $*$-algebra under the topology $\tau$.

**Question.** Under which conditions one could have a well-defined multiplication of elements in $\widetilde{A}_0[\tau]$ with elements in $\widetilde{A}_0[\| \cdot \|_0]$?

We consider the case that the locally convex topology $\tau$ defined by a directed family of seminorms, say $(p_\lambda)_{\lambda \in \Lambda}$, satisfies in addition to the conditions $(T_1)$ and $(T_2)$ an extra “good” condition for the $C^*$-norm $\| \cdot \|_0$, called regularity condition, denoted by $(R)$. That is,

$(R) \forall \lambda \in \Lambda, \exists \lambda' \in \Lambda$ and $\gamma_\lambda > 0 : p_\lambda(xy) \leq \gamma_\lambda \|x\|_0 p_{\lambda'}(y), \forall x, y \in A_0[\| \cdot \|_0]$. 

In this regard, we have the following

**Lemma 3.1.** Suppose $A_0[\| \cdot \|_0]$ is a $C^*$-normed algebra and $\tau$ a locally convex topology on $A_0$ satisfying the conditions $(T_1), (T_2)$ and the regularity condition $(R)$ for $\| \cdot \|_0$. Let $a$ be an arbitrary element in $\widetilde{A}_0[\tau]$ and $y$ an arbitrary element in $\widetilde{A}_0[\| \cdot \|_0]$. Then, the left resp. right multiplication of $a$ with $y$ is defined by

$$a \cdot y = \tau - \lim_{\alpha,n} x_\alpha y_n \text{ resp. } y \cdot a = \tau - \lim_{\alpha,n} y_n x_\alpha,$$

where $\{x_\alpha\}_{\alpha \in \Sigma}$ is a net in $A_0[\tau]$ converging to $a$, $\{y_n\}_{n \in \mathbb{N}}$ is a sequence in $A_0[\| \cdot \|_0]$ converging to $y$ and $\forall \lambda \in \Lambda, \exists \lambda' \in \Lambda$ and $\gamma_\lambda > 0$:

$$p_\lambda(a \cdot y) \leq \gamma_\lambda \|y\|_0 p_{\lambda'}(a), p_\lambda(y \cdot a) \leq \gamma_\lambda \|y\|_0 p_{\lambda'}(a).$$

Under this multiplication $\widetilde{A}_0[\tau]$ is a locally convex quasi $*$-algebra over the $C^*$-algebra $\widetilde{A}_0[\| \cdot \|_0]$.

The proof of Lemma 3.1 follows directly from the regularity condition $(R)$. If $A_0[\tau]$ is a locally convex $*$-algebra with jointly continuous multiplication and $\tau \preceq \| \cdot \|_0$, then it satisfies the regular condition $(R)$ for $\| \cdot \|_0$.

**Lemma 3.2.** Let $A_0[\| \cdot \|_0]$ be a $C^*$-normed algebra and $A_0[\tau]$ an $m^*$-convex algebra satisfying conditions $(T_2)$ and $(R)$. If $(p_\lambda)_{\lambda \in \Lambda}$ is a defining family of $m^*$-seminorms for $\tau$ (i.e., submultiplicative $*$-preserving seminorms) and there is $\lambda_0 \in \Lambda$ such that $p_{\lambda_0}$ is a norm, then $\tau \sim \| \cdot \|_0$, where $\sim$ means equivalence of the respective topologies. In particular, if $A_0[\| \cdot \|_0]$ is a normed $*$-algebra such that $\| \cdot \| \leq \| \cdot \|_0$ and $\| \cdot \|, \| \cdot \|_0$ are compatible, then $\| \cdot \| \sim \| \cdot \|_0$. 


Proof. By (T2) and (R) we have \( \widetilde{A}_0[\| \cdot \|_0] \hookrightarrow \widetilde{A}_0[\tau] \hookrightarrow \widetilde{A}_0[p_{\lambda_0}] \), which by the basic theory of \( C^* \)-algebras (see e.g., \cite[Proposition 5.3]{18}) implies that \( \| x \|_0 \leq p_{\lambda_0}(x) \), for all \( x \in A_0 \). Hence, \( \tau \sim \| \cdot \|_0 \).

By Lemma 3.2 there does not exist any normed \(*\)-algebra containing the \( C^* \)-algebra \( \widetilde{A}_0[\| \cdot \|_0] \) properly and densely.

We now consider whether a \( GB^* \)-algebra over the unit ball \( U(\widetilde{A}_0[\| \cdot \|_0]) \) exists in \( \widetilde{A}_0[\tau] \). If \( \widetilde{A}_0[\tau] \) has jointly continuous multiplication and \( U(\widetilde{A}_0[\| \cdot \|_0]) \) is \( \tau \)-closed in \( \widetilde{A}_0[\tau] \) then \( \widetilde{A}_0[\tau] \) is a \( GB^* \)-algebra over \( U(\widetilde{A}_0[\| \cdot \|_0]) \), (cf. \cite[Theorem 2.1]{13}).

**Theorem 3.3.** Let \( A_0[\| \cdot \|_0] \) be a unital \( C^* \)-normed algebra and \( A_0[\tau] \) a locally convex \(*\)-algebra such that \( \tau \) satisfies the conditions (T1), (T2), the regularity condition (R) for \( \| \cdot \|_0 \) and makes the unit ball \( U(\widetilde{A}_0[\| \cdot \|_0]) \) \( \tau \)-closed. Then every algebraically symmetric locally convex \(*\)-algebra \( A[\tau] \) such that \( \widetilde{A}_0[\| \cdot \|_0] \subset A[\tau] \subset \widetilde{A}_0[\tau] \) is a \( GB^* \)-algebra over \( U(\widetilde{A}_0[\| \cdot \|_0]) \).

Proof. The proof can be done in a similar way to that of \cite[Theorem 2.2]{7}. Here we give a simpler proof. Without loss of generality we may assume that \( A_0[\| \cdot \|_0] \) is a \( C^* \)-algebra. Then we have (see, e.g., proof of \cite[Lemma 2.1]{7}):

(1) \( (I + a^*a)^{-1} \in U(A_0), \forall a \in A \).

Moreover, we show that

(2) \( U(A_0) \) is the largest member in \( B^*(A) \).

It is clear that \( U(A_0) \in B^*(A) \). Suppose now that \( B \) is an arbitrary element in \( B^*(A) \) and take \( a = a^* \in B \). Let \( C(a) \) be the maximal commutative \(*\)-subalgebra of \( A \) containing \( a \) and

\[ C_1 \equiv (U(A_0) \cap C(a)) \cdot (B \cap C(a)) \]

Then, clearly \( C_1^* = C_1 \); by the regular condition (R) \( C_1 \) is \( \tau \)-bounded in \( C(a) \), while by the commutativity of \( U(A_0) \cap C(a) \) and \( B \cap C(a) \) one has that \( C_1^2 \subset C_1 \). It is now easily seen that \( C_1^* \in B^*(C(a)) \), where \( B^*(C(a)) = \{ B \cap C(a) : B \in B^*(A) \} \). Thus, there is \( B_1 \in B^*(A) \) such that \( C_1^* = B_1 \cap C(a) \).

Since \( C(a) \) is commutative and pseudo-complete, \( B^*(C(a)) \) is directed \cite[Theorem (2.10)]{1}. So for each \( B \in B^*(A) \) there is \( B_1 \in B^*(A) \) such that

\[ (B \cup U(A_0)) \cap C(a) \subset B_1 \cap C(a) \]

where \( A_0 \cap C(a) \) is a \( C^* \)-algebra and \( A[B_1] \cap C(a) \) a normed \(*\)-algebra. An application of Lemma 3.2 gives

\[ \| x \|_0 = \| x \|_{B_1}, \forall x \in A_0 \cap C(a) \]
Furthermore, it follows from (1) that \( x(I + \frac{1}{n}x^*x)^{-1} \in \mathcal{A}_0 \). Thus,
\[
\|x(I + \frac{1}{n}x^*x)^{-1} - x\|_{B_1} \leq \frac{1}{n}\|x^*x\|_{B_1}, \forall x \in A[B_1] \cap \mathcal{C}(a), n \in \mathbb{N},
\]
which implies that \( \mathcal{A}_0 \cap \mathcal{C}(a) \) is \( \| \cdot \|_{B_1} \)-dense in \( A[B_1] \cap \mathcal{C}(a) \). Therefore, from (3.1) and the fact that \( \mathcal{A}_0 \cap \mathcal{C}(a) \) is a C*-algebra we get \( \mathcal{A}_0 \cap \mathcal{C}(a) = A[B_1] \cap \mathcal{C}(a) \). It follows that \( B \cap \mathcal{C}(a) \subset B_1 \cap \mathcal{C}(a) = \mathcal{U}(\mathcal{A}_0) \cap \mathcal{C}(a) \), from which we conclude
\[
(3.2) \quad a \in \mathcal{U}(\mathcal{A}_0), \forall a \in B, \quad \text{with} \quad a^* = a.
\]
Now taking an arbitrary \( a \in B \) we clearly have \( a^*a \in B \), hence from (3.2) \( a^*a \in \mathcal{U}(\mathcal{A}_0) \), which gives \( a \in \mathcal{U}(\mathcal{A}_0) \). So, \( B \subset \mathcal{U}(\mathcal{A}_0) \) and the proof of (2) is complete. Now, since \( \mathcal{U}(\mathcal{A}_0) \) is the greatest member in \( B^* (\mathcal{A}) \), we have that \( A[\mathcal{U}(\mathcal{A}_0)] \) coincides with the C*-algebra \( \mathcal{A}_0 \), therefore it is complete. So [1, Proposition 2.7] implies that \( \mathcal{A}[\tau] \) is pseudo-complete, hence a GB*-algebra over \( \mathcal{U}(\mathcal{A}_0) \).

4. Locally convex quasi C*-normed algebras

Let \( \mathcal{A}_0[\| \cdot \|_0] \) be a C*-normed algebra and \( \tau \) a locally convex topology on \( \mathcal{A}_0 \) with \( \{p_\lambda\}_{\lambda \in \Lambda} \) a defining family of seminorms. Suppose that \( \tau \) satisfies the properties (T\(_1\)), (T\(_2\)). The regularity condition (R), considered in the previous Section 2, for \( \| \cdot \|_0 \), is too strong (see Section 6). So in the present Section we weaken this condition, and we use it together with the conditions (T\(_1\)), (T\(_2\)), in order to investigate the locally convex quasi *-algebra \( \widetilde{\mathcal{A}}_0[\tau] \). The weakened condition (R) will be denoted by (T\(_3\)) and it will read as follows:

\[
(T_3) \quad \forall \lambda \in \Lambda, \exists \lambda' \in \Lambda \text{ and } \gamma_\lambda > 0 : p_\lambda(xy) \leq \gamma_\lambda\|x\|_0p_{\lambda'}(y), \text{ for all } x, y \in \mathcal{A}_0 \text{ with } xy = yx.
\]

Then, we first consider the question stated in Section 3, just before Lemma 3.1, concerning a well-defined multiplication between elements of \( \widetilde{\mathcal{A}}_0[\tau] \) and \( \widetilde{\mathcal{A}}_0[\| \cdot \|_0] \).

If \( \mathcal{A}_0[\| \cdot \|_0] \) is commutative and \( \tau \) satisfies the conditions (T\(_1\)) – (T\(_3\)), then \( \tau \) fulfills clearly the regularity condition (R) for \( \| \cdot \|_0 \), and so by Lemma 3.1, for arbitrary \( a \in \widetilde{\mathcal{A}}_0[\tau] \) and \( y \in \widetilde{\mathcal{A}}_0[\| \cdot \|_0] \) the left and right multiplications \( a \cdot y \) and \( y \cdot a \) are defined, respectively, and \( \widetilde{\mathcal{A}}_0[\tau] \) is a locally convex quasi *-algebra over the C*-algebra \( \widetilde{\mathcal{A}}_0[\| \cdot \|_0] \).

We consider now the afore-mentioned question in the noncommutative case; for this we set the following

**Definition 4.1.** Let \( a \in \widetilde{\mathcal{A}}_0[\tau] \) and \( y \in \widetilde{\mathcal{A}}_0[\| \cdot \|_0] \). We shall say that \( y \) **commutes strongly** with \( a \) if there is a net \( \{x_\alpha\}_{\alpha \in \Sigma} \) in \( \mathcal{A}_0[\| \cdot \|_0] \) such that \( x_\alpha \xrightarrow{\tau} a \) and \( x_\alpha y = yx_\alpha \), for every \( \alpha \in \Sigma \).
• In the rest of the paper, \( \widetilde{A}_0[\| \cdot \|_0] \), denotes the completion of the \( C^\ast \)-algebra \( \widetilde{A}_0[\| \cdot \|_0] \) with respect to the locally convex topology \( \tau \). As a set it clearly coincides with \( \widetilde{A}_0[\tau] \), but there are cases that we need to distinguish them (see Remark 4.6).

**Remark 4.2.** Let \( a \in \widetilde{A}_0[\tau] \) and \( y \in \widetilde{A}_0[\| \cdot \|_0] \). Whenever \( y \in A_0 \), the multiplications \( ay \) and \( ya \) are always defined by

\[
ay = \lim_\alpha x_\alpha y \quad \text{and} \quad ya = \lim_\alpha yx_\alpha,
\]

where \( \{x_\alpha\}_{\alpha \in \Sigma} \) is a net in \( A_0 \) converging to \( a \) with respect to \( \tau \). Hence, we may define the notion \( y \) commutes with \( a \), as usually, i.e., when \( ay = ya \). But, even if \( y \) commutes with \( a \), one has, in general, that \( y \) does not commute strongly with \( a \). Thus, the notion of strong commutativity is clearly stronger than that of commutativity.

**Lemma 4.3.** Let \( A_0[\| \cdot \|_0] \) be a \( C^\ast \)-normed algebra and \( \tau \) a locally convex topology on \( A_0 \) that satisfies the properties (T1) – (T3). Let \( a \in \widetilde{A}_0[\tau] \) and \( y \in \widetilde{A}_0[\| \cdot \|_0] \) be strongly commuting. Then the multiplications \( a \cdot y \) resp. \( y \cdot a \) are defined by

\[
 a \cdot y = \tau - \lim_\alpha x_\alpha y \quad \text{resp.} \quad y \cdot a = \tau - \lim_\alpha yx_\alpha \quad \text{and} \quad a \cdot y = y \cdot a,
\]

where \( \{x_\alpha\}_{\alpha \in \Sigma} \) is a net in \( \widetilde{A}_0[\| \cdot \|_0] \), \( \tau \)-converging to \( a \) and commuting with \( y \). The preceding multiplications provide an extension of the multiplication of \( A_0 \). Moreover, an analogous condition to (T3) holds for the elements \( a, y \), i.e.,

\[
(T'_3) \quad \forall \lambda \in \Lambda \exists \lambda' \in \Lambda \quad \text{and} \quad \gamma_\lambda > 0 : p_\lambda(a \cdot y) \leq \gamma_\lambda \|y\|_0 p_\lambda'(a).
\]

**Proof.** Existence of the \( \tau - \lim_\alpha x_\alpha y \) in \( \widetilde{A}_0[\tau] \):

Note that \( \{x_\alpha y\}_{\alpha \in \Sigma} \) is a \( \tau \)-Cauchy net in \( \widetilde{A}_0[\| \cdot \|_0] \). Indeed, from (T3), for every \( \lambda \in \Lambda \), there are \( \lambda' \in \Lambda \) and \( \gamma_\lambda > 0 \) such that

\[
p_\lambda(x_\alpha y - x_\alpha' y) = p_\lambda((x_\alpha - x_\alpha') y) \leq \gamma_\lambda \|y\|_0 p_\lambda'(x_\alpha - x_\alpha') \xrightarrow[\alpha,\alpha']{} 0.
\]

Hence, \( \tau - \lim_\alpha x_\alpha y \) exists in \( \widetilde{A}_0[\| \cdot \|_0]^\sim[\tau] \), which, as already noticed, as a set clearly coincides with \( \widetilde{A}_0[\tau] \).

The existence of the \( \tau - \lim_\alpha yx_\alpha \) in \( \widetilde{A}_0[\| \cdot \|_0]^\sim[\tau] \) is similarly shown and clearly \( \tau - \lim_\alpha yx_\alpha = \tau - \lim_\alpha x_\alpha y \).

Independence of \( \tau - \lim_\alpha x_\alpha y \) from the choice of the net \( \{x_\alpha\}_{\alpha \in \Sigma} \):

Let \( \{x'_\beta\}_{\beta \in \Sigma'} \) be another net in \( A_0 \) such that \( x'_\beta \xrightarrow[\beta]{} a \) and \( x'_\beta y = yx'_\beta \), for all \( \beta \in \Sigma' \). Then,

\[
x_\alpha - x'_\beta \xrightarrow[\alpha]} 0 \quad \text{with} \quad (x_\alpha - x'_\beta)y = y(x_\alpha - x'_\beta), \forall (\alpha, \beta) \in \Sigma \times \Sigma'.
\]
Moreover, by (T3), for every $\lambda \in \Lambda$, there exist $\lambda' \in \Lambda$ and $\gamma_\lambda > 0$ such that

$$p_\lambda ((x_\alpha - x_\beta')y) \leq \gamma_\lambda \|y\|_0 p_\lambda (x_\alpha - x_\beta') \to 0;$$

this completes the proof of our claim. Thus, we set

$$a \cdot y := \tau - \lim x_\alpha y \quad \text{resp.} \quad y \cdot a := \tau - \lim yx_\alpha;$$

this clearly implies $a \cdot y = y \cdot a$. Furthermore, using again (T3) we conclude that

$$\forall \lambda \in \Lambda \exists \lambda' \in \Lambda \text{ and } \gamma_\lambda > 0 : p_\lambda (a \cdot y) \leq \gamma_\lambda \|y\|_0 p_\lambda (a), \forall a \in \tilde{A}_0[\tau] \text{ and } y \in \tilde{A}_0[\|\cdot\|_0],$$

and this proves (T3').

Now, following [8] we define notions of positivity for the elements of $\tilde{A}_0[\tau]$.

**Definition 4.4.** Let $a \in \tilde{A}_0[\tau]$. Consider the set

$$(A_0)_+ := \{x \in A_0 : x^* = x \text{ and } sp_{A_0}(x) \subseteq [0, \infty)\},$$

where $sp_{A_0}(x)$ means spectrum of $x$ in $A_0$. Clearly $(A_0)_+$ is contained in the positive cone of the C*-algebra $\tilde{A}_0[\|\cdot\|_0]$. The element $a$ is called quasi-positive if there is a net $\{x_\alpha\}_{\alpha \in \Sigma}$ in $(A_0)_+$ such that $x_\alpha \to a$. In particular, $a$ is called commutatively quasi-positive if there is a commuting net $\{x_\alpha\}_{\alpha \in \Sigma}$ in $(A_0)_+$ such that $x_\alpha \to a$.

Denote by $\tilde{A}_0[\tau]_{q+}$ the set of all quasi-positive elements of $\tilde{A}_0[\tau]$ and by $\tilde{A}_0[\tau]_{cq+}$ the set of all commutatively quasi-positive elements of $\tilde{A}_0[\tau]$.

An easy consequence of Definition 4.4 is the following

**Lemma 4.5.**

1. $$(A_0)_+ \cap \tilde{A}_0[\|\cdot\|_0]_{q+} \subseteq \tilde{A}_0[\tau]_{cq+}.$$ 
2. $\tilde{A}_0[\tau]_{q+}$ is a positive wedge, but it is not necessarily a positive cone. $\tilde{A}_0[\tau]_{cq+}$ is not even a positive wedge, in general.

**Remark 4.6.** As we have mentioned before, the equality $\tilde{A}_0[\|\cdot\|_0] \sim [\tau] = \tilde{A}_0[\tau]$ holds set-theoretically. We consider the following notation:

$$\tilde{A}_0[\|\cdot\|_0] \sim [\tau]_{q+} \equiv \{a \in \tilde{A}_0[\tau] : \exists \text{ a net } \{x_\alpha\}_{\alpha \in \Sigma} \text{ in } \tilde{A}_0[\|\cdot\|_0]_+ : x_\alpha \to a\},$$

$$\tilde{A}_0[\|\cdot\|_0] \sim [\tau]_{cq+} \equiv \{a \in \tilde{A}_0[\tau] : \exists \text{ a commuting net } \{x_\alpha\}_{\alpha \in \Sigma} \text{ in } \tilde{A}_0[\|\cdot\|_0]_+ : x_\alpha \to a\}.$$
Then,

\[(4.1) \quad \mathcal{A}_0[\parallel \cdot \parallel_0^\sim]_{\tau q+} = \mathcal{A}_0[\tau]_{q+}, \quad \text{but} \quad \mathcal{A}_0[\parallel \cdot \parallel_0^\sim]_{\tau q+} \supsetneq \mathcal{A}_0[\tau]_{q+}, \quad \text{in general.}\]

If \(\mathcal{A}_0\) is commutative, then

\[\mathcal{A}_0[\tau]_{q+} = \mathcal{A}_0[\parallel \cdot \parallel_0^\sim]_{\tau q+} = \mathcal{A}_0[\parallel \cdot \parallel_0^\sim]_{\tau q+} = \mathcal{A}_0[\tau]_{q+}.\]

The following Proposition 4.7 plays an important role in the present paper. It is a generalization of Proposition 3.2 in [8], stated for locally convex quasi \(C^*\)-algebras, to the case of locally convex quasi \(C^*\)-normed algebras.

**Proposition 4.7.** Let \(\mathcal{A}_0[\parallel \cdot \parallel_0]\) be a unital \(C^*\)-normed algebra and \(\tau\) a locally convex topology on \(\mathcal{A}_0\) that fulfils the conditions \((T_1) - (T_3)\). Suppose that the next condition \((T_4)\) holds:

\[(T_4) \quad \text{The set} \ U(\mathcal{A}_0[\parallel \cdot \parallel_0])_+ \equiv \{x \in \mathcal{A}_0[\parallel \cdot \parallel_0]_+ : \parallel x \parallel_0 \leq 1\} \text{ is} \ \tau \text{-closed in} \ \mathcal{A}_0[\tau] \quad \text{(or, equivalently, it is} \ \tau \text{-complete).} \]

Then, \(\mathcal{A}_0[\tau]\) is a locally convex quasi *-algebra over \(\mathcal{A}_0\) with the properties:

1. \(a \in \mathcal{A}_0[\tau]_{cq+}\) implies that \(1 + a\) is invertible with \((1 + a)^{-1}\) in \(U(\mathcal{A}_0[\parallel \cdot \parallel_0])_+\).
2. For \(a \in \mathcal{A}_0[\tau]_{cq+}\) and \(\varepsilon > 0\), the element \(a \varepsilon := a \cdot (1 + \varepsilon a)^{-1}\) is well-defined, \(a - a \varepsilon \in \mathcal{A}_0[\parallel \cdot \parallel_0^\sim]_{\tau cq+}\) and \(a = \tau - \lim a \varepsilon\).
3. \(\mathcal{A}_0[\tau]_{cq+} \cap (-\mathcal{A}_0[\tau]_{cq+}) = \{0\}\).
4. Furthermore, suppose that the following condition

\[(T_5) \quad \mathcal{A}_0[\tau]_{q+} \cap \mathcal{A}_0[\parallel \cdot \parallel_0] = \mathcal{A}_0[\parallel \cdot \parallel_0]_+\]

is satisfied. Then, if \(a \in \mathcal{A}_0[\tau]_{cq+}\) and \(y \in \mathcal{A}_0[\parallel \cdot \parallel_0]_+\) with \(y - a \in \mathcal{A}_0[\tau]_{q+}\), one has that \(a \in \mathcal{A}_0[\parallel \cdot \parallel_0]_+\).

**Proof.** 1. There exists a commuting net \(\{x_\alpha\}_{\alpha \in \Sigma}\) in \((\mathcal{A}_0)_+\) with \(x_\alpha \xrightarrow{\tau} a\) and \(x_\alpha x_{\alpha'} = x_{\alpha'}x_\alpha\), for all \(\alpha, \alpha' \in \Sigma\). Using properties of the positive elements in a \(C^*\)-algebra, and condition \((T_3)\) we get that for every \(\lambda \in A\) there are \(\lambda' \in A\) and \(\gamma_\lambda > 0\) such that:

\[\begin{align*}
p_\lambda((1 + x_\alpha)^{-1} - (1 + x_{\alpha'}^{-1})) &= p_\lambda((1 + x_\alpha)^{-1}(x_{\alpha'} - x_\alpha)(1 + x_{\alpha'}^{-1})) \\
&\leq \gamma_\lambda\parallel(1 + x_\alpha)^{-1}\parallel_0\parallel(1 + x_{\alpha'}^{-1})\parallel_0 p_{\lambda'}(x_{\alpha'} - x_\alpha) \\
&\leq \gamma_\lambda p_{\lambda'}(x_{\alpha'} - x_\alpha) \xrightarrow{a, a'} 0.
\end{align*}\]

Hence \(\{(1 + x_\alpha)^{-1}\}_{\alpha \in \Sigma}\) is a Cauchy net in \(\mathcal{A}_0[\tau]\) consisting of elements of \(U(\mathcal{A}_0[\parallel \cdot \parallel_0])_+\), the latter set being \(\tau\)-closed by \((T_4)\). Hence, there exists \(y \in U(\mathcal{A}_0[\parallel \cdot \parallel_0])_+\) such that

\[(4.2) \quad (1 + x_\alpha)^{-1} \xrightarrow{\tau} y.\]

We shall show that \((1 + a)^{-1}\) exists in \(U(\mathcal{A}_0[\parallel \cdot \parallel_0])_+\) and coincides with \(y\). It is easily seen that, for each index \(\alpha \in \Sigma\), \((1 + x_\alpha)^{-1}\) commutes strongly with \((1 + a)\), so that
\((I + a) \cdot (I + x_a)^{-1}\) is well-defined (Lemma 4.3). Similarly, \((x_a - a) \cdot (I + x_a)^{-1} = I - (I + a) \cdot (I + x_a)^{-1}\) is well-defined, therefore using \((T'_3)\) of Lemma 4.3, we have that for all \(\lambda \in A\) there are \(\lambda' \in A\) and \(\gamma_\lambda > 0\) with

\[
p_\lambda((I + a) \cdot (I + x_a)^{-1}) = p_\lambda((x_a - a) \cdot (I + x_a)^{-1}) \leq \gamma_\lambda p_{\lambda'}(x_a - a) \longrightarrow 0.
\]

Thus, \((I + a) \cdot (I + x_a)^{-1} \longrightarrow I\). By the above,

\[
I + x_a \longrightarrow I + a \quad \text{and} \quad (I + x_a)y = y(I + x_a), \quad \forall \alpha \in \Sigma.
\]

Hence, \(y\) commutes strongly with \(I + a\), therefore \((I + a) \cdot y\) is well-defined by Lemma 4.3. Now, since \(x_a \longrightarrow a\), we have that

\[
\forall \lambda \in A \quad \text{and} \quad \forall \varepsilon > 0, \exists \alpha_0 \in \Sigma : p_\lambda(x_{\alpha'} - a) < \varepsilon, \forall \alpha' \geq \alpha_0.
\]

Using \((T_3), (T'_3)\) of Lemma 4.3, and relations (4.3), (4.2) we obtain

\[
p_\lambda((I + a) \cdot (I + x_a)^{-1} - (I + a) \cdot y) \\
\leq p_\lambda((I + a) \cdot (I + x_a)^{-1} - (I + x_{\alpha_0})(I + x_a)^{-1}) \\
+ p_\lambda((I + x_{\alpha_0})(I + x_a)^{-1} - (I + x_{\alpha_0})y + p_\lambda((I + x_{\alpha_0})y - (I + a)y) \\
\leq \gamma_\lambda p_{\lambda'}(a - x_{\alpha_0}) + \gamma_\lambda \|I + x_{\alpha_0}\|p_{\lambda'}((I + x_a)^{-1} - y) + \gamma_\lambda p_{\lambda'}(x_{\alpha_0} - a) \\
< 2\varepsilon + \gamma_\lambda \|I + x_{\alpha_0}\|p_{\lambda'}((I + x_a)^{-1} - y), \quad \forall \varepsilon > 0.
\]

Hence,

\[
0 \leq \lim_\alpha p_\lambda((I + a) \cdot (I + x_a)^{-1} - (I + a) \cdot y) \leq 2\varepsilon, \quad \forall \varepsilon > 0,
\]

which implies

\[
\lim_\alpha p_\lambda((I + a) \cdot (I + x_a)^{-1} - (I + a) \cdot y) = 0.
\]

Consequently,

\[
(I + a) \cdot (I + x_a)^{-1} \longrightarrow_{\tau} (I + a) \cdot y.
\]

Similarly, \((I + x_a)^{-1} \cdot (I + a) \longrightarrow_{\tau} y \cdot (I + a)\). So from (4.3) and (4.4) we conclude that \((I + a) \cdot y = y \cdot (I + a) = 1\), therefore \(y = (I + a)^{-1}\).

(2) By (1), for every \(\varepsilon > 0\), the element \((I + \varepsilon a)^{-1}\) exists in \(U(\widetilde{A}_0[\| \cdot \|_0], +)\), and commutes strongly with \(a\). Hence (see Lemma 4.3), \(a_\varepsilon := a \cdot (I + \varepsilon a)^{-1}\) is well-defined. Moreover, applying \((T'_3)\) of Lemma 4.3, we have that for all \(\lambda \in A\), there exist \(\lambda' \in A\) and \(\gamma_\lambda > 0\) such that

\[
p_\lambda(I - (I + \varepsilon a)^{-1}) = \varepsilon p_\lambda(a \cdot (I + \varepsilon a)^{-1}) \leq \varepsilon \gamma_\lambda \|(I + \varepsilon a)^{-1}\|p_{\lambda'}(a) \leq \varepsilon \gamma_\lambda p_{\lambda'}(a).
\]
Therefore,

\[(4.5) \quad \tau - \lim_{\varepsilon \downarrow 0} (1 + \varepsilon a)^{-1} = 1.\]

On the other hand, since \((1 - (1 + \varepsilon a)^{-1})\) commutes strongly with \(a\) and \(a_{\varepsilon} = \varepsilon^{-1}(1 - (1 + \varepsilon a)^{-1})\), \(\varepsilon > 0\), we have

\[(4.6) \quad (1 - (1 + \varepsilon a)^{-1}) \cdot a = a \cdot (1 - (1 + \varepsilon a)^{-1}) = a - a_{\varepsilon} \in \widetilde{A}_0[\|\cdot\|_0]^{\sim}[\tau]_{cq+}.\]

Using (4.5), (4.6) and same arguments as in (4.4), we get that \(\tau - \lim_{\varepsilon \downarrow 0} a_{\varepsilon} = a.\)

(3) Let \(a \in \widetilde{A}_0[\tau]_{cq+} \cap (-\widetilde{A}_0[\tau]_{cq+})\) and \(\varepsilon > 0\) sufficiently small. By (2) (see also Remark 4.6), we have

\[\widetilde{A}_0[\|\cdot\|_0]^{\sim}[\tau]_{cq+} \ni a \cdot (1 + \varepsilon a)^{-1} \xrightarrow{\tau} a; \quad \text{in the same way} \quad -a \cdot (1 - \varepsilon a)^{-1} \xrightarrow{\tau} -a.\]

Now the element

\[x_{\varepsilon} \equiv a \cdot (1 + \varepsilon a)^{-1} - (-a) \cdot (1 - \varepsilon a)^{-1} = 2a \cdot (1 + \varepsilon a)^{-1}(1 - \varepsilon a)^{-1}\]

belongs to \(\widetilde{A}_0[\|\cdot\|_0]^+\) by (1) and the functional calculus of commutative \(C^*\)-algebras. Similarly, \(-x_{\varepsilon} = 2(-a) \cdot (1 - \varepsilon a)^{-1}(1 + \varepsilon a)^{-1} \in \widetilde{A}_0[\|\cdot\|_0]^{\sim}[\tau]_{cq+}.\) Hence,

\[x_{\varepsilon} \in \widetilde{A}_0[\|\cdot\|_0]^+ \cap (-\widetilde{A}_0[\|\cdot\|_0]^+) = \{0\}, \quad \text{so that} \quad a \cdot (1 + \varepsilon a)^{-1} = -a \cdot (1 - \varepsilon a)^{-1}.

Furthermore, by (2),

\[a = \tau - \lim_{\varepsilon \downarrow 0} a \cdot (1 + \varepsilon a)^{-1} = \tau - \lim_{\varepsilon \downarrow 0} (-a) \cdot (1 - \varepsilon a)^{-1} = -a, \quad \text{so} \quad a = 0.\]

(4) Note that \(y - a_{\varepsilon} = (y - a) + (a - a_{\varepsilon}) \in \widetilde{A}_0[\tau]_{cq+},\) since (by (4) and (2) resp.) the elements \(y - a, a - a_{\varepsilon}\) belong to \(\widetilde{A}_0[\tau]_{cq+}\) and the latter set is a positive wedge according to Lemma 4.5(2). On the other hand,

\[a_{\varepsilon} = a \cdot (1 + \varepsilon a)^{-1} = (1 + \varepsilon a)^{-1} \cdot a = \varepsilon^{-1}(1 - (1 + \varepsilon a)^{-1}) \in \widetilde{A}_0[\|\cdot\|_0].\]

Thus, taking under consideration the assumption (\(T_5\)) we conclude that

\[y - a_{\varepsilon} \in \widetilde{A}_0[\tau]_{cq+} \cap \widetilde{A}_0[\|\cdot\|_0] = \widetilde{A}_0[\|\cdot\|_0]^+,\]

which clearly gives \(\|a_{\varepsilon}\|_0 \leq \|y\|_0,\) for every \(\varepsilon > 0.\) Applying (\(T_4\)), we show that \(a \in \widetilde{A}_0[\|\cdot\|_0]^+.\) \(

\textbf{Definition 4.8.} Let \(A_0[\|\cdot\|_0]\) be a unital \(C^*\)-normed algebra, \(\tau\) a locally convex topology on \(A_0\) satisfying the conditions \((T_1) - (T_5)\) (for \((T_4),(T_5)\) see the previous proposition). Then,
• a quasi \( \ast \)-subalgebra \( A \) of the locally convex quasi \( \ast \)-algebra \( \widetilde{A}_0[\tau] \) over \( A_0 \) containing \( \widetilde{A}_0[\| \cdot \|_0] \) is said to be a locally convex quasi \( C^\ast \)-normed algebra over \( A_0 \).

A locally convex quasi \( C^\ast \)-normed algebra \( A \) over \( A_0 \) is said to be normal if \( a \cdot y \in A \) whenever \( a \in A \) and \( y \in \widetilde{A}_0[\| \cdot \|_0] \) commute strongly.

A locally convex quasi \( C^\ast \)-normed algebra \( A \) over \( A_0 \) is called a locally convex quasi \( C^\ast \)-algebra if \( \widetilde{A}_0[\| \cdot \|_0] \) is a \( C^\ast \)-algebra.

Note that the condition (T3) in the present paper is weaker than the condition

\[
(T_3) \forall \lambda \in A, \exists \lambda' \in A : p_\lambda(xy) \leq \|x\|_0p_\lambda(y), \forall x, y \in A_0 \text{ with } xy = yx
\]

in [8]. Nevertheless, results for locally convex quasi \( C^\ast \)-algebras in [8] are valid in the present paper for the wider class of locally convex \( C^\ast \)-normed algebras. It follows, by the very definitions, that a locally convex quasi \( C^\ast \)-algebra is a normal locally convex quasi \( C^\ast \)-normed algebra. A variety of examples of locally convex quasi \( C^\ast \)-algebras are given in [8], Sections 3 and 4. Examples of locally convex quasi \( C^\ast \)-normed algebras are presented in Sections 6 and 7.

An easy consequence of Definition 4.8 and Lemma 4.3 is the following

**Lemma 4.9.** Let \( A_0[\| \cdot \|_0] \) and \( \tau \) be as in Definition 4.8. Then the following hold:

1. \( \widetilde{A}_0[\tau] \) is a normal locally convex quasi \( C^\ast \)-normed algebra over \( A_0 \).

2. Suppose \( A \) is a commutative locally convex quasi \( C^\ast \)-normed algebra over \( A_0 \). Then \( A \cdot \widetilde{A}_0[\| \cdot \|_0] \equiv \text{linear span of } \{a \cdot y : a \in A, y \in \widetilde{A}_0[\| \cdot \|_0] \} \) is a commutative locally convex quasi \( C^\ast \)-algebra over \( \widetilde{A}_0[\| \cdot \|_0] \) under the multiplication \( a \cdot y \) (\( a \in A, y \in \widetilde{A}_0[\| \cdot \|_0] \)). In particular, if \( A \) is normal, then \( A \) is a commutative locally convex quasi \( C^\ast \)-algebra over \( \widetilde{A}_0[\| \cdot \|_0] \).

5. **Commutative locally convex quasi \( C^\ast \)-normed algebras**

In this Section, we discuss briefly some results on the structure of a commutative locally convex quasi \( C^\ast \)-normed algebra \( A[\tau] \) and on a functional calculus for its quasi-positive elements, that are similar to those in [8, Sections 5 and 6].

Let \( A[\tau] \) be a commutative locally convex quasi \( C^\ast \)-normed algebra over \( A_0 \) (see Definition 4.8). Then,

\[
\widetilde{A}_0[\| \cdot \|_0] \subset \widetilde{A}_0[\| \cdot \|_0] \subset A[\tau] \subset A[\tau] \cdot \widetilde{A}_0[\| \cdot \|_0] \subset \widetilde{A}_0[\tau],
\]

where \( A_0[\| \cdot \|_0] \) is a commutative unital \( C^\ast \)-normed algebra and \( A[\tau] \cdot \widetilde{A}_0[\| \cdot \|_0] \) is a commutative locally convex quasi \( C^\ast \)-algebra over the unital \( C^\ast \)-algebra \( \widetilde{A}_0[\| \cdot \|_0] \) according to Lemma 4.9(2). Thus, using some results of the Sections 5, 6 in [8] for the latter algebra we obtain information for the structure of \( A[\tau] \).
Let $W$ be a compact Hausdorff space, $C^* = \mathbb{C} \cup \{\infty\}$, and $\mathfrak{F}(W)_+$ a set of $C^*$-valued positive continuous functions on $W$, which take the value $\infty$ on at most a nowhere dense subset $W_0$ of $W$. The set
\[ \mathfrak{F}(W) = \{ fg_0 + h_0 : f \in \mathfrak{F}(W)_+ \text{ and } g_0, h_0 \in \mathcal{C}(W) \}, \]
where $\mathcal{C}(W)$ is the $C^*$-algebra of all continuous $\mathbb{C}$-valued functions on $W$, is called the set of $C^*$-valued continuous functions on $W$ generated by the wedge $\mathfrak{F}(W)_+$ and the $C^*$-algebra $\mathcal{C}(W)$. Using [8, Definition 5.6] and $\mathfrak{F}(W)$ we get the following theorem, which is an application of Theorem 5.8 of [8] for the commutative locally convex quasi $C^*$-algebra $\mathcal{A}[\tau] \cdot \mathfrak{A}_0[\| \cdot \|_0]$ over the unital commutative $C^*$-algebra $\mathfrak{A}_0[\| \cdot \|_0]$, with $\mathcal{A}[\tau]_+ \cdot \mathfrak{A}_0[\| \cdot \|_0]$, in the place of $\mathfrak{M}(\mathcal{A}_0, \mathcal{A}[\tau]_+)$.}

**Theorem 5.1.** There exists a map $\Phi$ from $\mathcal{A}[\tau]_+ \cdot \mathfrak{A}_0[\| \cdot \|_0]$ onto $\mathfrak{F}(W)$, where $W$ is the compact Hausdorff space corresponding to the Gel'fand space of the unital commutative $C^*$-algebra $\mathfrak{A}_0[\| \cdot \|_0]$, such that:

(i) $\Phi(\mathcal{A}[\tau]_+) = \mathfrak{F}(W)_+$ and $\Phi(\lambda a + b) = \lambda \Phi(a) + \Phi(b)$, $\forall a, b \in \mathcal{A}[\tau]_+$, $\lambda \geq 0$;

(ii) $\Phi$ is an isometric $*$-isomorphism from $\mathfrak{A}_0[\| \cdot \|_0]$ onto $\mathcal{C}(W)$;

(iii) $\Phi(ax) = \Phi(a)\Phi(x)$, $\Phi((\lambda a + b)x) = (\lambda \Phi(a) + \Phi(b))\Phi(x)$ and $\Phi(a(x_1 + x_2)) = \Phi(a)(\Phi(x_1) + \Phi(x_2))$, $\forall \lambda a, b \in \mathcal{A}[\tau]_+$, $x, x_1, x_2 \in \mathcal{A}_0$ and $\lambda \geq 0$.

Further we consider a functional calculus for the quasi-positive elements of the commutative locally convex quasi $C^*$-normed algebra $\mathcal{A}[\tau]$ over $\mathcal{A}_0$. For this, we must extend the multiplication of $\mathcal{A}[\tau]$.

Let $a, b \in \mathcal{A}[\tau]_+$. Then (see also [8, Definition 6.1]), $a$ is called left multiplier of $b$ if there are nets $\{ x_\alpha \}_{\alpha \in \Sigma}, \{ y_\beta \}_{\beta \in \Sigma'}$ in $(\mathcal{A}_0)_+$ such that $x_\alpha \xrightarrow{\tau} a$, $y_\beta \xrightarrow{\tau} b$ and $x_\alpha y_\beta \xrightarrow{\tau} c$, where the latter means that the double indexed net $\{ x_\alpha y_\beta \}_{(\alpha, \beta) \in \Sigma \times \Sigma'}$ converges to $c \in \mathcal{A}[\tau]$. Then, we set
\[ a \cdot b := c = \tau - \lim_{\alpha, \beta} x_\alpha y_\beta, \]
where the multiplication $a \cdot b$ is well defined, in the sense that it is independent of the choice of the nets $\{ x_\alpha \}_{\alpha \in \Sigma}, \{ y_\beta \}_{\beta \in \Sigma'}$, as follows from the proof of Lemma 6.2 in [8] applying arguments of the proof of Proposition 4.7. In the sequel, we simply denote $a \cdot b$ by $ab$. In analogy to Definition 6.3 of [8], if $x, y \in \mathfrak{A}_0[\| \cdot \|_0]$ and $a, b \in \mathcal{A}[\tau]_+$ with $a$ left multiplier of $b$, we may define the product of the elements $ax$ and $by$ as follows:
\[ (ax)(by) := (ab)xy. \]

The spectrum of an element $a \in \mathcal{A}[\tau]_+$, denoted by $\sigma_{\mathfrak{A}_0[\| \cdot \|_0]}(a)$, is defined as in Definition 6.4 of [8].
So using Theorem 5.1, it is shown (cf., for instance, Lemma 6.5 in [8]) that for every \( a \in \mathcal{A}[\tau] \), one has that \( \sigma_{\mathcal{A}_0[\|\cdot\|_0]}(a) \) is a locally compact subset of \( \mathbb{C}^* \) and\( \sigma_{\mathcal{A}_0[\|\cdot\|_0]}(a) \subset \mathbb{R}_+ \cup \{ \infty \} \).

According to the above, and taking into account the comments after Lemma 6.5 in [8] with \( \mathcal{A}_0[\|\cdot\|_0] \) in the place of \( \mathcal{A}_0 \), the next Theorem 5.2 provides a generalization of [8, Theorem 6.6] in the setting of commutative locally convex quasi \( \mathcal{C}^* \)-normed algebras. In particular, Theorem 5.2 supplies us with a functional calculus for the quasi-positive elements of the commutative locally convex quasi \( \mathcal{C}^* \)-normed algebra \( \mathcal{A}[\tau] \).

**Theorem 5.2.** Let \( a \in \mathcal{A}[\tau] \). Let \( a^n \) be well-defined for some \( n \in \mathbb{N} \). Then there is a unique \(*\)-isomorphism \( f \to f(a) \) from \( \bigcup_{k=1}^{n} \mathcal{C}_k(\sigma_{\mathcal{A}_0[\|\cdot\|_0]}(a)) \) into \( \mathcal{A}[\tau] \cdot \mathcal{A}_0[\|\cdot\|_0] \) such that:

(i) If \( u_0(\lambda) = 1 \), with \( u_0 \in \bigcup_{k=1}^{n} \mathcal{C}_k(\sigma_{\mathcal{A}_0[\|\cdot\|_0]}(a)) \) and \( \lambda \in \sigma_{\mathcal{A}_0[\|\cdot\|_0]}(a) \), then \( u_0(a) = 1 \).

(ii) If \( u_1(\lambda) = \lambda \) with \( u_1 \in \bigcup_{k=1}^{n} \mathcal{C}_k(\sigma_{\mathcal{A}_0[\|\cdot\|_0]}(a)) \) and \( \lambda \in \sigma_{\mathcal{A}_0[\|\cdot\|_0]}(a) \), then \( u_1(a) = a \).

(iii) \( (\lambda_1 f_1 + f_2)(a) = \lambda_1 f_1(a) + f_2(a) \), \( \forall f_1, f_2 \in \mathcal{C}_k(\sigma_{\mathcal{A}_0[\|\cdot\|_0]}(a)) \) and \( \lambda_1 \in \mathbb{C} \); \( (f_1 f_2)(a) = f_1(a) f_2(a) \), \( \forall f_j \in \mathcal{C}_{k_j}(\sigma_{\mathcal{A}_0[\|\cdot\|_0]}(a)) \), \( j = 1, 2 \), with \( k_1 + k_2 \leq n \).

(iv) Denoting with \( \mathcal{C}_b \) the set of the bounded and continuous functions, the map \( f \to f(a) \) restricted to \( \mathcal{C}_b(\sigma_{\mathcal{A}_0[\|\cdot\|_0]}(a)) \) is an isometric \(*\)-isomorphism of the \( \mathcal{C}^* \)-algebra \( \mathcal{C}_b(\sigma_{\mathcal{A}_0[\|\cdot\|_0]}(a)) \) on the closed \(*\)-subalgebra of \( \mathcal{A}_0[\|\cdot\|_0] \) generated by 1 and \((1 + a)^{-1}\).

Applying Theorem 5.2 and Proposition 4.7 in the proof of [8, Corollary 6.7] we get the following

**Corollary 5.3.** Let \( a \in \mathcal{A}[\tau] \) and \( n \in \mathbb{N} \). Then, there exists unique \( b \) in \( \mathcal{A}[\tau] \cdot \mathcal{A}_0[\|\cdot\|_0] \) such that \( a = b^n \). The unique element \( b \) is called quasi \( n \)-th root of \( a \) and we write \( b = a^{\frac{1}{n}} \).

6. **Structure of noncommutative locally convex quasi \( \mathcal{C}^* \)-normed algebras**

Using the notation of [8, Section 4] (see also [3]), let \( \mathcal{H} \) be a Hilbert space, \( \mathcal{D} \) a dense subspace of \( \mathcal{H} \) and \( \mathcal{M}_0[\|\cdot\|_0] \) a unital \( \mathcal{C}^* \)-normed algebra on \( \mathcal{H} \), such that

\( \mathcal{M}_0 \mathcal{D} \subset \mathcal{D} \), but \( \mathcal{M}_0[\|\cdot\|_0] \mathcal{D} \nsubseteq \mathcal{D} \).

Then, the restriction \( \mathcal{M}_0 \upharpoonright \mathcal{D} \) of \( \mathcal{M}_0 \) to \( \mathcal{D} \) is an \( \mathcal{O}^* \)-algebra on \( \mathcal{D} \), so that an element \( X \) of \( \mathcal{M}_0 \) may be regarded as an element \( X \upharpoonright \mathcal{D} \) of \( \mathcal{M}_0[\mathcal{D}] \). Moreover, let

\( \mathcal{M}_0 \subset \mathcal{M} \subset \mathcal{L}^1(\mathcal{D}, \mathcal{H}) \).
where $\mathcal{M}$ is an $O^*$-vector space on $\mathcal{D}$, that is, a $*$-invariant subspace of $\mathcal{L}^1(\mathcal{D}, \mathcal{H})$. Denote by $\mathcal{B}(\mathcal{M})$ the set of all bounded subsets of $\mathcal{D}[\tau_{s^*}]$ ($\tau_{s^*}$ is the graph topology on $\mathcal{M}$; see [14, p.9]) and by $\mathcal{B}_f(\mathcal{D})$ the set of all finite subsets of $\mathcal{D}$. Then $\mathcal{B}_f(\mathcal{D}) \subset \mathcal{B}(\mathcal{M})$ and both of them are admissible in the sense of [8, p. 522].

We recall the topologies $\tau_{s^*}$, $\tau_s^u(\mathcal{B})$, $\tau_s^u(\mathcal{M})$ defined in [8, pp. 522-523]. More precisely, for an arbitrary admissible subset $\mathcal{B}$ of $\mathcal{B}(\mathcal{M})$, and any $\mathfrak{M} \in \mathcal{B}$ consider the following seminorm:

$$p^\mathfrak{M}_s(X) := \sup_{\xi \in \mathfrak{M}} \{\|X\xi\| + \|X^\dagger \xi\|\}, \quad X \in \mathcal{M}.$$ 

We call the corresponding locally convex topology on $\mathcal{M}$ induced by the preceding family of seminorms, strongly $s^*$ $\mathcal{B}$-uniform topology and denote it by $\tau_s^u(\mathcal{B})$. In particular, the strongly $s^*$ $\mathcal{B}(\mathcal{M})$-uniform topology will be simply called strongly $s^*$ $\mathcal{M}$-uniform topology and will be denoted by $\tau_s^u(\mathcal{M})$. In Schmüdgen's book [17], this topology is called bounded topology. The strongly $s^*$ $\mathcal{B}_f(\mathcal{D})$-uniform topology is called strong $s^*$-topology on $\mathcal{M}$, denoted by $\tau_{s^*}$. All three topologies are related in the following way:

$$\tau_{s^*} \leq \tau_s^u(\mathcal{B}) \leq \tau_s^u(\mathcal{M}).$$

Then, one gets that

$$\mathcal{M}_0[\| \cdot \|_0] \subset \widehat{\mathcal{M}}_0[\| \cdot \|_0] \subset \widehat{\mathcal{M}}_0[\tau_s^u] \subset \widehat{\mathcal{M}}_0[\tau_{s^*}] \subset \mathcal{L}^1(\mathcal{D}, \mathcal{H}).$$

In this regard, we have now the following

**Proposition 6.1.** Let $\mathcal{M}_0[\| \cdot \|_0]$, $\mathcal{M}$ be as before. Let $\mathcal{B}$ be any admissible subset of $\mathcal{B}(\mathcal{M})$. Then $\widehat{\mathcal{M}}_0[\tau_s^u(\mathcal{B})]$ is a locally convex quasi $C^*$-normed algebra over $\mathcal{M}_0$, which is contained in $\mathcal{L}^1(\mathcal{D}, \mathcal{H})$. In particular, $\widehat{\mathcal{M}}_0[\tau_{s^*}]$ is a locally convex quasi $C^*$-normed algebra over $\mathcal{M}_0$. Furthermore, if $A \in \widehat{\mathcal{M}}_0[\tau_s^u(\mathcal{B})]$ and $Y \in \widehat{\mathcal{M}}_0[\| \cdot \|_0]$ commute strongly, then $A \Box Y$ is well-defined and

$$A \Box Y = A \cdot Y = Y \cdot A = Y \triangleleft A.$$ 

**Proof.** It is easily checked that $\widehat{\mathcal{M}}_0[\tau_s^u(\mathcal{B})]$ and $\widehat{\mathcal{M}}_0[\tau_{s^*}]$ are locally convex quasi $C^*$-normed algebras over $\mathcal{M}_0$. Suppose now that $A \in \widehat{\mathcal{M}}_0[\tau_s^u(\mathcal{B})]$ and $Y \in \widehat{\mathcal{M}}_0[\| \cdot \|_0]$ commute strongly. Then, there is a net $\{X_\alpha\}_{\alpha \in \Sigma}$ in $\mathcal{M}_0$ such that $X_\alpha Y = Y X_\alpha$, for all $\alpha \in \Sigma$ and $A = \tau_s^u(\mathcal{B}) - \lim_{\alpha} X_\alpha$. Since

$$(A^\dagger \xi)Y\eta = \lim_{\alpha} (X_\alpha^\dagger \xi)Y\eta = \lim_{\alpha} (\xi | X_\alpha Y \eta) = \lim_{\alpha} (\xi | Y X_\alpha \eta) = (\xi | Y A \eta)$$

for all $\xi, \eta \in \mathcal{D}$, it follows that $A \Box Y$ is well-defined and $A \Box Y = YA$. Furthermore, since

$$A \cdot Y = \tau_s^u(\mathcal{B}) - \lim_{\alpha} X_\alpha Y = \tau_{s^*} - \lim_{\alpha} X_\alpha Y,$$
we have

\[(A \cdot Y)\xi = \lim_{\alpha} X_{\alpha} Y \xi = \lim_{\alpha} Y X_{\alpha} \xi = Y A \xi = (A \square Y)\xi\]

for each \(\xi \in D\). Hence, \(A \cdot Y = A \square Y\).

Proposition 6.2. \(\mathcal{L}^1(D, \mathcal{H})[\tau_{s^*}]\) is a locally convex quasi \(C^*\)-normed algebra over \(\mathcal{L}^1(D)_b \equiv \{X \in \mathcal{L}^1(D) : X \in \mathcal{B}(\mathcal{H})\}\).

Proof. Indeed, as shown in [3, Section 2.5], \(\mathcal{L}^1(D)_b\) is a \(C^*\)-normed algebra which is \(\tau_{s^*}\) dense in \(\mathcal{L}^1(D, \mathcal{H})\). Hence, \(\mathcal{L}^1(D, \mathcal{H})\) is a locally convex quasi \(C^*\)-normed algebra over \(\mathcal{L}^1(D)_b\).

Remark 6.3. The following questions arise naturally:

(1) What is exactly the \(C^*\)-algebra \(\mathcal{L}^1(D)_b[\| \cdot \|_0]\)?

Under what conditions may one have the equality \(\mathcal{L}^1(D)_b[\| \cdot \|_0] = \mathcal{B}(\mathcal{H})\)?

(2) Is \(\mathcal{L}^1(D, \mathcal{H})\) a locally convex quasi \(C^*\)-algebra under the strong\(^*\) uniform topology \(\tau_{s^*}\)?

More precisely, does the equality \(\mathcal{L}^1(D)_b[\tau_{s^*}] = \mathcal{L}^1(D, \mathcal{H})\) hold?

We expect the answer to these questions to depend on the properties of the topology \(t_1 \equiv t_{\mathcal{L}^1(D, \mathcal{H})}\) given on \(D\) and we conjecture positive answers in the case where \(D \equiv D^\infty(T)\), with \(T\) a positive self-adjoint operator in a Hilbert space \(\mathcal{H}\), and \(\| \cdot \|_0\) the operator norm in \(\mathcal{B}(\mathcal{H})\). We leave these questions open.

In the rest of this Section we consider conditions under which a locally convex quasi \(C^*\)-normed algebra is continuously embedded in a locally convex quasi \(C^*\)-normed algebra of operators.

So let \(\mathcal{A}[\tau]\) be a locally convex quasi \(C^*\)-normed algebra over \(\mathcal{A}_0\) and \(D\) a dense subspace in a Hilbert space \(\mathcal{H}\). Let \(\pi : \mathcal{A} \longrightarrow \mathcal{L}^1(D, \mathcal{H})\) be a \(*\)-representation. Then we have the following:

Lemma 6.4. Let \(\mathcal{A}[\tau]\) be a locally convex quasi \(C^*\)-normed algebra over \(\mathcal{A}_0\) and \(\pi : \mathcal{A} \longrightarrow \mathcal{L}^1(D, \mathcal{H})\) a \((\tau, \tau_{s^*}(\mathcal{B}))\)-continuous \(*\)-representation of \(\mathcal{A}\). Then,

(1) \(\pi\) is a \(*\)-representation of the \(C^*\)-algebra \(\tilde{\mathcal{A}}_0[\| \cdot \|_0]\);

(2) \(\pi(\mathcal{A})[\tau_{s^*}^u] \) resp. \(\pi(\mathcal{A})[\tau_{s^*}^u]\) are locally convex quasi \(C^*\)-normed algebras over \(\pi(\mathcal{A}_0)\).

Proof. (1) Since \(\mathcal{A}_0 \subset \tilde{\mathcal{A}}_0[\| \cdot \|_0] \subset \mathcal{A}\) and \(\pi\) is a \(*\)-representation of \(\mathcal{A}\), it follows that

\[\pi(ay) = \pi(a) \circ \pi(y), \quad \forall a \in \tilde{\mathcal{A}}_0[\| \cdot \|_0], \forall y \in \mathcal{A}_0.\]
Now we show that

\[(6.3) \quad \pi(ab) = \pi(a)\circ \pi(b), \quad \forall a, b \in \widetilde{A}_0[\| \cdot \|_0].\]

Indeed, let \(a, b\) be arbitrary elements of \(\widetilde{A}_0[\| \cdot \|_0]\). Then, there exists a sequence \(\{y_n\}\) in \(A_0\) such that \(b = \| \cdot \|_0 - \lim_{n \to \infty} y_n\). Hence, \(ab = \| \cdot \|_0 - \lim_{n \to \infty} ay_n\).

Moreover, it is easily seen that \(\pi\) is also \((\tau, \tau_s^*)\)-continuous and so, by (6.2),

\[
\langle \pi(b)\xi|\pi(a^*)\eta \rangle = \lim_{n \to \infty} \langle \pi(y_n)\xi|\pi(a^*)\eta \rangle = \lim_{n \to \infty} \langle \pi(a)\circ \pi(y_n)\xi|\eta \rangle = \lim_{n \to \infty} \langle \pi(ay_n)\xi|\eta \rangle = \langle \pi(ab)\xi|\eta \rangle,
\]

for every \(\xi, \eta \in D\). Thus, (6.3) holds.

For any \(\xi \in D\), we put

\[
f(a) = (\pi(a)\xi), \quad a \in \widetilde{A}_0[\| \cdot \|_0].
\]

Then, by (6.3), \(f\) is a positive linear functional on the unital \(C^*\)-algebra \(\widetilde{A}_0[\| \cdot \|_0]\). Hence, we have

\[
\| \pi(a)\xi \|^2 = f(a^*a) \leq f(I)\|a\|_0^2 = \|\xi\|^2\|a\|_0^2
\]

for all \(a \in \widetilde{A}_0[\| \cdot \|_0]\), which implies that \(\pi\) is bounded. This completes the proof of (1).

(2) \(\pi(A)\) is a quasi \(*\)-subalgebra of the locally convex quasi \(*\)-algebras \(\pi(\widetilde{A})[\tau_s^u(B)]\) and \(\pi(\widetilde{A})[\tau_s^u] \) over \(\pi(A_0)\). Furthermore, by (1), \(\pi(\widetilde{A}_0[\| \cdot \|_0])\) is a \(C^*\)-algebra and

\[
\pi(\widetilde{A}_0)[\| \cdot \|_0] = \pi(\widetilde{A}_0[\| \cdot \|_0]) \subset \pi(A).
\]

\(\square\)

**Remark 6.5.** Let \(A[\tau]\) be a locally convex quasi \(C^*\)-normed algebra over \(A_0\), and \(\pi\) a \((\tau, \tau_s^u(B))\)-continuous \(*\)-representation of \(A\), where \(B\) is an admissible subset in \(B(\pi(A))\). Let \(a \in A\) be strongly commuting with \(y \in \widetilde{A}_0[\| \cdot \|_0]\). Then \(\pi(a)\) commutes strongly with \(\pi(y)\). The converse does not necessarily hold. So even if \(A[\tau]\) is normal, the locally convex quasi \(C^*\)-normed algebra \(\pi(A)\) over \(\pi(A_0)\) is not necessarily normal.

We are going now to discuss the faithfulness of a \((\tau, \tau_s^u)\)-continuous \(*\)-representation of \(A\). For this, we need some facts on sesquilinear forms, for which the reader is referred to [8, p. 544]. We only recall that if

\[
S(A_0) := \{\tau\text{-continuous positive invariant sesquilinear forms } \varphi \text{ on } A_0 \times A_0\},
\]

we say that the set \(S(A_0)\) is **sufficient**, whenever

\[
a \in A \text{ with } \tilde{\varphi}(a, a) = 0, \forall \varphi \in S(A_0), \text{ implies } a = 0,
\]
where ˜ϕ is the extension of ϕ to a τ-continuous positive invariant sesquilinear form on \( \mathcal{A} \times \mathcal{A} \).

From the next results, Theorem 6.6 and Corollary 6.7 can be regarded as generalizations of the analogues of the Gel’fand-Naimark theorem, in the case of locally convex quasi \( C^* \)-algebras proved in [8, Section 7]. Theorem 6.6 is proved in the same way as [8, Theorem 7.3].

**Theorem 6.6.** Let \( A[\tau] \) be a locally convex quasi \( C^* \)-normed algebra over a unital \( C^* \)-normed algebra \( A_0 \). The following statements are equivalent:

(i) There exists a faithful \((\tau, \tau_s^*)\)-continuous \(*\)-representation of \( A \).

(ii) The set \( S(A_0) \) is sufficient.

**Corollary 6.7.** Suppose \( S(A_0) \) is sufficient. Then, the locally convex quasi \( C^* \)-normed algebra \( A[\tau] \) over \( A_0 \) is continuously embedded in a locally convex quasi \( C^* \)-normed algebra of operators.

We end this Section with the study of a functional calculus for the commutatively quasi-positive elements (see Definition 4.4) of \( A[\tau] \).

Let \( A[\tau] \) be a locally convex quasi \( C^* \)-normed algebra over a unital \( C^* \)-normed algebra \( A_0 \). If \( a \in A[\tau]_{cq^+} \), then by Proposition 4.7(1), the element \((1 + a)^{-1}\) exists and belongs to \( \mathcal{U}(\tilde{A}_0[\| \cdot \|_0]) \). Denote by \( \tilde{C}^*(a)[\tau] \) the maximal commutative \( C^* \)-subalgebra of the \( C^* \)-algebra \( \tilde{A}_0[\| \cdot \|_0] \) containing the elements \( 1 \) and \((1 + a)^{-1}\).

**Lemma 6.8.** \( \tilde{C}^*(a)[\tau] \) is a commutative unital locally convex quasi \( C^* \)-algebra over \( C^*(a) \) and \( a \in \tilde{C}^*(a)[\tau]_{cq^+} \).

**Proof.** Since \( C^*(a) \) is a unital \( C^* \)-algebra, we have only to check the properties \((T_1) - (T_5)\). We show \((T_1)\); the rest of them, as well as the fact that \( a \in \tilde{C}^*(a)[\tau]_{cq^+} \) are proved by the same way as in [8, Proposition 7.6 and Corollary 7.7]. From the condition \((T_3)\) for \( A_0[\tau] \), we have that for all \( \lambda \in \Lambda \), there exist \( \lambda' \in \Lambda \) and \( \gamma_\lambda > 0 \) such that

\[
p_\lambda(x y) \leq \gamma_\lambda \| x \|_0 p_{\lambda'}(y), \forall x, y \in C^*(a).
\]

So, \( \tilde{C}^*(a)[\tau] \) is a locally convex \(*\)-algebra with separately continuous multiplication. \( \Box \)

By Lemma 6.8 and Theorem 5.2 we can now obtain a functional calculus for the commutatively quasi-positive elements of the noncommutative locally convex quasi \( C^* \)-normed algebra \( A[\tau] \) (see also [8, Theorem 7.8, Corollary 7.9]).

**Theorem 6.9.** Let \( A[\tau] \) be an arbitrary locally convex quasi \( C^* \)-normed algebra over a unital \( C^* \)-normed algebra \( A_0 \) and \( a \in A[\tau]_{cq^+} \). Suppose that \( a^n \) is well-defined for some \( n \in \mathbb{N} \). Then, there is a unique \(*\)-isomorphism \( f \rightarrow f(a) \) from \( \bigcup_{k=1}^n C_k(\sigma_{C^*(a)}(a)) \) into \( A[\tau] \cdot C^*(a) \) such that:
If \( u_0(\lambda) = 1 \), with \( u_0 \in \bigcup_{k=1}^{n} C_k(\sigma_{C^*}(a)) \) and \( \lambda \in \sigma_{C^*}(a) \), then \( u_0(a) = 1 \).

(ii) If \( u_1(\lambda) = \lambda \) with \( u_1 \in \bigcup_{k=1}^{n} C_k(\sigma_{C^*}(a)) \) and \( \lambda \in \sigma_{C^*}(a) \), then \( u_1(a) = a \).

(iii) \( (\lambda_1 f_1 + f_2)(a) = \lambda_1 f_1(a) + f_2(a) \), \( \forall f_1, f_2 \in \bigcup_{k=1}^{n} C_k(\sigma_{C^*}(a)) \) and \( \lambda_1 \in \mathbb{C} \);

\[
(f_1 f_2)(a) = f_1(a) f_2(a), \quad \forall f_j \in C_{k_j}(\sigma_{C^*}(a)), \; j = 1, 2, \text{ with } k_1 + k_2 \leq n.
\]

(iv) The map \( f \to f(a) \) restricted to \( C_b(\sigma_{C^*}(a)) \) is an isometric \(*\)-isomorphism of the \( C^*\)-algebra \( C_b(\sigma_{C^*}(a)) \) on the \( C^*\)-algebra \( C^*(a) \).

Using Theorem 6.9 and applying Corollary 5.3 for the commutative unital locally convex quasi \( C^* \)-algebra \( C^*(a)[\tau] \), we conclude the following

**Corollary 6.10.** Let \( A[\tau] \) and \( A_0 \) be as in Theorem 6.9. If \( a \in A[\tau]_{cq^+} \) and \( n \in \mathbb{N} \), there is a unique element \( b \in A[\tau]_{cq^+} \cdot C^*(a) \), such that \( a = b^n \). The element \( b \) is called commutatively quasi \( n \)th-root of \( a \) and is denoted by \( a^{\frac{1}{n}} \).

## 7. Applications

Locally convex quasi \( C^* \)-normed algebras arise, as we have discussed throughout this paper, as completions of a \( C^* \)-normed algebra with respect to a locally convex topology which satisfies a series of requirements. Completions of this sort actually occur in quantum statistics.

In statistical physics, in fact, one has to deal with systems consisting of a very large number of particles, so large that one usually considers this number to be *infinite*. One begins by considering systems living in a local region \( V \) (\( V \) is, for instance, a bounded region of \( \mathbb{R}^3 \) for gases or liquids, or a finite subset of the lattice \( \mathbb{Z}^3 \) for crystals) and requires that the set of local regions is directed, i.e., if \( V_1, V_2 \) are two local regions, then there exists a third local region \( V_3 \) containing both \( V_1 \) and \( V_2 \). The observables on a given bounded region \( V \) are supposed to constitute a \( C^* \)-algebra \( A_V \), where all \( A_V \)'s have the same norm, and so the \(*\)-algebra \( A_0 \) of local observables, \( A_0 = \bigcup_V A_V \), is a \( C^* \)-normed algebra. Its uniform completion is, obviously, a \( C^* \)-algebra (more precisely, a quasi local \( C^* \)-algebra) that in the original algebraic approach was taken as the observable algebra of the system. As a matter of fact, this \( C^* \)-algebraic formulation reveals to be insufficient, since for many models there is no way of including in this framework the thermodynamical limit of the local Heisenberg dynamics [6]. Then a possible procedure to follow in order to circumvent this difficulty is to define in \( A_0 \) a new locally convex topology, \( \tau \), called, for obvious reasons, physical topology, in such a way that the dynamics in the thermodynamical limit belongs to the completion of \( A_0 \) with respect to \( \tau \). For that purpose, a class of topologies for the \(*\)-algebra \( A_0 \) of local observables of a quantum system was proposed by Lassner in [15, 16]. We will sketch in
what follows this construction. Let $\mathcal{A}_0$ be a $C^*$-normed algebra to be understood as the algebra of local observables described above; thus we will suppose that $\mathcal{A}_0 = \bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda$, where $\{\mathcal{A}_\lambda\}_{\lambda} \in \Lambda$ is a family of $C^*$-algebras labeled by a directed set of indices $\Lambda$. Assume that, for every $\alpha \in \Sigma$ ($\Sigma$ a given set of indices), $\pi_\alpha$ is a $\ast$-representation of $\mathcal{A}_0$ on a dense subspace $D_\alpha$ of a Hilbert space $\mathcal{H}_\alpha$, i.e. each $\pi_\alpha$ is a $\ast$-homomorphism of $\mathcal{A}_0$ into the partial $O^*$-algebra $L^1(D_\alpha, \mathcal{H}_\alpha)$ endowed, for instance, with the topology $\tau_\pi^\dagger(L^1(D_\alpha, \mathcal{H}_\alpha))$. We shall assume that $\pi_\alpha(x)D_\alpha \subset D_\alpha$, for every $\alpha \in \Sigma$ and $x \in \mathcal{A}_0$. Since every $\mathcal{A}_\lambda$ is a $C^*$-algebra, each $\pi_\alpha$ is a bounded and continuous $\ast$-representation, i.e. $\pi_\alpha(x) \in B(\mathcal{H}_\alpha)$, $\|\pi_\alpha(x)\| \leq \|x\|_0$, for every $x \in \mathcal{A}_0$. So each $\pi_\alpha$ can be extended to the $C^*$-algebra $\tilde{\mathcal{A}}_0[\| \cdot \|_0]$ (we denote the extension by the same symbol). The family is supposed to be faithful, in the sense that if $x \in \tilde{\mathcal{A}}_0[\| \cdot \|_0]$, $x \neq 0$, then there exists $\alpha \in \Sigma$ such that $\pi_\alpha(x) \neq 0$. Let us further suppose that $\mathcal{D}_\alpha = D^\infty(M_\alpha) = \bigcap_{n \in \mathbb{N}} D(M_\alpha^n)$, where $M_\alpha$ is a selfadjoint operator. Without loss of generality we may assume that $M_\alpha \geq I_\alpha$, with $I_\alpha$ the identity operator in $B(\mathcal{H}_\alpha)$. Under these assumptions, a physical topology $\tau$ can be defined on $\mathcal{A}_0$ by the family of seminorms

$$p_\alpha^f(x) = \|\pi_\alpha(x)f(M_\alpha)\| + \|\pi_\alpha(x^*)f(M_\alpha)\|, \ x \in \mathcal{A}_0,$$

where $\alpha \in \Sigma$ and $f$ runs over the set $\mathcal{F}$ of all positive, bounded and continuous functions $f(t)$ on $\mathbb{R}^+$ such that

$$\sup_{t \in \mathbb{R}^+} t^k f(t) < \infty, \ \forall \ k = 0, 1, 2, \ldots.$$

Then, $\mathcal{A}_0[\tau]$ is a locally convex $\ast$-algebra with separately continuous multiplication (i.e. (T1) holds). In order to prove that $\tilde{\mathcal{A}}_0[\tau]$ is a locally convex quasi $C^*$-normed algebra, we need to prove that (T2)-(T3) also hold. As for (T2), we have, for every $\alpha \in \Sigma$,

$$p_\alpha^f(x) = \|\pi_\alpha(x)f(M_\alpha)\| + \|\pi_\alpha(x^*)f(M_\alpha)\| \leq 2\|f(M_\alpha)\|\|\pi_\alpha(x)\| \leq 2\|f(M_\alpha)\||x||_0, \ x \in \mathcal{A}_0.$$

The compatibility of $\tau$ with $\| \cdot \|_0$ follows easily from the closedness of the operators $f(M_\alpha)^{-1}$ and the faithfulness of the family $\{\pi_\alpha\}_{\alpha \in \Sigma}$ of $\ast$-representations.

The condition (R) does not hold, in general, but, on the other hand, if $x, y \in \mathcal{A}_0$ with $xy = yx$, we have

$$p_\alpha^f(xy) = \|\pi_\alpha(xy)f(M_\alpha)\| + \|\pi_\alpha((xy)^*)f(M_\alpha)\|$$

$$= \|\pi_\alpha(xy)f(M_\alpha)\| + \|\pi_\alpha(x^*y^*)f(M_\alpha)\|$$

$$\leq \|\pi_\alpha(x)||\pi_\alpha(y)\|\|f(M_\alpha)\| + \|\pi_\alpha(y^*)f(M_\alpha)\|$$

$$= \|\pi_\alpha(x)||p_\alpha^f(y)\| \leq \|x\|_0 p_\alpha^f(y).$$

Hence (T3) holds. As for (T4), we begin with noticing that for every $\alpha \in \Sigma$, $\pi_\alpha(\mathcal{A}_0)$ is an $O^*$-algebra of bounded operators in $\mathcal{D}_\alpha$. Hence, its closure in $L^1(D_\alpha, \mathcal{H}_\alpha)[\tau_\pi^\dagger(L^1(D_\alpha, \mathcal{H}_\alpha))]$
is a locally convex C*-normed algebra of operators, by Proposition 6.1. Moreover, every \( \pi_\alpha \) can be extended by continuity to \( \tilde{A}_0 || \cdot ||_0 \). The extension, that we denote by the same symbol, takes values in \( \mathcal{L}(\mathcal{D}_\alpha, \mathcal{H}_\alpha)[\tau^\alpha(\mathcal{L}(\mathcal{D}_\alpha, \mathcal{H}_\alpha))] \), since this space is complete. Now, if \( \{x_\lambda\} \) is a net in \( \mathcal{U}(\tilde{A}_0 || \cdot ||_0)_+ \), \( \tau \)-converging to \( x \in \tilde{A}_0 || \cdot ||_0 \), then \( x = x' \) and \( \pi_\alpha(x_\lambda) \rightarrow \pi_\alpha(x) \) in \( \mathcal{L}(\mathcal{D}_\alpha, \mathcal{H}_\alpha)[\tau^\alpha(\mathcal{L}(\mathcal{D}_\alpha, \mathcal{H}_\alpha))] \), for every \( \alpha \in \Sigma \). Thus \( \pi_\alpha(x) \geq 0 \) and \( ||\pi_\alpha(x)|| \leq 1 \), for every \( \alpha \in \Sigma \), since the same is true for every \( x_\lambda \). By constructing a faithful representation \( \pi \) by direct sum of the \( \pi_\alpha \)'s, one easily realizes that \( x \geq 0 \) and \( ||x||_0 \leq 1 \). The inclusion \( \tilde{A}_0[\tau]_q + \cap \tilde{A}_0 || \cdot ||_0 \subset \tilde{A}_0 || \cdot ||_0 \) in Condition \( (T_5) \) can be proved in similar fashion. The converse inclusion comes from Lemma 4.5. Thus Condition \( (T_5) \) holds.

Then we conclude that

**Statement 7.1.** \( A = \tilde{A}_0[\tau] \) is a locally convex quasi C*-normed algebra, which can be understood as the quasi *-algebra of the observables of the physical system.

A more concrete realization of the situation discussed above is obtained for the so-called BCS model. Let \( V \) be a finite region of a \( d \)-dimensional lattice \( \Lambda \) and \( |V| \) the number of points in \( V \). The local C*-algebra \( A_V \) is generated by the Pauli operators \( \vec{\sigma}_p = (\sigma^1_p, \sigma^2_p, \sigma^3_p) \) and by the unit \( 2 \times 2 \) matrix \( e_p \) at every point \( p \in V \). The \( \vec{\sigma}_p \)'s are copies of the Pauli matrices localized in \( p \).

If \( V \subset V' \) and \( A_V \in A_{V'} \), then \( A_V \rightarrow A_{V'} = A_V \otimes (\mathcal{1}_{V \backslash V'} \otimes e_p) \) defines the natural imbedding of \( A_V \) into \( A_{V'} \).

Let \( \vec{n} = (n_1, n_2, n_3) \) be a unit vector in \( \mathbb{R}^3 \), and put \( (\vec{\sigma} \cdot \vec{n}) = n_1\sigma^1 + n_2\sigma^2 + n_3\sigma^3 \). Then, denoting as \( Sp(\vec{\sigma} \cdot \vec{n}) \) the spectrum of \( \vec{\sigma} \cdot \vec{n} \), we have \( Sp(\vec{\sigma} \cdot \vec{n}) = \{1, -1\} \). Let \( |\vec{n}| \in \mathbb{C}^2 \) be a unit eigenvector associated with 1.

Let now denote by \( \mathbf{n} := (\vec{n}_p)_{p \in \Lambda} \) an infinite sequence of unit vectors in \( \mathbb{R}^3 \) and \( |\mathbf{n}| = \bigotimes_p |\vec{n}_p| \) the corresponding unit vector in the infinite tensor product \( \mathcal{H}_\infty = \bigotimes_p \mathbb{C}^2 \).

We put \( A_0 = \bigcup_V A_V \) and \( D^0_\mathbf{n} = A_0|\mathbf{n}| \) and we denote the closure of \( D^0_\mathbf{n} \) in \( \mathcal{H}_\infty \) by \( \mathcal{H}_\mathbf{n} \).

As we saw above, to any sequence \( \mathbf{n} \) of three-vectors there corresponds a state \( \rho_{\mathbf{n}} \) of the system. Such a state defines a realization \( \pi_{\mathbf{n}} \) of \( A_0 \) in the Hilbert space \( \mathcal{H}_\mathbf{n} \). This representation is faithful, since the norm completion \( \mathcal{A}_S \) of \( A_0 \) is a simple C*-algebra.

A special basis for \( \mathcal{H}_\mathbf{n} \) is obtained from the ground state \( |\mathbf{n}| \) by flipping a finite number of spins using the following strategy;

Let \( \vec{n} \) be a unit vector in \( \mathbb{R}^3 \), as above, and \( |\vec{n}| \) the corresponding vector of \( \mathbb{C}^2 \). Let us choose two other unit vectors \( \vec{n}^1, \vec{n}^2 \) so that \( (\vec{n}, \vec{n}^1, \vec{n}^2) \) form an orthonormal basis of \( \mathbb{R}^3 \). We put \( \vec{n}_\pm = \frac{1}{2}(\vec{n}^1 \pm i\vec{n}^2) \) and define \( |m, \vec{n}| := (\vec{\sigma} \cdot \vec{n}_-)^m|\vec{n}| \) \( (m = 0, 1) \). Then we have

\[
(\vec{\sigma} \cdot \vec{n})|m, \vec{n}| = (-1)^m|m, \vec{n}| \quad (m = 0, 1).
\]
Thus, the set \( \{ |m, n⟩ = \bigotimes_p |m_p, \bar{n}_p⟩; \ m_p = 0, 1, \ \sum_p m_p < ∞ \} \) forms an orthonormal basis in \( \mathcal{H}_n \).

In this space we define the unbounded self-adjoint operator \( M_n \) by

\[(7.1) \quad M_n |m, n⟩ = (1 + \sum_p m_p) |m, n⟩.\]

\( M_n \) counts the number of the flipped spins in \( |m, n⟩ \) with respect to the ground state \( |n⟩ \). Now we put \( \mathcal{D}_n = \bigcap_k \mathcal{D}(M_n^k) \).

The representation \( \pi_n \) is defined on the basis vectors \( \{ |m, n⟩ \} \) by

\[\pi_n (\sigma_p^i) |m, n⟩ = \sigma_p^i \big| m_p, \bar{n}_p⟩ \bigotimes (\prod_{p' \neq p} | m_{p'}, \bar{n}_{p'}⟩) \quad (i = 1, 2, 3).\]

This definition is then extended in obvious way to the whole space \( \mathcal{H}_n \). It turns out that \( \pi_n \) is a bounded representation of \( \mathcal{A}_0 \) into \( \mathcal{H}_n \). For more details we refer to [20, 11]. Hence, the procedure outlined above applies, showing that a natural framework for discussing the BCS model is, indeed, provided by locally convex quasi C*-normed algebras considered in this paper. We argue that an analysis similar to that of [11] can be carried out also in the present context, so that for suitable finite volume hamiltonians, the thermodynamical limit of the local dynamics can be appropriately defined in \( \tilde{\mathcal{A}}_0[τ] \).

References


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