Representations and derivations of quasi \(*\)-algebras induced by \textit{local modifications} of states

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Abstract
The relationship between the GNS representations associated to states on a quasi \(*\)-algebra, which are \textit{local modifications} of each other (in a sense which we will discuss) is examined. The role of local modifications on the spatiality of the corresponding induced derivations describing the dynamics of a given quantum system with infinite degrees of freedom is discussed.
I Introduction and preliminaries

In two recent papers, [1, 2], we have investigated the role of derivations of quasi *-algebras and the possibility of finding a certain symmetric operator which implements the derivation, in the sense that in a suitable representation the derivation can be written as a commutator with an operator which in the physical literature is usually called the effective hamiltonian. This is useful for physical applications and produces an algebraic framework in which the time evolution of some physical model can be analyzed, [3].

Here we continue our analysis, taking inspiration again from physical motivations: it is known [4] that in a physical context local modifications do not affect much the main physical results. Our interest here is to understand this statement more in detail, mainly in the framework of quasi *-algebras which, as we have discussed in several other places, see [5, 3, 6], in our opinion play an important role in the mathematical description of quantum mechanical systems with infinite degrees of freedom.

Just as an introductory example, let us consider a C*-algebra $A$ with unit $e$, and let $\omega$ and $\omega'$ be two (different) positive linear functionals on $A$. Let further $(\pi_\omega, \xi_\omega, H_\omega)$ and $(\pi_{\omega'}, \xi_{\omega'}, H_{\omega'})$ be their associated GNS-representations. An interesting problem is the following: under which conditions on $\omega$ and $\omega'$ are the representations $\pi_\omega$ and $\pi_{\omega'}$ unitarily equivalent?

It is somehow more convenient to consider first the following preliminary problem: how must $\omega$ and $\omega'$ be related for $\pi_{\omega'}$ to be unitarily equivalent to a sub *-representation of $\pi_\omega$? An easy proof shows that

$\pi_{\omega'}$ is unitarily equivalent to a sub *-representation of $\pi_\omega$ if, and only if, there exists a sequence $\{b_n\}$ of elements of $A$ such that $\omega'(a) = \lim_{n \to \infty} \omega(b_n^* a b_n) \forall a \in A$, and the sequence $\{\pi_\omega(b_n) \xi_\omega\}$ converges in $H_\omega$.

We refer to [4] for the physical implications of this result. Here we observe that, in particular, if $\omega$ is a positive linear functional on $A$, and $b \in A$ a fixed element such that $\omega(b^* b) \neq 0$, then the GNS-representation associated to $\omega_b(\cdot) = \omega(b^* \cdot b)$ is unitarily equivalent to a sub *-representation of $\pi_\omega$. This means that there exists a subspace $H^b_\omega$ of $H_\omega$ and a unitary operator $U : H_{\omega_b} \to H^b_\omega$ such that $\pi_{\omega_b}(a) = U^* \pi^b_\omega(a) U$ for all $a \in A$, where $\pi^b_\omega(a) := \pi_\omega(a) |_{H^b_\omega}$.

Going back to our original question, i.e. to the unitary equivalence of $\pi_\omega$ and $\pi_{\omega'}$, we will postpone this analysis to the next section, where the more relevant case of quasi *-algebras is discussed.

Let now $\delta$ be a *-derivation on $A$ and let us define $\delta_{\pi_\omega}(a) = \pi_\omega(\delta(a))$ and $\delta_{\pi_{\omega'}}(a) =$
πωb(δ(a)), a ∈ A. The first obvious remark is that, under our assumptions,

\[ \delta_{πωb}(a) = πωb(δ(a)) = U^*πω_b^h(δ(a))U = U^*δ_{πωb}(a)U. \]

Secondly, if \( δ_{πωb}(a) \) is spatial, i.e. there exists an element \( H_{πωb} ∈ B(\mathcal{H}_ω^b) \) such that \( δ_{πωb}(a) = i[H_{πωb}, πω_b(a)], a ∈ A, \) then \( δ_{πωb} \) is also spatial and the implementing operator is \( H_{πωb} = U^*H_{πωb}U, \) which belongs to \( B(\mathcal{H}_ω). \)

From a physical point of view we can interpret this result as follows: it is well known that no hamiltonian operator exists in general which implements the dynamics of an infinitely extended system, [4]. For this reason one has to consider a finite-volume approximation of the system, for which a self-adjoint energy operator \( H_V \) can be defined. Associated to \( H_V \) we can introduce a finite-volume derivation \( \delta_V(X) = i[H_V, X] \), for each observable \( X \) localized in \( V \), and a time evolution \( \alpha_t^V(X) = e^{iH_Vt}Xe^{-iH_Vt}. \) However, usually, neither \( \delta_V(X) \) nor \( \alpha_t^V(X) \) converge in the uniform, strong or weak topology. One usually has to consider some representation of the abstract algebra and, as in [1], the corresponding family of effective derivations, i.e. derivations in the given representation. This net of derivations may now be converging and, under suitable conditions, it still defines a derivation whose implementing operator is the effective hamiltonian. Therefore the choice of the representations in this procedure is crucial. Our results show that, in fact, there is no essential difference between the effective hamiltonians that we obtain starting from two different representations, at least if they are GNS generated by a fixed positive linear functional \( ω \) and by a different positive linear functional \( ω' = ω_b, \) for each possible choice of \( b ∈ A. \) In particular this implies that, if \( b \) is a local observable (meaning by this that it belongs to some of the \( A_V \)’s which produce the quasi local C*-algebra, [4, 3]), then the two related derivations are unitarily equivalent and, consequently, the two effective hamiltonians are unitarily equivalent as well. Hence their physical content is essentially the same, as claimed before.

II The case of quasi *-algebras

We begin this section with recalling briefly the definitions of *-representations and sub *-representations of quasi *-algebras. More details can be found in [5].

Let \( D \) be a dense subspace of a Hilbert space \( \mathcal{H}. \) We denote by \( L^d(D, \mathcal{H}) \) the set of all (closable) linear operators \( X \) such that \( D(X) = D, \) \( D(X^*) ⊇ D. \)

The set \( L^d(D, \mathcal{H}) \) is a partial *-algebra with respect to the following operations: the usual sum \( X_1 + X_2, \) the scalar multiplication \( λX, \) the involution \( X ↦ X^t = X^*[D \) and the (weak) partial multiplication \( X_1 □ X_2 = X_1^*X_2, \) defined whenever \( X_2 \) is a weak right
multiplier of $X_1$ (we shall write $X_2 \in R^n(X_1)$ or $X_1 \in L^n(X_2)$), that is, $X_2D \subset D(X_1^*)$ and $X_1^*D \subset D(X_2^*)$.

Let $\mathcal{L}^1(D)$ be the subspace of $\mathcal{L}^1(D, \mathcal{H})$ consisting of all its elements which leave, together with their adjoints, the domain $D$ invariant. Then $\mathcal{L}^1(D)$ is a *-algebra with respect to the usual operations.

Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi *-algebra with identity $e$ and $D_\pi$ a dense domain in a certain Hilbert space $\mathcal{H}_\pi$. A linear map $\pi$ from $\mathcal{A}$ into $\mathcal{L}^1(D_\pi, \mathcal{H}_\pi)$ such that:

(i) $\pi(a^*) = \pi(a)^*$, $\forall a \in \mathcal{A}$,

(ii) if $a \in \mathcal{A}$, $x \in \mathcal{A}_0$, then $\pi(a)x$ is well defined and $\pi(ax) = \pi(a)\pi(x)$,

is called a *-representation of $\mathcal{A}$. Moreover, if

(iii) $\pi(\mathcal{A}_0) \subset \mathcal{L}^1(D_\pi)$,

then $\pi$ is said to be a *-representation of the quasi *-algebra $(\mathcal{A}, \mathcal{A}_0)$.

If $\pi$ is a *-representation of $(\mathcal{A}, \mathcal{A}_0)$, then the closure $\overline{\pi}$ of $\pi$ is defined, for each $x \in \mathcal{A}$, as the restriction of $\overline{\pi(x)}$ to the domain $\overline{D}_\pi$, which is the completion of $D_\pi$ under the graph topology $t_\pi$ [5] defined by the seminorms $\xi \in D_\pi \rightarrow \|\pi(a)\xi\|$, $a \in \mathcal{A}$. If $\pi = \overline{\pi}$ the representation is said to be closed.

The adjoint of a *-representation $\pi$ of a quasi *-algebra $(\mathcal{A}, \mathcal{A}_0)$ is defined as follows:

$$D_\pi^* = \bigcap_{x \in \mathcal{A}} D(\pi(x)^*)$$ and $\pi^*(x) = (\pi(x)^*)^* \mid D_\pi^*$, $x \in \mathcal{A}$.

The representation $\pi$ is said to be self-adjoint if $\pi = \pi^*$.

The representation $\pi$ is said to be ultra-cyclic if there exists $\xi_0 \in D_\pi$ such that $D_\pi = \pi(\mathcal{A}_0)\xi_0$, while is said to be cyclic if there exists $\xi_0 \in D_\pi$ such that $\pi(\mathcal{A}_0)\xi_0$ is dense in $D_\pi$ w.r.t. $t_\pi$.

**Definition 1** Let $\pi$ be a *-representation of $\mathcal{A}$. A subset $\mathcal{M} \subset D_\pi$ is a reducing subspace for $\pi$ if $\pi(\mathcal{A}_0)\mathcal{M} \subset \mathcal{M}$ and $\pi(\mathcal{A})\mathcal{M} \subset \overline{\mathcal{M}}$, the closure of $\mathcal{M}$ in the Hilbert norm of $\mathcal{H}_\pi$. Moreover the reducing subspace $\mathcal{M}$ is called ultra-cyclic if there exists $\xi_0 \in \mathcal{M}$ such that $\mathcal{M} = \pi(\mathcal{A}_0)\xi_0$. $\mathcal{M}$ is called cyclic if there exists $\xi_0 \in \mathcal{M}$ such that $\pi(\mathcal{A}_0)\xi_0$ is dense in $\mathcal{M}$ w.r.t. $t_\pi$.

**Proposition 2** Let $\pi$ be a *-representation of $\mathcal{A}$ and $\mathcal{M}$ a reducing subspace of $D_\pi$ for $\pi$. We put

$$\begin{cases} D_{\pi\mid\mathcal{M}} := \mathcal{M}, \\ (\pi\mid\mathcal{M})(x) := \pi(x)\mid\mathcal{M}, & x \in \mathcal{A}. \end{cases}$$

Then $\pi\mid\mathcal{M}$ is a *-representation of $\mathcal{A}$ with domain $\mathcal{M}$ in $\overline{\mathcal{M}}$. Let $\pi\mathcal{M}$ denote the closure of $\pi\mid\mathcal{M}$. Then...
(i) if $\mathcal{M}$ is ultra-cyclic then $\pi|\mathcal{M}$ is ultra-cyclic and $\pi_{\mathcal{M}}$ is cyclic;

(ii) if $\mathcal{M}$ is cyclic then $\pi|\mathcal{M}$ and $\pi_{\mathcal{M}}$ are cyclic.

In the sequel we will also need the following definitions:

**Definition 3** Let $\rho$ and $\pi$ be $^*$-representations of $\mathcal{A}$ respectively on $\mathcal{D}_\rho \subset \mathcal{H}_\rho$ and $\mathcal{D}_\pi \subset \mathcal{H}_\pi$. Then $\rho$ and $\pi$ are unitarily equivalent if there exists a unitary operator $U : \mathcal{H}_\rho \rightarrow \mathcal{H}_\pi$ such that $U\mathcal{D}_\rho = \mathcal{D}_\pi$ and $\rho(x) = U^*\pi(x)U$, for all $x \in \mathcal{A}$.

**Definition 4** Let $\pi$ be a $^*$-representation of $\mathcal{A}$. Then $\pi'$ is a sub $^*$-representation of $\pi$ if and only if $\pi' = \pi|\mathcal{M}$, for a certain reducing subspace $\mathcal{M}$ of $\mathcal{D}_\pi$. Furthermore $\pi'$ is a closed sub $^*$-representation of $\pi$ if and only if $\pi' = \pi_{\mathcal{M}}$, for a certain reducing subspace $\mathcal{M}$ of $\mathcal{D}_\pi$.

The following proposition, proved by one of us in [7], extends the GNS construction to quasi $^*$-algebras.

**Proposition 5** Let $\omega$ be a linear functional on $\mathcal{A}$ satisfying the following requirements:

(L1) $\omega(a^*a) \geq 0$ for all $a \in \mathcal{A}_0$;
(L2) $\omega(b^*x^*a) = \overline{\omega(a^*xb)}$, $\forall a, b \in \mathcal{A}_0$, $x \in \mathcal{A}$;
(L3) $\forall x \in \mathcal{A}$ there exists $\gamma_x > 0$ such that $|\omega(x^*a)| \leq \gamma_x \omega(a^*a)^{1/2}$.

Then there exists a triple $(\pi_\omega, \lambda_\omega, \mathcal{H}_\omega)$ such that

- $\pi_\omega$ is a ultra-cyclic $^*$-representation of $\mathcal{A}$ with ultra-cyclic vector $\xi_\omega$;
- $\lambda_\omega$ is a linear map of $\mathcal{A}$ into $\mathcal{H}_\omega$ with $\lambda_\omega(\mathcal{A}_0) = \mathcal{D}_{\pi_\omega}$, $\xi_\omega = \lambda_\omega(e)$ and $\pi_\omega(x)\lambda_\omega(a) = \lambda_\omega(xa)$, for every $x \in \mathcal{A}$, $a \in \mathcal{A}_0$;
- $\omega(x) = \langle \pi_\omega(x)\xi_\omega | \xi_\omega \rangle$, for every $x \in \mathcal{A}$.

The representation $\pi_\omega$ satisfies the properties: (1) $\pi_{\omega_0} = \pi_\omega |_{\mathcal{A}_0}$; (2) $\pi_\omega(x)\lambda_\omega(a) = \lambda_\omega(xa)$, $x \in \mathcal{A}$, $a \in \mathcal{A}_0$ and (3) $\pi_\omega^*(a)\lambda_\omega(x) = \lambda_\omega(ax)$, $x \in \mathcal{A}$, $a \in \mathcal{A}_0$. Here $\pi_\omega^*$ denotes the adjoint representation of $\pi$, see [5].

For shortness, a linear functional $\omega$ on $\mathcal{A}$ satisfying (L1)-(L3) will be called a representable functional on $\mathcal{A}$. If $\omega$ is representable, $(\pi_\omega, \lambda_\omega, \mathcal{H}_\omega)$ will be called, as usual, the GNS construction for $\omega$.

It is possible to check that conditions (L1)-(L3) are stable under the map $\omega \rightarrow \omega_b$, with $b \in \mathcal{A}_0$. This means that, if $\omega$ is representable, then $\omega_b$ is representable, for every $b \in \mathcal{A}_0$. We only prove (L3) since (L1) and (L2) are trivial. We have

$$|\omega_b(x^*a)| = |\omega((xb)^*)ab| \leq \gamma_{xb}\omega((ab)^*ab)^{1/2} = \gamma_{xb}\omega_b(a^*a)^{1/2}.$$
Hence $\omega_b$ produces a GNS representation as well, so that it is worth comparing the two representations arising from $\omega$ and $\omega_b$, in view of extending to quasi $*$-algebras what we discussed in the first section for $C^*$-algebras.

We start with considering the following question: \textit{when a representable linear functional $\omega'$ can be written as $\omega' = \omega_b$, for some $b \in A_0$?} To answer this question we give the following

**Proposition 6** Let $\omega'$ and $\omega$ be representable linear functionals on $A$. Then $\omega' = \omega_b$ for some $b \in A_0$ if and only if $\pi_{\omega'}$ is unitarily equivalent to a sub $*$-representation of $\pi_{\omega}$.

**Proof:** Suppose first that $\omega' = \omega_b$ for some $b \in A_0$. For every $x \in A$ and $a, c \in A_0$, we have

$$\omega_b(c^*xa) = \langle \pi_{\omega_b}(x)\lambda_{\omega_b}(a)|\lambda_{\omega_b}(c)\rangle. \tag{2.1}$$

On the other hand,

$$\omega_b(c^*xa) = \omega(b^*c^*xab) = \langle \pi_{\omega}(x)\pi_{\omega}(a)\lambda_{\omega}(b)|\pi_{\omega}(c)\lambda_{\omega}(b)\rangle. \tag{2.2}$$

Now put $H^b_{\omega} := \pi_{\omega}(A_0)\lambda_{\omega}(b)$. Then, from equality (2.1), it follows that there exists a unitary operator $U : H^b_{\omega} \to H_{\omega_b}$ such that

$$U\pi_{\omega}(a)\lambda_{\omega}(b) = \lambda_{\omega_b}(a), \quad \forall a \in A_0.$$ From (2.2) we deduce that, for every $a \in A$ and $a, c \in A_0$, \n
$$\langle \pi_{\omega}(x)\pi_{\omega}(a)\lambda_{\omega}(b)|\pi_{\omega}(c)\lambda_{\omega}(b)\rangle = \langle \pi_{\omega_b}(x)\lambda_{\omega_b}(a)|\lambda_{\omega_b}(c)\rangle = \langle \pi_{\omega_b}(x)U\pi_{\omega}(a)\lambda_{\omega}(b)|U\pi_{\omega}(c)\lambda_{\omega}(b)\rangle = \langle U^*\pi_{\omega_b}(x)U\pi_{\omega}(a)\lambda_{\omega}(b)|\pi_{\omega}(c)\lambda_{\omega}(b)\rangle.$$

This implies that \n
$$\pi^b_{\omega}(x) := \pi_{\omega}(x)|\pi_{\omega}(A_0)\lambda_{\omega}(b) = U^*\pi_{\omega_b}(x)U|\pi_{\omega}(A_0)\lambda_{\omega}(b).$$

Hence, $\pi_{\omega}(A_0)\lambda_{\omega}(b)$ is a reducing subspace for $\pi_{\omega}$, that is, $\pi_{\omega}(A)\pi_{\omega}(A_0)\lambda_{\omega}(b) \subseteq \overline{\pi_{\omega}(A_0)\lambda_{\omega}(b)}$ and so $\pi^b_{\omega}$ is a sub $*$-representation of $\pi_{\omega}$ with ultra-cyclic vector $\lambda_{\omega}(b)$, and it is unitarily equivalent to $\pi_{\omega_b}$.

Conversely, suppose that $\pi_{\omega'}$ is unitarily equivalent to a sub $*$-representation of $\pi_{\omega}$. Then there exists a reducing subspace $\mathcal{M}$ of $D_{\pi_{\omega}}$, and a unitary operator $U : H_{\omega'} \to \overline{\mathcal{M}} \subset H_{\omega}$ such that $U\lambda_{\omega'}(A_0) = \mathcal{M} \subset \lambda_{\omega}(A_0) = D_{\pi_{\omega}}$ and $\pi_{\omega'}(x) = U^*(\pi[\mathcal{M}](x))U$, $\forall x \in A$. \n
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Let \( \omega' \) be such that \( U \omega' = \omega \). Then, for every \( x \in A \),
\[
\omega'(x) = \langle \pi_{\omega'}(x) \lambda_{\omega'}(e) | \lambda_{\omega'}(e) \rangle = \langle \pi_{\omega'}(x) U \lambda_{\omega'}(e) | U \lambda_{\omega'}(e) \rangle = \langle \pi_{\omega'}(x) \lambda_{\omega}(b) | \lambda_{\omega}(b) \rangle = \omega_b(x).
\]

\[\square\]

We now consider a slightly generalized problem, looking for conditions under which a representable linear functional \( \omega' \) on \( A \) can be written as \( \omega' = \lim_n \omega_{b_n} \) for some net \( \{b_n\} \) in \( A_0 \).

**Proposition 7** Let \( \omega' \) and \( \omega \) be representable linear functionals on \( A \). Then \( \omega' = \lim_n \omega_{b_n} \) for some net \( \{b_n\} \) in \( A_0 \) such that \( \{\pi_{\omega_{b_n}}(b_n) \xi_b \} \) converges w.r. to \( t_{\pi_{\omega}} \) if, and only if, \( \pi_{\omega'} \) is unitarily equivalent to a sub *-representation of \( \pi_{\omega} \).

**Proof:** Suppose that \( \omega' = \lim_n \omega_{b_n} \), for some net \( \{b_n\} \) in \( A_0 \) such that \( \{\pi_{\omega_{b_n}}(b_n) \xi_b \} \) converges w.r. to \( t_{\pi_{\omega}} \). Then, it is easily shown that \( M := \pi_{\omega}(A_0) \xi_0 \) is a reducing subspace of \( \mathcal{D}_{\pi_{\omega}} \), where \( \xi_0 := t_{\pi_{\omega}} - \lim_n \pi_{\omega}(b_n) \xi_b \). For every \( x \in A \) and every \( a, c \in A_0 \), we have
\[
\langle \pi_{\omega'}(x) \lambda_{\omega'}(a) | \lambda_{\omega'}(c) \rangle = \omega'(c^* x a) = \lim_n \omega(b_n^* e^* x a b_n) = \lim_n \langle \pi_{\omega}(x a) \lambda_{\omega}(b_n) | \pi_{\omega}(c) \lambda_{\omega}(b_n) \rangle = \langle \pi_{\omega} \mathcal{M}(x) \pi_{\omega}(a) \xi_0 | \pi_{\omega}(c) \xi_0 \rangle = \langle (\pi_{\omega} \mathcal{M})(x) \pi_{\omega}(a) | \pi_{\omega}(c) \rangle. \tag{2.3}
\]

Here we put
\[
U \pi_{\omega}(a) \xi_0 = \lambda_{\omega}(a), \quad a \in A_0.
\]

Then \( U \) extends to a unitary operator of \( \overline{\mathcal{M}} \) onto \( \mathcal{H}_{\omega'} \), which we denote with the same symbol, such that \( UM = \lambda_{\omega}(A_0) = \mathcal{D}_{\pi_{\omega}} \). Furthermore, by (2.3), we have
\[
\langle \pi_{\omega'}(x) \lambda_{\omega'}(a) | \lambda_{\omega'}(c) \rangle = \langle (\pi_{\omega} \mathcal{M})(x) \pi_{\omega}(a) \xi_0 | \pi_{\omega}(c) \xi_0 \rangle = \langle (\pi_{\omega} \mathcal{M})(x) U^* \lambda_{\omega'}(a) | U^* \lambda_{\omega'}(c) \rangle = \langle U(\pi_{\omega} \mathcal{M})(x) U^* \lambda_{\omega'}(a) | \lambda_{\omega}(c) \rangle,
\]
for each \( a, c \in A_0 \) and \( x \in A \), which implies that
\[
\pi_{\omega'}(x) = U(\pi_{\omega} \mathcal{M})(x) U^*, \quad \forall x \in A.
\]
Thus $\pi'_\omega$ is unitarily equivalent to a sub *-representation $\tilde{\pi}_\omega[M]$ of $\tilde{\pi}_\omega$. Conversely, suppose $\pi'_{\omega'}$ is unitarily equivalent to a sub *-representation of $\tilde{\pi}_\omega$. Then, there exists a reducing subspace of $D_{\pi'_\omega}$, $M$, and a unitary operator $U : H_{\omega'} \to \overline{M}$ such that $U\lambda'_{\omega'}(A_0) = M \subset D_{\tilde{\pi}_\omega}$ and $\pi'_{\omega}(x) = U^*(\pi'_\omega[M])(x)U$, $\forall x \in \mathcal{A}$. Since $U\lambda'_{\omega'}(e) \in M \subset D_{\tilde{\pi}_\omega}$, there exists $\{b_\alpha\} \subset \mathcal{A}_0$ such that $\lambda'_\omega(b_\alpha) = \pi'_\omega(b_\alpha)\xi_\omega \to U\lambda'_{\omega'}(e)$, in the topology $t_{\pi'_\omega}$. Hence,

$$\omega'(x) = \langle \pi'_\omega(x)\lambda'_{\omega'}(e)|\lambda'_{\omega'}(e) \rangle = \langle \pi'_\omega(x)U\lambda'_{\omega'}(e)|U\lambda'_{\omega'}(e) \rangle = \lim_\alpha \langle \pi'_\omega(x)\lambda'_\omega(b_\alpha)|\lambda'_\omega(b_\alpha) \rangle = \lim_\alpha \omega_{b_\alpha}(x),$$

for every $x \in \mathcal{A}$.

$\square$

The previous propositions, and in particular Proposition 6, show that, for every $b \in \mathcal{A}_0$ such that $\omega(b^*b) \neq 0$, $\omega$ and $\omega_b$ produce close GNS representations and the same physical considerations given in Section I can also be repeated here, with no major change. In particular we consider now some consequences of our results on the theory of spatial derivations in the quasi *-algebraic setting discussed in [1, 2]. To keep the paper self-contained, let us first recall few definitions. Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi *-algebra. A *-derivation of $\mathcal{A}_0$ is a map $\delta : \mathcal{A}_0 \to \mathcal{A}$ with the following properties:

(i) $\delta(a^*) = \delta(a)^*$, $\forall a \in \mathcal{A}_0$;

(ii) $\delta(\alpha a + \beta b) = \alpha \delta(a) + \beta \delta(b)$, $\forall a, b \in \mathcal{A}_0, \forall \alpha, \beta \in \mathbb{C}$;

(iii) $\delta(ab) = \alpha \delta(b) + \delta(a)b$, $\forall a, b \in \mathcal{A}_0$.

Further, let $\pi$ be a *-representation of $(\mathcal{A}, \mathcal{A}_0)$. As in [1] we will always assume that whenever $a \in \mathcal{A}_0$ is such that $\pi(a) = 0$, then $\pi(\delta(a)) = 0$ as well. Under this assumption, the linear map $\delta_\pi(\pi(a)) = \pi(\delta(a))$, $a \in \mathcal{A}_0$, is well-defined on $\pi(\mathcal{A}_0)$ with values in $\pi(\mathcal{A})$ and it is a *-derivation of $\pi(\mathcal{A}_0)$. We call $\delta_\pi$ the *-derivation induced by $\pi$. Given such a representation $\pi$ and its dense domain $D_\pi$, we consider the usual graph topology $t_1$ generated by the seminorms $\xi \in D_\pi \to \|A\xi\|$, $A \in \mathcal{L}(D_\pi)$.

If $D'_\pi$ denotes the conjugate dual of $D_\pi$, we get the usual rigged Hilbert space $D_\pi[t_1] \subset H_\pi \subset D'_\pi[t'_1]$, where $t'_1$ is the strong dual topology of $D'_\pi$. As usual, we denote by $\mathcal{L}(D_\pi, D'_\pi)$ the space of all continuous linear maps from $D_\pi[t_1]$ into $D'_\pi[t'_1]$. In this case, $\mathcal{L}(D_\pi, D'_\pi) \subset \mathcal{L}(D_\pi, D'_\pi)$. Each operator $A \in \mathcal{L}(D_\pi)$ can be extended to the whole $D'_\pi$ by putting

$$\langle A\xi', \eta \rangle = \langle \xi', A^\dagger \eta \rangle, \forall \xi', \eta \in D'_\pi, \eta \in D_\pi.$$
where \(< \cdot, \cdot >\) denotes the form which puts \(\mathcal{D}_\pi\) and \(\mathcal{D}'_\pi\) in conjugate duality. Hence the multiplication of \(X \in \mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)\) and \(A \in \mathcal{L}^1(\mathcal{D}_\pi)\) can always be defined. Indeed, [1], \((X \circ A)\xi = X(A\xi)\), and \((A \circ X)\xi = \hat{A}(X\xi)\), \(\forall \xi \in \mathcal{D}_\pi\). With these definitions \((\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi), \mathcal{L}^1(\mathcal{D}_\pi))\) is a quasi *-algebra.

Given a derivation \(\delta\) of \((\mathcal{A}, \mathcal{A}_0)\) and a *-representation \(\pi\) of \((\mathcal{A}, \mathcal{A}_0)\), that we suppose to be cyclic with cyclic vector \(\xi_0\), the induced derivation \(\delta_\pi\) is spatial if there exists \(H_\pi = H^1_\pi \in \mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)\) such that \(H_\pi \xi_0 \in \mathcal{H}_\pi\) and

\[
\delta_\pi(\pi(x)) = i\{H_\pi \circ \pi(x) - \pi(x) \circ H_\pi\}, \quad \forall x \in \mathcal{A}_0.
\]

Let \((\mathcal{A}, \mathcal{A}_0)\) be a locally convex quasi *-algebra with locally convex topology \(\tau\). In [1] we have found necessary and sufficient conditions for an induced derivation to be spatial. One of these conditions is the following:

defines a positive linear functional \(f\) on \(\mathcal{A}_0\) such that:

\[
f(a^*a) \leq p(a)^2, \quad \forall a \in \mathcal{A}_0,\tag{2.4}
\]

for some continuous seminorm \(p\) of \(\tau\) and, denoting with \(\tilde{f}\) the continuous extension of \(f\) to \(\mathcal{A}\), the following inequality holds:

\[
|\tilde{f}(\delta(a))| \leq C(\sqrt{f(a^*a)} + \sqrt{f(aa^*)}), \quad \forall a \in \mathcal{A}_0,\tag{2.5}
\]

for some positive constant \(C\).

Suppose now that \(\omega_0\) is a positive linear representable functional on \(\mathcal{A}_0\) satisfying condition (2.4). Let \(\omega := \tilde{\omega}_0\) be the continuous extension of \(\omega_0\) to \(\mathcal{A}\), that is

\[
\omega(x) = \lim_\alpha \omega_0(a_\alpha), \quad x \in \mathcal{A},
\]

where \(a_\alpha\) is a net in \(\mathcal{A}_0\) which converges to \(x\) w. r. to \(\tau\). Then \(\omega\) automatically satisfies conditions (L1), (L2) and (L3). Indeed, (L1) is clear since \(\omega_0\) is positive by assumption. As for (L2), let \(x \in \mathcal{A}\) and \(\{x_\alpha\} \subset \mathcal{A}_0\) be a net \(\tau\)-converging to \(x\). Since \(\omega_0\) is hermitian we have \(\omega_0(b^*x_\alpha^*a) = \overline{\omega_0(a^*x_\alpha b)}\), for all \(a, b \in \mathcal{A}_0\). Because of (2.4), taking the limit on \(\alpha\) of this equality we get (L2). To prove (L3) we first use the Schwarz inequality on \(\mathcal{A}_0\):

\[
|\omega_0(x_\alpha a)| \leq \omega_0(x_\alpha^*x_\alpha)^{1/2} \omega_0(a^*a)^{1/2}. \quad \text{But } \omega_0(x_\alpha^*x_\alpha)^{1/2} \leq p(x_\alpha)^2 \to p(x)^2 \quad \text{so that}
\]

\[
|\omega(xa)| = \lim_\alpha |\omega_0(x_\alpha a)| \leq p(x) \omega(a^*a)^{1/2}
\]

which is (L3).

Suppose that \(\omega_0\) is a positive linear representable functional on \(\mathcal{A}_0\) satisfying both conditions (2.4) and (2.5). Then we consider the question as to whether \((\omega_0)_b\) satisfies
these same conditions. This is important for the following reason. If both $\omega_0$ and $(\omega_0)_b$ satisfy (2.4) and (2.5), then they have continuous extensions $\omega$ and $(\omega_0)_b$ respectively to $\mathcal{A}$ and it turns out that $(\omega_0)_b = \omega_b$. Thus $\omega_b$ satisfies conditions (L1), (L2) and (L3) and both $\delta_{\pi_\omega}$ and $\delta_{\pi_\omega_b}$ are spatial. Hence a relation between the effective hamiltonians can be found.

First we notice that, because of the continuity of the multiplication, we have

\[(\omega_0)_b(a^*a) = \omega_0((ab)^*ab) \leq p(ab)^2 \leq q(a)^2, \quad a \in \mathcal{A}_0\]

for some continuous seminorm $q$ of $\tau$. Moreover we have

**Lemma 8** If $b \in \mathcal{A}_0$ is such that $\pi_\omega(b)$ is bounded, then $\omega_b$ satisfies (2.5).

**Proof:** Since for all $a \in \mathcal{A}_0$ the equality $b^*\delta(a)b = \delta(b^*ab) - \delta(b^*)ab - b^*a\delta(b)$ holds, we have

\[|\omega_b(\delta(a))| = |\omega(b^*\delta(a)b)| \leq |\omega(\delta(b^*ab))| + |\omega(\delta(b^*)ab)| + |\omega(b^*a\delta(b))|.

Using (2.5) for the first and introducing $\pi_\omega$ for the second and the third contributions above, we find that, for every $a \in \mathcal{A}_0$,

\[
|\omega_b(\delta(a))| \leq C \left( \omega(b^*a^*b^*ab)^{1/2} + \omega(b^*abb^*a^*b)^{1/2} \right) \\
+ |\langle \omega(ab) | \omega(\delta(b)) \rangle| + |\langle \omega(\delta(b)) | \omega(ab) \rangle| \\
= C \left( \| \pi_\omega(b) \| \omega(\delta(b)) \right) + \| \pi_\omega(b)^* \omega(ab) \| \\
+ |\langle \omega(ab) | \omega(\delta(b)) \rangle| + |\langle \omega(\delta(b)) | \omega(ab) \rangle| \\
\leq \left( C \| \pi_\omega(b) \| + \| \omega(\delta(b)) \| \right) \left( \omega_b(a^*a)^{1/2} + \omega_b(aa^*)^{1/2} \right),
\]

Thus we have the following

**Proposition 9** Let $(\mathcal{A}, \mathcal{A}_0)$ be a locally convex quasi *-algebra with locally convex topology $\tau$, $\delta$ a *-derivation of $(\mathcal{A}, \mathcal{A}_0)$ and $\omega_0$ a positive linear functional on $\mathcal{A}_0$.

1. Suppose that $\omega_0$ satisfies the condition

\[\omega_0(a^*a) \leq p(a)^2, \quad \forall a \in \mathcal{A}_0\]

for some continuous seminorm $p$ of $\tau$. Then the continuous extension $\omega := \omega_0$ of $\omega_0$ to $\mathcal{A}$ and every $\omega_b, b \in \mathcal{A}_0$, produce the ultra-cyclic GNS-representations $\pi_\omega$ and $\pi_{\omega_b}$.

2. Furthermore, suppose that

\[|\omega(\delta(a))| \leq C \left( \sqrt{\omega(a^*a)} + \sqrt{\omega(aa^*)} \right), \quad \forall a \in \mathcal{A}_0\]
for some positive constant $C$. Then the $*$-derivation $\delta_{\pi_\omega}$ induced by $\pi_\omega$ is spatial. If $\pi_\omega(\mathcal{A}_0)$ is bounded, in particular in case $\mathcal{A}_0$ is a C*-algebra, then the $*$-derivation $\delta_{\pi_\omega b}$ induced by $\pi_\omega b$ is also spatial for every $b \in \mathcal{A}_0$.

The conclusion is therefore that, under mild conditions on $\pi_\omega$, and therefore on $\omega$, both $\delta_{\pi_\omega}$ and $\delta_{\pi_\omega b}$ turn out to be spatial so that two different effective hamiltonians $H_\omega$ and $H_{\omega b}$ do exist, and they are related as in Section I. Once again, the physical contents of the two representations is essentially the same.

We end this section with some further results on the GNS representations of a quasi $*$-algebra $(\mathcal{A}, \mathcal{A}_0)$.

Let $(\mathcal{A}, \mathcal{A}_0)$ be a locally convex quasi $*$-algebra, $\omega_0$ a positive linear functional on $\mathcal{A}_0$ satisfying (2.4) and $\omega = \tilde{\omega}_0$ its continuous extension on $\mathcal{A}$. As we have shown, both $\omega$ and $\omega_0$, $b \in \mathcal{A}_0$, satisfy conditions (L1), (L2) and (L3), and so the GNS-constructions $(\pi_\omega, \lambda_\omega, H_\omega)$ and $(\pi_\omega b, \lambda_\omega b, H_{\omega b})$ are defined. Let $\tilde{\pi}_\omega$ and $\tilde{\pi}_\omega b$ be the closures of $\pi_\omega$ and $\pi_\omega b$, respectively. In this section we find conditions which imply that $\tilde{\pi}_\omega$ is unitarily equivalent to the direct sum of a family of $\tilde{\pi}_\omega b$, $b \in \mathcal{A}_0$.

Lemma 10 Let $x \in \mathcal{A}$ and $\{x_\alpha\} \subset \mathcal{A}_0$ such that $\tau - \lim_\alpha x_\alpha = x$, then $\lambda_\omega(x_\alpha) = \lambda_{\omega_0}(x_\alpha) \to \lambda_\omega(x)$.

Proof: We begin with proving that $\{\lambda_\omega(x_\alpha)\}$ is a Cauchy net in the Hilbert space $\mathcal{H}_\omega$:

$$\|\lambda_\omega(x_\alpha) - \lambda_\omega(x_\beta)\|^2 = \omega((x_\alpha - x_\beta)^*(x_\alpha - x_\beta)) \leq p(x_\alpha - x_\beta)^2 \to 0.$$ 

Therefore there exists a vector $\xi \in \mathcal{H}_\omega$ such that $\lambda_\omega(x_\alpha) \to \xi$. We now prove that $\xi = \lambda_\omega(x)$. Indeed we have, for every $c \in \mathcal{A}_0$, $\langle \lambda_\omega(x_\alpha)|\lambda_\omega(c)\rangle \to \langle \xi|\lambda_\omega(c)\rangle$ and, on the other hand, $\langle \lambda_\omega(x_\alpha)|\lambda_\omega(c)\rangle = \omega(c^*x_\alpha) \to \tilde{\omega}(c^*x) = \langle \lambda_\omega(x)|\lambda_\omega(c)\rangle$, due to the definition of $\tilde{\omega}$. Therefore $\xi = \lambda_\omega(x)$. $\Box$

We recall that the weak commutant $\mathcal{M}_w'$ of a $*$-invariant subset $\mathcal{M}$ of $\mathcal{L}(\mathcal{D}, \mathcal{H})$ is defined as

$$\mathcal{M}_w' = \{C \in \mathcal{B}(\mathcal{H}) : \langle X\xi|C^*\eta\rangle = \langle C\xi|X^\dagger\eta\rangle, \forall X \in \mathcal{M}, \xi, \eta \in \mathcal{D}\}.$$ 

Then we can prove the following

Proposition 11 $\pi_\omega(\mathcal{A})_w' = \pi_\omega(\mathcal{A}_0)_w'$. 

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Proof: The inclusion \( \pi_\omega(A)'_w \subset \pi_\omega(A_0)'_w \) is clear. To prove the converse inclusion we take \( C \in \pi_\omega(A_0)'_w \) and \( x \in A, c_1, c_2 \in A_0 \). Then we have, using the previous Lemma,

\[
\langle C\pi_\omega(x)\lambda_\omega(c_1)|\lambda_\omega(c_2) \rangle = \lim_{\alpha} \langle C\pi_\omega(x^\alpha)\lambda_\omega(c_1)|\lambda_\omega(c_2) \rangle = \langle C\lambda_\omega(c_1)|\pi_\omega(x^\alpha)\lambda_\omega(c_2) \rangle.
\]

Let \( b \in A_0 \). We denote by \( P^b_\omega \) the projection of \( \mathcal{H}_\omega \) onto \( \mathcal{H}^b_\omega = \pi_\omega(A_0)\lambda_\omega(b) \). By Lemma 10 we deduce the following

**Lemma 12** Suppose that \( \pi_\omega(a) \) is bounded for every \( a \in A_0 \). Then \( \pi_\omega(A)'_w \) is a von Neumann algebra and \( P^b_\omega \in \pi_\omega(A)'_w \).

Even if \( \pi_\omega|_A \) is bounded, \( P^b_\omega D(\tilde{\pi}_\omega) \neq D(\tilde{\pi}_\omega) \) in general. Hence we introduce the following notion:

**Definition 13** Let \( b \in A_0 \). We say that \( b \) is a self-adjoint element for \( \pi_\omega \) if \( \tilde{\pi}^b_\omega \) is a self-adjoint *-representation of \( A \).

By (\[5\], Theorem 74.4) we have the following

**Lemma 14** Let \( b \) be a self-adjoint element for \( \pi_\omega \). Then

1. \( P^b_\omega \in \pi_\omega(A_0)'_w = \pi_\omega(A)'_w \)
2. \( P^b_\omega D(\tilde{\pi}_\omega) = D(\tilde{\pi}^b_\omega) \)
3. \( \tilde{\pi}_\omega = (\tilde{\pi}_\omega)P^b_\omega := P^b_\omega \tilde{\pi}_\omega(\cdot)P^b_\omega \).

By Proposition 1, \( \tilde{\pi}^b_\omega \) is unitarily equivalent to \( \tilde{\pi}_{\omega b} \), and by the above Lemma we have the following result, which answer our original question

**Proposition 15** Suppose that \( \tilde{\pi}_\omega \) is self-adjoint. If there exist a family \( \{b_\gamma\}_{\gamma \in \Gamma} \) of self-adjoint elements for \( \pi_\omega \), such that \( \{P^b_\omega\} \) consists of mutually orthogonal projections and \( \sum_{\gamma \in \Gamma} P^b_\omega = I \), then \( \tilde{\pi}_\omega \) is unitarily equivalent to \( \bigoplus_{\gamma \in \Gamma} \tilde{\pi}_{\omega b_\gamma} \).

### III Local modifications of states

We consider now the particular case in which the C*-algebra \( A \) is endowed with a local structure. Following [8] we construct the local C*-algebra as follows.

Let \( \mathcal{F} \) be a set of indexes directed upward and with an orthonormality relation \( \perp \) such that (i.) \( \forall \alpha \in \mathcal{F} \) there exists \( \beta \in \mathcal{F} \) such that \( \alpha \perp \beta \); (ii.) if \( \alpha \leq \beta \) and \( \beta \perp \gamma \),

\(\alpha, \beta, \gamma \in \mathcal{F}\), then \(\alpha \perp \gamma\); (iii.) if, for \(\alpha, \beta, \gamma \in \mathcal{F}\), \(\alpha \perp \beta\) and \(\alpha \perp \gamma\), there exists \(\delta \in \mathcal{F}\) such that \(\alpha \perp \delta\) and \(\delta \geq \beta, \gamma\).

Let now \(\{A_\alpha (\|\cdot\|_\alpha), \alpha \in \mathcal{F}\}\) be a family of \(\mathcal{C}^*\)-algebras with \(\mathcal{C}^*\)-norm \(\|\cdot\|_\alpha\), indexed by \(\mathcal{F}\), such that (a.) if \(\alpha \geq \beta\) then \(A_\alpha \supset A_\beta\); (b.) there exists a unique identity \(e\) for all \(A_\alpha\)’s; (c.) if \(\alpha \perp \beta\) then \(xy = yx\) for all \(x \in A_\alpha, y \in A_\beta\). Let further \(A_0 := \bigcup_\alpha A_\alpha\).

The uniform completion of \(A_0\) is, as it is well known, the quasi-local \(\mathcal{C}^*\)-algebra with the norm \(\|\cdot\|\) inherited by the \(\|\cdot\|_\alpha\)’s. If we take instead the completion of \(A_0\) w.r.t. a locally convex topology \(\tau\) which makes the involution and the multiplications continuous we get, in general, a locally convex quasi \(\ast\)-algebra \(A\) which we call a quasi-local quasi \(\ast\)-algebra.

Given \(x \in A_0\), there will be some \(\beta \in \mathcal{F}\) such that \(x \in A_\beta\). But of course, \(x\) also belongs to many other \(A_\beta\)’s, for instance to all those algebras which contains \(A_\beta\) as a sub-algebra. For this reason we introduce a set \(J_x\), related to \(x \in A_0\), which is defined as follows:

\[
J_x = \{\alpha \in \mathcal{F} \mid x \in A_\alpha\}.
\]

If we now define \(A_\infty = \bigcap_{\alpha \in \mathcal{F}} A_\alpha\), then we will work here under the assumption, which is verified for very general discrete and continuous models [4], that \(\forall x \in A_0, x \notin A_\infty\), there exists \(\alpha_x \in \mathcal{F}\) such that \(\bigcap_{\beta \in J_x} A_\beta = A_{\alpha_x}\). We call \(\alpha_x\) the support of \(x\).

The following definition selects states on \(A\) with a reasonable asymptotic behavior. These states, indeed, factorize on regions far enough from the support of a given element.

**Definition 16** A state \(\omega\) over \(A\) is said to be almost clustering (AC) if, \(\forall b \in A_0\) and \(\forall \epsilon > 0\), there exists \(\alpha \in \mathcal{F}\), \(\alpha \geq \alpha_b\), such that, \(\forall \gamma \perp \alpha\) we have \(|\omega(ab) - \omega(a)\omega(b)| \leq \epsilon \|a\|, \forall a \in A_\gamma\).

Similar definitions are given in many textbooks, like [4], [8] and [9], where the physical motivations are discussed in detail. Related to the notion of factorization is also that of local modification of a given state. Of course, several definitions of local modifications can be introduced. The most natural one is perhaps the following: \(\omega'\) is a local modification of \(\omega\) if there exists \(\alpha \in \mathcal{F}\) such that \(\forall \gamma \in \mathcal{F}, \gamma \perp \alpha, \omega'(a) = \omega(a)\) for all \(a \in A_\gamma\). This simply implies that, outside a fixed region \(\alpha\), the two states coincide. However this condition is rather strong and has no counterpart in the existing literature on this subject and for this reason will not be considered here. To stay in touch with the existing literature, we rather consider the following definitions.

**Definition 17** Given two states \(\omega\) and \(\omega'\) over \(A\), \(\omega'\) is said to be a local modification of type 1 (1LM) of \(\omega\) if, calling \(\pi_\omega\) and \(\pi_{\omega'}\) their associated GNS-representations, \(\pi_{\omega'}\) is unitarily equivalent to a sub \(\ast\)-representation of \(\pi_{\omega}\).

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1 This terminology is due to the fact that, in concrete applications, \(\alpha\) is quite often a given bounded open region in a \(d\)-dimensional space.
Also, $\omega'$ is said to be a local modification of type 2 (2LM) of $\omega$ if $\forall \epsilon > 0$, there exists $\alpha_\epsilon \in \mathcal{F}$ such that, $\forall \gamma \in \mathcal{F}$, $\gamma \perp \alpha_\epsilon$, $|\omega'(x) - \omega(x)| \leq \epsilon \|x\|$, $\forall x \in \mathcal{A}_\gamma$.

These definitions are physically motivated essentially from what is discussed in [4]. Just to clarify the situation if, for instance, $\omega'$ is a 2LM of $\omega$ then they coincide, but for an error of order $\epsilon$, outside a region whose size is, in general, proportional to $1/\epsilon$.

There is an apparent difference between the conditions 1LM and 2LM: if $\omega'$ is a 2LM of $\omega$, then $\omega$ is a 2LM of $\omega'$. This symmetry is not shared by 1LM. We argue that 2LM could be used for the mathematical description of reversible local operations on a given state while 1LM seems to be more appropriate for describing the action of irreversible operations (like a quantum mechanical measurement).

One immediate consequence of the results of Section II and of these definitions is that if $b \in \mathcal{A}_\alpha$ for some $\alpha \in \mathcal{F}$ then the state $\omega_b(.)$ is a 1LM of $\omega$. Less trivial is the proof of the following statement: let the state $\omega$ be AC and $b \in \mathcal{A}_0$ with $\omega(b^\dagger b) = 1$. Then $\omega_b$ is a 2LM of $\omega$. This is not the end of the story. Indeed, let us suppose that $\omega$ is AC and that $\omega'$ is a 1LM of $\omega$. Therefore there exists a sequence $\{b_n\}$ of elements of $\mathcal{A}_0$ such that $\omega'(a) = \lim_{n \to \infty} \omega(b_n^\dagger a b_n)$, $\forall a \in \mathcal{A}$, and the sequence $\{\pi_\omega(b_n)\xi_\omega\}$ converges in $\mathcal{H}_\omega$. We suppose now that there exists $n_0 \in \mathbb{N}$ and $\lambda \in \mathcal{F}$ such that, for all $n \geq n_0$, $b_n \in \mathcal{A}_\lambda$. Then $\omega'$ is also a 2LM of $\omega$. The proof of these statements are easy and will be omitted here.

We end this section, and the paper, with the following example of what a concrete local modification of a state could be.

**Discrete system:** Let $V$ be a finite region of a $d$-dimensional lattice $\Lambda$ and $|V|$ the number of points in $V$. The local $C^*$-algebra $\mathcal{A}_V$ is generated by the Pauli operators $\sigma_p = (\sigma_p^1, \sigma_p^2, \sigma_p^3)$ and by the unit $2 \times 2$ matrix $e_p$ at every point $p \in V$. The $\sigma_p$’s are copies of the Pauli matrices localized in $p$.

If $V \subset V'$ and $\mathcal{A}_V \in \mathcal{A}_{V'}$, then $\mathcal{A}_V \to \mathcal{A}_{V'} = \mathcal{A}_V \otimes (\otimes_{p \in V \setminus V} e_p)$ defines the natural imbedding of $\mathcal{A}_V$ into $\mathcal{A}_{V'}$.

Let $\mathbf{n} = (n_1, n_2, n_3)$ be a unit vector in $\mathbb{R}^3$, and put $(\sigma \cdot \mathbf{n}) = n_1\sigma^1 + n_2\sigma^2 + n_3\sigma^3$. Then, denoting as $\text{Sp}(\sigma \cdot \mathbf{n})$ the spectrum of $\sigma \cdot \mathbf{n}$, we have $\text{Sp}(\sigma \cdot \mathbf{n}) = \{1, -1\}$. Let $|n\rangle$ be a unit eigenvector associated with 1. Let $\{n\} = \{\mathbf{n}_1, \mathbf{n}_2, \ldots\}$ be an infinite sequence of unit vectors in $\mathbb{R}^3$ and $\{|n\rangle\} = \bigotimes_p |n_p\rangle$ the corresponding unit vector in the infinite tensor product $\mathcal{H}_\infty = \bigotimes_p \mathbb{C}_{p}^2$. We put $\mathcal{A}_0 = \bigcup_V \mathcal{A}_V$ and $\mathcal{D}_\infty^0 = \mathcal{A}_0\{n\}$ and we denote the closure of $\mathcal{D}_\infty^0$ in $\mathcal{H}_\infty$ by $\mathcal{H}_\{n\}$. As we saw above, to any sequence $\{n\}$ of three-vectors there corresponds a state $|\{n\}\rangle$ of the system. Such a state defines a realization $\pi_{\{n\}}$ of $\mathcal{A}_0$ in the Hilbert space $\mathcal{H}_\{n\}$. This representation is faithful, since the norm completion $\mathcal{A}_S$ of $\mathcal{A}_0$ is a simple $C^*$-algebra. A special basis for $\mathcal{H}_\{n\}$ is obtained from the ground
state $|\{n\}\rangle$ by flipping a finite number of spins using the following strategy:

let $(n, n^1, n^2)$ be an orthonormal basis of $\mathbb{R}^3$. We put $n^\pm = \frac{1}{2}(n^1 \pm in^2)$ and $|m, n\rangle = (\sigma \cdot n^\pm)^m|n\rangle \quad (m = 0, 1)$. Then we have

$$(\sigma \cdot n)|m, n\rangle = (-1)^m|m, n\rangle \quad (m = 0, 1).$$

Thus the set $\{\{|m\}, \{n\}\rangle = \otimes_p |m_p, n_p\rangle; \quad m_p = 0, 1, \quad \sum_p m_p < \infty\}$ forms an orthonormal basis in $\mathcal{H}_{\{n\}}$, \[10\].

Now we consider the natural representation $\pi_{\{n\}}$ of $A_0$ into some class of operators in the Hilbert space $\mathcal{H}_{\{n\}}$. The representation $\pi_{\{n\}}$ is defined on the basis vectors $\{\{|m\}, \{n\}\rangle\}$ by

$$\pi_{\{n\}}(\sigma^i_p)\{|m\}, \{n\}\rangle = \sigma^i_p|m_p, n_p\rangle \otimes (\prod_{p' \neq p} |m_{p'}, n_{p'}\rangle) \quad (i = 1, 2, 3).$$

This definition is then extended in obvious way to the whole space $\mathcal{H}_{\{n\}}$. It turns out that $\pi_{\{n\}}$ is a bounded representation of $A_0$ into $\mathcal{H}_{\{n\}}$. More details on this construction, particularly in connection with quasi *-algebras, can be found in \[6, 11\].

Let now $\varphi = \otimes_{j \in \Lambda} \varphi_j$ be a fixed normalized vector in $\mathcal{H}_{\{n\}}$ and $\omega$ the related vector state: if $a \in A_0$ then $\omega(a) = \langle \varphi, \pi_{\{n\}}(a)\varphi \rangle$. Let now $x = \prod_{p \in \lambda} \otimes x_p$, for some bounded subset $\lambda$ in $\Lambda$. Here $x_p$ acts on $\mathbb{C}^2_p$ and $\lambda$ is the support of $x$. Let furthermore $\gamma$ be another bounded subset of $\Lambda$, orthogonal to $\lambda$: this means that the sets $\lambda$ and $\gamma$ have empty intersection. Then, we fix $b = \prod_{p \in \gamma} \otimes b_p$, where as before $b_p$ acts on $\mathbb{C}^2_p$. We further assume that $\langle \pi_{\{n\}}(b)\varphi, \pi_{\{n\}}(b)\varphi \rangle = 1$. Then we can check that $\omega(a)$ coincides with $\omega_b(a) = \langle \pi_{\{n\}}(b)\varphi, \pi_{\{n\}}(a)\pi_{\{n\}}(b)\varphi \rangle$, and this is true for all possible choices of $a$ and $b$ which are supported in separated regions. So $\omega_b$ is a local modification of $\omega$ in the strongest sense and, in particular, is a 2LM of $\omega$.

This example shows that the definitions of local modification given here are really physically motivated. States sharing the same properties in the case of continuous physical systems, \[4\], could also be constructed with no major difficulty. To \[4\] we also refer for a more physically-minded discussion on 1LM of states.

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