QUASI *-ALGEBRAS OF MEASURABLE OPERATORS

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Abstract. Non-commutative $L^p$-spaces are shown to constitute examples of a class of Banach quasi *-algebras called CQ*-algebras. For $p \geq 2$ they are also proved to possess a sufficient family of bounded positive sesquilinear forms satisfying certain invariance properties. CQ*-algebras of measurable operators over a finite von Neumann algebra are also constructed and it is proven that any abstract CQ*-algebra $(X, A_0)$ possessing a sufficient family of bounded positive tracial sesquilinear forms can be represented as a CQ*-algebra of this type.

1. Introduction and preliminaries

A quasi *-algebra is a couple $(X, A_0)$, where $X$ is a vector space with involution $\ast$, $A_0$ is a *-algebra and a vector subspace of $X$ and $X$ is an $A_0$-bimodule whose module operations and involution extend those of $A_0$. Quasi *-algebras were introduced by Lassner [1, 2, 12] to provide an appropriate mathematical framework where discussing certain quantum physical systems for which the usual algebraic approach made in terms of C*-algebras revealed to be insufficient. In these applications they usually arise by taking the completion of the C*-algebra of observables in a weaker topology satisfying certain physical requirements. The case where this weaker topology is a norm topology has been considered in a series of previous papers [3]-[7], where CQ*-algebras were introduced: a CQ*-algebra is, indeed, a quasi *-algebra $(X, A_0)$ where $X$ is a Banach space with respect to a norm $\| \cdot \|$ possessing an isometric involution and $A_0$ is a C*-algebra with respect to a norm $\| \cdot \|_0$, which is dense in $X[\| \cdot \|]$. Since any C*-algebra $A_0$ has a faithful *-representation $\pi$, it is natural to pose the question if this completion also can be realized as a quasi *-algebra of operators affiliated to $\pi(A_0)'$. The Segal-Nelson theory [10, 11] of non-commutative integration provides a number of mathematical tools for dealing with this problem.

The paper is organized as follows. In Section 2 we consider non-commutative $L^p$-spaces constructed starting from a von Neumann algebra $\mathcal{M}$ and a normal, semifinite, faithful trace $\tau$ as Banach quasi *-algebras. In particular if $\varphi$ is finite, then it is shown that $(L^p(\varphi), \mathcal{M})$ is a CQ*-algebra. If $p \geq 2$, they even possess a sufficient family of positive sesquilinear forms enjoying certain invariance properties.

In Section 3, starting from a family $\mathcal{F}$ of normal, finite traces on a von Neumann algebra $\mathcal{M}$, we prove that the completion of $\mathcal{M}$ with respect to a norm defined in natural...
way by the family $\mathcal{F}$ is indeed a CQ*-algebra consisting of measurable operators, in
Segal’s sense, and therefore affiliated with $\mathcal{M}$.

Finally, in Section 4, we prove that any CQ*-algebra $(\mathcal{X}, \mathcal{A}_0)$ possessing a sufficient
family of bounded positive tracial sesquilinear forms can be continuously embedded into
the CQ*-algebra of measurable operators constructed in Section 3.

In order to keep the paper sufficiently self-contained, we collect below some prelimi-
nary definitions and propositions that will be used in what follows.

Let $(\mathcal{X}, \mathcal{A}_0)$ be a quasi *-algebra. The unit of $(\mathcal{X}, \mathcal{A}_0)$ is an element $e \in \mathcal{A}_0$ such that
$xe = ex = x$, for every $x \in \mathcal{X}$. A quasi *-algebra $(\mathcal{X}, \mathcal{A}_0)$ is said to be locally convex if
$\mathcal{X}$ is endowed with a topology $\tau$ which makes of $\mathcal{X}$ a locally convex space and such that
the involution $a \mapsto a^*$ and the multiplications $a \mapsto ab$, $a \mapsto ba$, $b \in \mathcal{A}_0$, are continuous.
If $\tau$ is a norm topology and the involution is isometric with respect to the norm, we
say that $(\mathcal{X}, \mathcal{A}_0)$ is a normed quasi *-algebra and, if it is complete, we say it is a Banach
quasi*-algebra.

**Definition 1.1.** Let $(\mathcal{X}, \mathcal{A}_0)$ be a Banach quasi *-algebra with norm $\| \cdot \|_0$ defined on $\mathcal{A}_0$, satisfying the following conditions:

1. $\|a^*a\|_0 = \|a\|_0^2$, $\forall a \in \mathcal{A}_0$;
2. $\|a\| \leq \|a\|_0$, $\forall a \in \mathcal{A}_0$;
3. $\|ax\| \leq \|a\|_0 \|x\|$, $\forall a \in \mathcal{A}_0, x \in \mathcal{X}$;
4. $\mathcal{A}_0[\| \cdot \|_0]$ is complete.

Then we say that $(\mathcal{X}, \mathcal{A}_0)$ is a CQ*-algebra.

**Remark 1.2.** (1) If $\mathcal{A}_0[\| \cdot \|_0]$ is not complete, we say that $(\mathcal{X}, \mathcal{A}_0)$ is a pre CQ*-algebra.
(2) In previous papers the name CQ*-algebra was given to a more complicated structure
where two different involutions were considered on $\mathcal{A}_0$. When these involutions coincide,
we spoke of a proper CQ *-algebra. In this paper only this case will be considered and
so we systematically omit the term proper.

The following basic definitions and results on non-commutative measure theory are
also needed in what follows.

Let $\mathcal{M}$ be a von Neumann algebra and $\varphi$ a normal faithful semifinite trace defined
on $\mathcal{M}_+$. Put

$$J = \{X \in \mathcal{M} : \varphi(|X|) < \infty\}.$$ 

$J$ is a *-ideal of $\mathcal{M}$.

We denote with $\text{Proj}(\mathcal{M})$, the lattice of projections of $\mathcal{M}$.

**Definition 1.3.** A vector subspace $\mathcal{D}$ of $\mathcal{H}$ is said to be strongly dense (resp., strongly
$\varphi$-dense) if

- $U'\mathcal{D} \subset \mathcal{D}$ for any unitary $U'$ in $\mathcal{M}'$;
- there exists a sequence $P_n \in \text{Proj}(\mathcal{M})$: $P_n \mathcal{H} \subset \mathcal{D}$, $P_n \perp 0$ and $(P_n^\perp)$ is a finite
  projection (resp., $\varphi(P_n^\perp) < \infty$).
Clearly, every strongly \( \varphi \)-dense domain is strongly dense.

Throughout this paper, when we say that an operator \( T \) is affiliated with a von Neumann algebra, written \( T \eta \mathcal{M} \), we always mean that \( T \) is closed, densely defined and \( TU \supseteq UT \) for every unitary operator \( U \in \mathcal{M}' \).

**Definition 1.4.** An operator \( T \eta \mathcal{M} \) is called

- measurable (with respect to \( \mathcal{M} \)) if its domain \( D(T) \) is strongly dense;
- \( \varphi \)-measurable if its domain \( D(T) \) is strongly \( \varphi \)-dense.

¿From the definition itself it follows that, if \( T \) is \( \varphi \)-measurable, then there exists \( P \in \text{Proj}(\mathcal{M}) \) such that \( TP \) is bounded and \( \varphi(P^\perp) < \infty \).

We remind that any operator affiliated with a finite von Neumann algebra is measurable [10, Cor. 4.1] but it is not necessarily \( \varphi \)-measurable.

2. Non-commutative \( L^p \)-spaces as CQ *-algebras

In this Section we will discuss the structure of the non-commutative \( L^p \)-spaces as quasi *-algebras. We begin with recalling the basic definitions.

Let \( \mathcal{M} \) be a von Neumann algebra and \( \varphi \) a normal faithful semifinite trace defined on \( \mathcal{M}_+ \). For each \( p \geq 1 \), let

\[
\mathcal{J}_p = \{ X \in \mathcal{M} : \varphi(|X|^p) < \infty \}.
\]

Then \( \mathcal{J}_p \) is a *-ideal of \( \mathcal{M} \). Following [11], we denote with \( L^p(\varphi) \) the Banach space completion of \( \mathcal{J}_p \) with respect to the norm \( \| X \|_p := \varphi(|X|^p)^{1/p} \), \( X \in \mathcal{J}_p \).

One usually defines \( L^\infty(\varphi) = \mathcal{M} \). Thus, if \( \varphi \) is a finite trace, then \( L^\infty(\varphi) \subseteq L^p(\varphi) \) for every \( p \geq 1 \). As shown in [11], if \( X \in L^p(\varphi) \), then \( X \) is a measurable operator.

**Proposition 2.1.** Let \( \mathcal{M} \) be a von Neumann algebra and \( \varphi \) a normal faithful semifinite trace on \( \mathcal{M}_+ \). Then \( (L^p(\varphi), L^\infty(\varphi) \cap L^p(\varphi)) \) is a Banach quasi *-algebra.

If \( \varphi \) is a finite trace and \( \varphi(\mathbb{I}) = 1 \), then \( (L^p(\varphi), L^\infty(\varphi)) \) is a CQ*-algebra.

**Proof.** Indeed, it is easily seen that the norms \( \| \cdot \|_\infty \) of \( L^\infty(\varphi) \cap L^p(\varphi) \) and \( \| \cdot \|_p \) on \( L^p(\varphi) \) satisfy the conditions (a.1)-(a.2) of Definition 1.1. Moreover, if \( \varphi \) is finite, then \( L^\infty(\varphi) \subseteq L^p(\varphi) \) and thus \( (L^p(\varphi), L^\infty(\varphi)) \) is a CQ*-algebra. \( \square \)

**Remark 2.2.** Of course the condition \( \varphi(\mathbb{I}) = 1 \) can be easily removed by rescaling the trace.

**Definition 2.3.** Let \( (\mathfrak{X}, \mathfrak{A}_0) \) be a Banach quasi *-algebra. We denote with \( S(\mathfrak{X}) \) the set of all sesquilinear forms \( \Omega \) on \( \mathfrak{X} \times \mathfrak{X} \) with the following properties

(i) \( \Omega(x, x) \geq 0 \) \( \forall x \in \mathfrak{X} \)
(ii) \( \Omega(xa, b) = \Omega(a, x^*b) \) \( \forall x \in \mathfrak{X}, \forall a, b \in \mathfrak{A}_0 \)
(iii) \( \| \Omega(x, y) \| \leq \| x \| \| y \| \) \( \forall x, y \in \mathfrak{X} \).

A subfamily \( \mathcal{A} \) of \( S(\mathfrak{X}) \) is called sufficient if \( x \in \mathfrak{X} \), \( \Omega(x, x) = 0 \), for every \( \Omega \in \mathcal{A} \), implies \( x = 0 \).
If \((\mathfrak{X}, \mathfrak{A}_0)\) is a Banach quasi \*\,-algebra, then the Banach dual space \(\mathfrak{X}^d\) of \(\mathfrak{X}\) can be made into a Banach \(\mathfrak{A}_0\)-bimodule with norm
\[
\|f\|^2 = \sup_{\|x\| \leq 1} |\langle x, f \rangle|, \quad f \in \mathfrak{X}^d,
\]
by defining, for \(f \in \mathfrak{X}^d\), \(a \in \mathfrak{A}_0\), the module operations in the following way:
\[
\langle x, f \circ a \rangle := \langle ax, f \rangle, \quad x \in \mathfrak{X}
\]
\[
\langle x, a \circ f \rangle := \langle xa, f \rangle, \quad x \in \mathfrak{X}.
\]
As usual, an involution \(f \mapsto f^*\) can be defined on \(\mathfrak{X}^d\) by \(\langle x, f^* \rangle = \langle x^*, f \rangle\), \(x \in \mathfrak{X}\).

With these notations we can easily prove the following (see, also [8]):

**Proposition 2.4.** \((\mathfrak{X}, \mathfrak{A}_0)\) be a Banach quasi \*\,-algebra and \(\Omega\) a positive sesquilinear form on \(\mathfrak{X} \times \mathfrak{X}\). The following statements are equivalent:

(i) \(\Omega \in \mathcal{S}(\mathfrak{X})\);

(ii) there exists a bounded conjugate linear operator \(T : \mathfrak{X} \to \mathfrak{X}^d\) with the properties

(ii.1) \(\langle x, Tx \rangle \geq 0, \quad \forall x \in \mathfrak{X}\);

(ii.2) \(\|Tx\|_{\mathfrak{B}(\mathfrak{X}, \mathfrak{X}^d)} \leq 1\);

(ii.3) \(\Omega(x, y) = \langle x, Ty \rangle, \quad \forall x, y \in \mathfrak{X}\).

We will now focus our attention on the question as to whether for the Banach quasi \*\,-algebra \((L^p(\varphi), L^\infty(\varphi) \cap L^p(\varphi))\), the family \(\mathcal{S}(L^p(\varphi))\), that we are going to describe, is or is not sufficient.

Before going forth, we remind that many of the familiar results of the ordinary theory of \(L^p\)-spaces hold in the very same form for the non-commutative \(L^p\)-spaces. This is the case, for instance, of Hölder’s inequality and also of the statement that characterizes the dual of \(L^p\): the form defining the duality is the extension of \(\varphi\) (this extension will be denoted with the same symbol) to products of the type \(XY\) with \(X \in L^p(\varphi)\), \(Y \in L^p(\varphi)\) with \(p^{-1} + p^{-1} = 1\) and one has \((L^p(\varphi))^d \simeq L^p(\varphi)\).

In order to study \(\mathcal{S}(L^p(\varphi))\), we introduce, for \(p \geq 2\), the following notation
\[
\mathcal{B}_p^+ = \{X \in L^{p/(p-2)}(\varphi), \ X \geq 0, \ X \|_{p/(p-2)} \leq 1\}
\]
where \(p/(p-2) = \infty\) if \(p = 2\).

For each \(W \in \mathcal{B}_p^+\), we consider the right multiplication operator
\[
R_W : L^p(\varphi) \to L^{\frac{p}{p-1}}(\varphi); \quad R_W X = XW, \quad X \in L^p(\varphi).
\]
Since \(L^\infty(\varphi) \cap L^p(\varphi) = \mathcal{J}_p\), we use, for shortness, the latter notation.

**Lemma 2.5.** The following statements hold.

(i) Let \(p \geq 2\). For every \(W \in \mathcal{B}_p^+\), the sesquilinear form \(\Omega(X, Y) = \varphi[X(R_W Y)^*]\) is an element of \(\mathcal{S}(L^p(\varphi))\)

(ii) If \(\varphi\) is finite, then for each \(\Omega \in \mathcal{S}(L^p(\varphi))\), there exists \(W \in \mathcal{B}_p^+\) such that
\[
\Omega(X, Y) = \varphi[X(R_W Y)^*], \quad \forall X, Y \in L^p(\varphi).
\]
Proof. (i): We check that the sesquilinear $\Omega(X,Y) = \varphi[X(R_\omega Y)^*]$, $X, Y \in L^p(\varphi)$ satisfies the conditions (i),(ii),(iii) of Definition 2.3.
For every $X \in L^p(\varphi)$ we have
$$\Omega(X,X) = \varphi[X(R_\omega Y)^*] = \varphi[(XW)^*X] = \varphi[W|X|^2] \geq 0.$$ 
For every $X \in L^p(\varphi)$, $A, B \in \mathcal{F}_p$, we get
$$\Omega(XA,B) = \varphi(XA(BW)^*) = \varphi(WB^*XA) = \varphi(A(X^*BW)^*) = \Omega(A, X^*B).$$
Finally, for every $X, Y \in L_p(\varphi)$,
$$|\Omega(X,Y)| \leq \|X\|_p \|Y\|_p \|W\|_{p/p-2} \leq \|X\|_p \|Y\|_p.$$
(ii) Let $\Omega \in S(L^p(\varphi))$. Let $T : L^p(\varphi) \to L^p(\varphi)$ be the operator which represents $\Omega$ in the sense of Proposition 2.4. The finiteness of $\varphi$ implies that $\mathcal{F}_p = \mathcal{M}$; thus we can put $W = T(1)$. It is easy to check that $R_W = T$. This concludes the proof.

Proposition 2.6. If $p \geq 2$, $S(L^p(\varphi))$ is sufficient.

Proof. Let $X \in L^p(\varphi)$ be such that $\Omega(X,X) = 0$ for every $\Omega \in S(L_p(\varphi))$. By the previous lemma, since $|X|^{p-2} \in L^{p/(p-2)}(\varphi)$, the right multiplication operator $R_\omega$ with $W = \frac{|X|^{p-2}}{\alpha}$, $\alpha \in \mathbb{R}$ satisfying $\|\frac{|X|^{p-2}}{\alpha}\|_{p/(p-2)} \leq 1$, represents a sesquilinear form $\Omega \in S(L_p(\varphi))$. By the assumption, $\Omega(X,X) = 0$. We then have
$$\Omega(X,X) = \varphi[X(R_\omega X)^*] = \varphi[\frac{|X|^2}{\alpha}] = \varphi[|X|^2] = \varphi[|X|^p] = 0 \Rightarrow X = 0,$$
by the faithfulness of $\varphi$.

3. CQ*-algebras over finite von Neumann algebras

Let $\mathcal{M}$ be a von Neumann algebra and $\mathfrak{F} = \{\varphi_\alpha; \alpha \in \mathcal{I}\}$ be a family of normal, finite traces on $\mathcal{M}$. As usual, we say that the family $\mathfrak{F}$ is sufficient if for $X \in \mathcal{M}$, $X \geq 0$ and $\varphi_\alpha(X) = 0$ for every $\alpha \in \mathcal{I}$, then $X = 0$ (clearly, if $\mathfrak{F} = \{\varphi\}$, then $\mathfrak{F}$ is sufficient if, and only if, $\varphi$ is faithful). In this case, $\mathcal{M}$ is a finite von Neumann algebra [15, ch.7]. We assume, in addition, that the following condition (P) is satisfied:

(P) \quad $\varphi_\alpha(1) \leq 1$, \quad \forall \alpha \in \mathcal{I}.$

Then we define
$$\|X\|_{p,\mathcal{I}} = \sup_{\alpha \in \mathcal{I}} \|X\|_{p,\varphi_\alpha} = \sup_{\alpha \in \mathcal{I}} \varphi_\alpha(|X|^p)^{1/p}.$$ 
Since $\mathfrak{F}$ is sufficient, $\|\cdot\|_{p,\mathcal{I}}$ is a norm on $\mathcal{M}$.

In the sequel we will need the following Lemmas whose simple proofs will be omitted.

Lemma 3.1. Let $\mathcal{M}$ be a von Neumann algebra in Hilbert space $\mathcal{H}$, $\{P_\alpha\}_{\alpha \in \mathcal{I}}$ a family of projections of $\mathcal{M}$ with
$$\bigvee_{\alpha \in \mathcal{I}} P_\alpha = \mathcal{P}.$$ 
If $A \in \mathcal{M}$ and $AP_\alpha = 0$ for every $\alpha \in \mathcal{I}$, then $AP = 0$. 
Lemma 3.2. Let $\mathcal{F} = \{\varphi_\alpha\}_{\alpha \in I}$ be a sufficient family of normal, finite traces on the von Neumann algebra $\mathcal{M}$ and let $P_\alpha$ be the support of $\varphi_\alpha$. Then, $\forall P_\alpha = I$, where $I$ denotes the identity of $\mathcal{M}$.

It is well-known that the support of each $\varphi_\alpha$ enjoy the following properties

(i) $P_\alpha \in Z(\mathcal{M})$, the center of $\mathcal{M}$, for each $\alpha \in I$;
(ii) $\varphi_\alpha(X) = \varphi_\alpha(XP_\alpha)$, for each $\alpha \in I$.

From the two preceding lemmas it follows that, if the $P_\alpha$’s are as in Lemma 3.2, then

$$AP_\alpha = 0 \quad \forall \alpha \in I \Rightarrow A = 0.$$  

If Condition (P) is fulfilled, then

$$\|X\|_{p,I} = \sup_{\alpha \in I} \|XP_\alpha\|_{p,\alpha} \quad \forall X \in \mathcal{M}.$$  

Clearly, the sufficiency of the family of traces and Condition (P) imply that $\|\cdot\|_{p,I}$ is a norm $\mathcal{M}$.

Proposition 3.3. Let $\mathcal{M}(p,I)$ denote the Banach space completion of $\mathcal{M}$ with respect to the norm $\|\cdot\|_{p,I}$. Then $(\mathcal{M}(p,I)[\|\cdot\|_{p,I}], \mathcal{M}[\|\cdot\|_{B(H)}])$ is a CQ*-algebra.

Proof. Indeed, we have

(1) $\|X^*\|_{p,I} = \sup_{\alpha \in I} \|X^*P_\alpha\|_{p,\alpha} = \sup_{\alpha \in I} \|(XP_\alpha)^*\|_{p,\alpha} = \|X\|_{p,I}, \quad \forall X \in \mathcal{M}.$

Furthermore, for every $X, Y \in \mathcal{M}$,

(2) $\|XY\|_{p,I} = \sup_{\alpha \in I} \|XYP_\alpha\|_{p,\alpha} \leq \|X\|_{B(H)} \sup_{\alpha \in I} \|YP_\alpha\|_{p,\alpha} = \|X\|_{B(H)} \|Y\|_{p,I}.$

Finally, condition (P) implies that

$$\|X\|_{p,I} \leq \|X\|_{B(H)}, \quad \forall X \in \mathcal{M}.$$  

From (1) and (2) it follows that $\mathcal{M}(p,I)$ is a Banach quasi *-algebra. It is clear that $\|\cdot\|_{B(H)}$ satisfies the conditions (a.1)-(a.3) of Section 1. Therefore $(\mathcal{M}(p,I), \mathcal{M})$ is a CQ *-algebra. \[\square\]

The next step consists in investigating the Banach space $\mathcal{M}(p,I)[\|\cdot\|_{p,I}]$. In particular we are interested in the question as to whether $\mathcal{M}(p,I)[\|\cdot\|_{p,I}]$ can be identified with a space of operators affiliated with $\mathcal{M}$. For shortness, whenever no ambiguity can arise, we write $\mathcal{M}_p$ instead of $\mathcal{M}(p,I)$.

Let $\mathcal{F} = \{\varphi_\alpha\}_{\alpha \in I}$ be a sufficient family of normal, finite traces on the von Neumann algebra $\mathcal{M}$ satisfying Condition (P). The traces $\varphi_\alpha$ are not necessarily faithful. Put $\mathcal{M}_\alpha = MP_\alpha$, where, as before, $P_\alpha$ denotes the support of $\varphi_\alpha$. Each $\mathcal{M}_\alpha$ is a von Neumann algebra and $\varphi_\alpha$ is faithful in $\mathcal{M}P_\alpha$ [14, Proposition V. 2.10].

More precisely,

$$\mathcal{M}_\alpha := \mathcal{M}P_\alpha = \{Z = XP_\alpha, \text{ for some } X \in \mathcal{M}\}.$$  

The positive cone $\mathcal{M}_\alpha^+$ of $\mathcal{M}_\alpha$ equals the set

$$\{Z = XP_\alpha, \text{ for some } X \in \mathcal{M}^+\}.$$
For $Z = XP_\alpha \in \mathfrak{M}_\alpha^+$, we put:

$$\sigma_\alpha(Z) := \varphi_\alpha(XP_\alpha).$$

The definition of $\sigma_\alpha(Z)$ does not depend on the particular choice of $X$. Each $\sigma_\alpha$ is a normal, finite, faithful trace on $\mathfrak{M}_\alpha$. It is then possible to consider the spaces $L^p(\mathfrak{M}_\alpha, \sigma_\alpha)$, $p \geq 1$, in the usual way. The norm of $L^p(\mathfrak{M}_\alpha, \sigma_\alpha)$ is indicated as $\| \cdot \|_{p, \alpha}$.

Let now $(X_k)$ be a Cauchy sequence in $\mathfrak{M}[\| \cdot \|_{p, Z}]$. For each $\alpha \in \mathcal{I}$, we put $Z^{(\alpha)}_k = X_kP_\alpha$. Then, for each $\alpha \in \mathcal{I}$, $(Z^{(\alpha)}_k)$ is a Cauchy sequence in $\mathfrak{M}_\alpha[\| \cdot \|_{p, \alpha}]$. Indeed, since $|Z^{(\alpha)}_k - Z^{(\alpha)}_h|^p = |X_k - X_h|^p P_\alpha$,

$$\|Z^{(\alpha)}_k - Z^{(\alpha)}_h\|_{p, \alpha} = \sigma_\alpha(|Z^{(\alpha)}_k - Z^{(\alpha)}_h|^p)^{1/p} = \varphi_\alpha(|X_k - X_h|^p P_\alpha)^{1/p} = \varphi_\alpha(|X_k - X_h|^p)^{1/p} \to 0.$$

Therefore, for each $\alpha \in \mathcal{I}$, there exists an operator $Z^{(\alpha)} \in L^p(\mathfrak{M}_\alpha, \sigma_\alpha)$ such that:

$$Z^{(\alpha)} = \| \cdot \|_{p, \alpha} - \lim_{k \to \infty} Z^{(\alpha)}_k.$$

It is now natural to ask the question as to whether there exists an operator $X$ closed, densely defined, affiliated with $\mathfrak{M}$ which reduces to $Z^{(\alpha)}$ on $\mathfrak{M}_\alpha$. To begin with, we assume that the projections $\{P_\alpha\}$ are mutually orthogonal. In this case, putting $\mathcal{H}_\alpha = P_\alpha \mathcal{H}$, we have

$$\mathcal{H} = \bigoplus_{\alpha \in \mathcal{I}} \mathcal{H}_\alpha = \{(f_\alpha) : f_\alpha \in \mathcal{H}_\alpha, \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|^2 < \infty \}.$$

We put

$$D(X) = \{(f_\alpha) \in \mathcal{H} : f_\alpha \in D(Z^{(\alpha)}); \sum_{\alpha \in \mathcal{I}} \|Z^{(\alpha)} f_\alpha\|^2 < \infty \}$$

and for $f = (f_\alpha) \in D(X)$ we define

$$Xf = (Z^{(\alpha)} f_\alpha).$$

Then

(i) $D(X)$ is dense in $\mathcal{H}$.

Indeed, $D(X)$ contains all $f = (f_\alpha)$ with $f_\alpha = 0$ except that for a finite subset of indeces.

(ii) $X$ is closed in $\mathcal{H}$.

Indeed, let $f_n = (f_{n, \alpha})$ be a sequence of elements of $D(X)$ with $f_n \to g = (g_\alpha) \in \mathcal{H}$ and $Xf_n \to h$. Since

$$f_n \to g \iff f_{n, \alpha} \to g_\alpha \in \mathcal{H}_\alpha, \forall \alpha \in \mathcal{I}$$

and

$$Xf_n \to h \iff (Xf_n)_\alpha \to h_\alpha \in \mathcal{H}_\alpha, \forall \alpha \in \mathcal{I},$$

by $(Xf_n)_\alpha = Z^{(\alpha)} f_{n, \alpha}$ and from the closedness of each $Z^{(\alpha)}$ in $\mathcal{H}_\alpha$, we get

$$g_\alpha \in D(Z^{(\alpha)}) \quad \text{and} \quad h_\alpha = Z^{(\alpha)} g_\alpha.$$
It remains to check that $\sum_{\alpha \in I} \|Z^{(\alpha)}g_\alpha\|^2 < \infty$ but this is clear, since both 
$(Z^{(\alpha)}g_\alpha)$ and $h = (h_\alpha) \in \mathcal{H}$.

(iii) $X \eta \mathcal{M}$.
Let $Y \in \mathcal{M}'$. Then, $\forall f \in \mathcal{H}$, $Yf = (YP_\alpha f)$ and $YP_\alpha \in (\mathcal{M}P_\alpha)' = \mathcal{M}'P_\alpha$.
Therefore 
$$XYf = ((XY)P_\alpha f) = (YP_\alpha f) = Xf.$$

In conclusion, $X$ is a measurable operator.

Thus, we have proved the following

**Proposition 3.4.** Let $\tilde{\mathcal{F}} = \{\varphi_\alpha\}_{\alpha \in I}$ be a sufficient family of normal, finite traces on the von Neumann algebra $\mathcal{M}$. Assume that Condition (P) is fulfilled and that the $\varphi_\alpha$’s have mutually orthogonal supports. Then $\mathcal{M}_p$, $p \geq 1$, consists of measurable operators.

The analysis of the general case would really be simplified if, from a given sufficient family $\tilde{\mathcal{F}}$ of normal finite traces, one could extract (or construct) a sufficient subfamily $\mathcal{G}$ of traces with mutually orthogonal supports. Apart from quite simple situations (for instance when $\tilde{\mathcal{F}}$ is finite or countable), we do not know if this is possible or not. There is however a relevant case where this can be fairly easily done. This occurs when $\tilde{\mathcal{F}}$ is a convex and $w^*$-compact family of traces on $\mathcal{M}$.

**Lemma 3.5.** Let $\tilde{\mathcal{F}}$ be a convex $w^*$-compact family of normal, finite traces on a von Neumann algebra $\mathcal{M}$; assume that, for each central operator $Z$, with $0 \leq Z \leq 1$, and each $\eta \in \tilde{\mathcal{F}}$ the functional $\eta_\alpha(X) := \eta(XZ)$ belongs to $\tilde{\mathcal{F}}$. Let $\mathcal{E}\tilde{\mathcal{F}}$ be the set of extreme elements of $\tilde{\mathcal{F}}$. If $\eta_1, \eta_2 \in \mathcal{E}\tilde{\mathcal{F}}$, $\eta_1 \neq \eta_2$, and $P_1$ and $P_2$ are their respective supports, then $P_1$ and $P_2$ are orthogonal.

**Proof.** Let $P_1, P_2$ be, respectively, the supports of $\eta_1$ and $\eta_2$. We begin with proving that either $P_1 = P_2$ or $P_1 P_2 = 0$. Indeed, assume that $P_1 P_2 \neq 0$. We define 
$$\eta_{1,2}(X) = \eta_1(XP_2) \quad X \in \mathcal{M}.$$ 
Were $\eta_{1,2} = 0$, then, in particular $\eta_{1,2}(P_2) = 0$, i.e. $\eta_1(P_2) = 0$ and therefore, by definition of support, $P_2 \leq 1 - P_1$. This implies that $P_1 P_2 = 0$, which contradicts the assumption. We now show that the support of $\eta_{1,2}$ is $P_1 P_2$. Let, in fact, $Q$ be a projection such that $\eta_{1,2}(Q) = 0$. Then 
$$\eta_1(QP_2) = 0 \Rightarrow QP_2 \leq 1 - P_1 \Rightarrow QP_2(1 - P_1) = QP_2 \Rightarrow QP_2 P_1 = 0.$$ 
Then the largest $Q$ for which this happens is $1 - P_2 P_1$. We conclude that the support of the trace $\eta_{1,2}$ is $P_1 P_2$. Finally, by definition, one has $\eta_{1,2}(X) = \eta_1(XP_2)$, and, since $XP_2 \leq X$, 
$$\eta_{1,2}(X) = \eta_1(XP_2) \leq \eta_1(X) \quad \forall X \in \mathcal{M}.$$ 
Thus $\eta_1$ majorizes $\eta_{1,2}$. But $\eta_1$ is extreme in $\tilde{\mathcal{F}}$. Therefore $\eta_{1,2}$ has the form $\lambda \eta_1$ with $\lambda \in [0, 1]$. This implies that $\eta_{1,2}$ has the same support as $\eta_1$; therefore $P_1 P_2 = P_1$ i.e. $P_1 \leq P_2$. Starting from $\eta_{1,2}(X) = \eta_2(XP_1)$, we get, in similar way, $P_2 \leq P_1$. Therefore, $P_1 P_2 \neq 0$ implies $P_1 = P_2$. However, two different traces of $\mathcal{E}\tilde{\mathcal{F}}$ cannot have the same support. Indeed, assume that there exist $\eta_1, \eta_2 \in \tilde{\mathcal{F}}$ having the same support $P$. Since $P$ is central, we can consider the von Neumann algebra $\mathcal{M}P$. The restrictions of $\eta_1, \eta_2$
to $\mathcal{M}P$ are normal faithful semifinite traces. By [14, Prop. V.2.31] there exist a central element $Z$ in $\mathcal{M}P$ with $0 \leq Z \leq P$ ($P$ is here considered as the unit of $\mathcal{M}P$) such that

$$(3) \quad \eta_1(X) = (\eta_1 + \eta_2)(ZX) \quad \forall X \in (\mathcal{M}P)_+.$$ 

Then $Z$ also belongs to the center of $\mathcal{M}$, since for every $V \in \mathcal{M}$

$$ZV = Z(VP + VP^\perp) = ZVP = VZ.$$

Therefore the functionals

$$\eta_{1,z}(X) := \eta_1(XZ) \quad \eta_{2,z}(X) := \eta_2(XZ) \quad X \in \mathcal{M}$$

belong to the family $\mathcal{F}$ and are majorized, respectively, by the extreme elements $\eta_1, \eta_2$. Then, there exist $\lambda, \mu \in [0,1]$ such that

$$\eta_1(XZ) = \lambda \eta_1(X) \quad \eta_2(XZ) = \mu \eta_1(X), \quad \forall X \in \mathcal{M}.$$ 

If $\lambda = 1$ we would have, from (3), $\eta_2(ZX) = 0$, for every $X \in (\mathcal{M}P)_+$; in particular, $\eta_2(Z^2) = 0$; this implies that $Z = 0$. Thus $\lambda \neq 1$. Analogously, $\mu \neq 0$; indeed, if $\mu = 0$, then $\eta_1(X) = \lambda \eta_1(X)$ and thus $\lambda = 1$. Therefore there exist $\lambda, \mu \in (0,1)$ such that

$$\eta_1(X) = \lambda \eta_1(X) + \mu \eta_2(X) \quad \forall X \in \mathcal{M};$$

which, in turn, implies

$$\eta_1(X) = \lambda \eta_1(X) + \mu \eta_2(X) \quad \forall X \in \mathcal{M}$$

Hence,

$$(1 - \lambda)\eta_1(X) = \mu \eta_2(X) \quad \forall X \in \mathcal{M}.$$ 

From the last equality, dividing by $\max\{1 - \lambda, \mu\}$ one gets that one of the two elements is a convex combination of the other and of 0; which is absurd. In conclusion, different supports of extreme traces of $\mathcal{F}$ are orthogonal. 

Since, for every $X \in \mathcal{M}$, $\|X\|_{p,\mathcal{Z}}$ remains the same if computed either with respect to $\mathcal{F}$ or to $\mathcal{E}\mathcal{F}$, we can deduce the following

**Theorem 3.6.** Let $\mathcal{F}$ be a convex and $w^*$-compact sufficient family of normal, finite traces on the von Neumann algebra $\mathcal{M}$. Assume that $\mathcal{F}$ satisfies Condition (P) and that for each central operator $Z$, with $0 \leq Z \leq I$, and each $\eta \in \mathcal{F}$ the functional $\eta_Z(X) := \eta(XZ)$ belongs to $\mathcal{F}$. Then the completion $\mathcal{M}_p[\|\cdot\|_{p,\mathcal{Z}}]$, consists of measurable operators.

Families of traces satisfying the assumptions of Theorem 3.6 will be constructed in the next section.
4. A representation theorem

Once we have constructed in the previous section some CQ*-algebras of operators affiliated to a given von Neumann algebra, it is natural to pose the question under which conditions can an abstract CQ*-algebra \((X, \mathcal{A}_0)\) be realized as a CQ*-algebra of this type.

Let \((X[\| \cdot \|], \mathcal{A}_0[\| \cdot \|_0])\) be a CQ*-algebra with unit \(e\) and let
\[
\mathcal{T}(X) = \{ \Omega \in \mathcal{S}(X) : \Omega(x, x) = \Omega(x^*, x^*), \forall x \in X \}.
\]
We remark that if \(\Omega \in \mathcal{T}(X)\) then, by polarization, \(\Omega(y^*, x^*) = \Omega(x, y), \forall x, y \in X\).

It is easy to prove that the set \(\mathcal{T}(X)\) is convex.

For each \(\Omega \in \mathcal{T}(X)\), we define a linear functional \(\omega_{\Omega}\) on \(\mathcal{A}_0\) by
\[
\omega_{\Omega}(a) := \Omega(a, e) \quad a \in \mathcal{A}_0.
\]
We have
\[
\omega_{\Omega}(a^*a) = \Omega(a^*a, e) = \Omega(a, a) = \Omega(a^*, a^*) = \omega_{\Omega}(aa^*) \geq 0.
\]
This shows at once that \(\omega_{\Omega}\) is positive and tracial.

We put
\[
\mathcal{M}_\Omega(\mathcal{A}_0) = \{ \omega_{\Omega}; \Omega \in \mathcal{T}(X) \}.
\]
From the convexity of \(\mathcal{T}(X)\) it follows easily that \(\mathcal{M}_\Omega(\mathcal{A}_0)\) is convex too. If we denote with \(\| f \|_2\) the norm of the bounded functional \(f\) on \(\mathcal{A}_0\), we also get
\[
\| \omega_{\Omega} \|_2 = \omega_{\Omega}(e, e) = \| e \|^2.
\]
Therefore
\[
\mathcal{M}_\Omega(\mathcal{A}_0) \subseteq \{ \omega \in \mathcal{A}_0^\# : \| \omega \|^2 \leq \| e \|^2 \},
\]
where \(\mathcal{A}_0^\#\) denotes the topological dual of \(\mathcal{A}_0[\| \cdot \|_0]\).

Setting
\[
f_{\Omega}(a) := \frac{\omega_{\Omega}(a)}{\| e \|^2}
\]
we get
\[
f_{\Omega} \in \{ \omega \in \mathcal{A}_0^\# : \| \omega \|^2 \leq 1 \}.
\]
By the Banach - Alaglou theorem, the set \(\{ \omega \in \mathcal{A}_0^\# : \| \omega \|^2 \leq 1 \}\) is a \(w^*\)-compact subset of \(\mathcal{A}_0^\#\). Then, the set \(\{ \omega \in \mathcal{A}_0^\# : \| \omega \|^2 \leq \| e \|^2 \}\) is also \(w^*\)-compact.

**Proposition 4.1.** \(\mathcal{M}_\Omega(\mathcal{A}_0)\) is \(w^*\)-closed and, therefore, \(w^*\)-compact.

**Proof.** Let \((\omega_{\Omega_\alpha})\) be a net in \(\mathcal{M}_\Omega(\mathcal{A}_0)\) \(w^*\)-converging to a functional \(\omega \in \mathcal{A}_0^\#\). We will show that \(\omega = \omega_{\Omega}\) for some \(\Omega \in \mathcal{T}(X)\).

Let us begin with defining \(\Omega_\alpha(a, b) = \omega(b^*a), a, b \in \mathcal{A}_0\). By the definition itself, \((\omega_{\Omega_\alpha})(a) \rightarrow \omega(a) = \Omega_\alpha(a, e)\). Moreover, for every \(a, b \in \mathcal{A}_0\),
\[
\Omega_\alpha(a, b) = \omega(b^*a) = \lim_{\alpha} \omega_{\Omega_\alpha}(b^*a) = \lim_{\alpha} \Omega_\alpha(a, b).
\]
Therefore
\[
\Omega_\alpha(a, a) = \lim_{\alpha} \Omega_\alpha(a, a) \geq 0.
\]
We also have
\[
\| \Omega_\alpha(a, b) \| \leq \lim_{\alpha} \| \Omega_\alpha(a, b) \| \leq \| a \| \| b \| .
\]
Hence \( \Omega_o \) can be extended by continuity to \( \mathcal{X} \times \mathcal{X} \). Indeed, let
\[
x = \| \cdot \| - \lim_{n} a_n \quad y = \| \cdot \| - \lim_{n} b_n \quad (a_n), (b_n) \subseteq \mathfrak{A}_o
\]
then
\[
| \Omega_o (a_n, b_n) - \Omega_o (a_m, b_m) | = | \Omega_o (a_n, b_n) - \Omega_o (a_m, b_n) + \Omega_o (a_m, b_n) - \Omega_o (a_m, b_m) | \leq \\
\leq \| \Omega_o (a_n - a_m, b_n) \| + \| \Omega_o (a_m, b_n - b_m) \| \leq \| a_n - a_m \| \| b_n \| + \| a_m \| \| b_n - b_m \| \to 0,
\]
since \( \| a_n \| \) and \( \| b_n \| \) are bounded sequences. Therefore we can define
\[
\Omega (x, y) = \lim_n \Omega_o (a_n, b_n).
\]

Clearly, \( \Omega (x, x) \geq 0 \ \forall x \in \mathcal{X} \).

It is easily checked that \( \Omega \in \mathcal{T} (\mathcal{X}) \). This concludes the proof.

Since \( \mathcal{M}_T (\mathfrak{A}_o) \) is convex and \( w^* \)-compact, by the Krein-Milman theorem it follows that it has extreme points and it coincides with the \( w^* \)-closure of the convex hull of the set \( \mathcal{E}\mathcal{M}_T (\mathfrak{A}_o) \) of its extreme points.

By the Gelfand - Naimark theorem each \( C^* \)-algebra is isometrically \( * \)-isomorphic to a \( C^* \)-algebra of bounded operators in Hilbert space. This isometric \( * \)-isomorphism is called the universal \( * \)-representation.

Thus, let \( \pi \) be the universal \( * \)-representation of \( \mathfrak{A}_o \) and \( \pi (\mathfrak{A}_o)'' \) the von Neumann algebra generated by \( \pi (\mathfrak{A}_o) \).

For every \( \Omega \in \mathcal{T} (\mathcal{X}) \) and \( a \in \mathfrak{A}_o \), we put
\[
\varphi_\Omega (\pi (a)) = \omega_\Omega (a).
\]

Then, for each \( \Omega \in \mathcal{T} (\mathcal{X}) \), \( \varphi_\Omega \) is a positive bounded linear functional on the operator algebra \( \pi (\mathfrak{A}_o) \).

Clearly,
\[
\varphi_\Omega (\pi (a)) = \omega_\Omega (a) = \Omega (a, e)
\]

\[
| \varphi_\Omega (\pi (a)) | = | \omega_\Omega (a) | = | \Omega (a, e) | \leq \| a \| \| e \| \leq \| a \|_o \| e \|^2 = \| \pi (a) \| \| e \|^2.
\]

Thus \( \varphi_\Omega \) is continuous on \( \pi (\mathfrak{A}_o) \).

By [16, Theorem 10.1.2], \( \varphi_\Omega \) is weakly continuous and so it extends uniquely to \( \pi (\mathfrak{A}_o)'' \). Moreover, since \( \varphi_\Omega \) is a trace on \( \pi (\mathfrak{A}_o) \), the extension \( \tilde{\varphi}_\Omega \) is a trace on \( \mathcal{M} := \pi (\mathfrak{A}_o)'' \) too.

The norm \( \| \tilde{\varphi}_\Omega \| \) of \( \tilde{\varphi}_\Omega \) as a linear functional on \( \mathcal{M} \) equals the norm of \( \varphi_\Omega \) as a functional on \( \pi (\mathfrak{A}_o) \).

We have:
\[
\| \tilde{\varphi}_\Omega \| = \tilde{\varphi}_\Omega (\pi (e)) = \varphi_\Omega (\pi (e)) = \omega_\Omega (e) \leq \| e \|^2.
\]

The set
\[
\mathfrak{N}_T (\mathfrak{A}_o) = \{ \tilde{\varphi}_\Omega ; \Omega \in \mathcal{T} (\mathcal{X}) \}
\]
is convex and \( w^* \)-compact in \( \mathfrak{M}_T \), as can be easily seen by considering the map
\[
\omega_\Omega \in \mathfrak{M}_T (\mathfrak{A}_o) \rightarrow \tilde{\varphi}_\Omega \in \mathfrak{N}_T (\mathfrak{A}_o)
\]
which is linear and injective and taking into account the fact that, if \( a_\alpha \to a \) in \( \mathfrak{A}_o [\| \cdot \|] \), then
\[
\tilde{\varphi}_\Omega (\pi (a_\alpha) - \pi (a)) = \omega_\Omega (a_\alpha - a) \to 0.
\]
Let $\mathcal{E}M_T(\mathfrak{A}_0)$ be the set of extreme points of $\mathcal{M}_T(\mathfrak{A}_0)$; then $\mathcal{M}_T(\mathfrak{A}_0)$ coincides with $w^*$-closure of the convex hull of $\mathcal{E}M_T(\mathfrak{A}_0)$. The extreme elements of $\mathcal{M}_T(\mathfrak{A}_0)$ are easily characterized by the following

**Proposition 4.2.** $\tilde{\varphi}_\Omega$ is extreme in $\mathcal{M}_T(\mathfrak{A}_0)$ if, and only if, $\omega_\Omega$ is extreme in $\mathcal{M}_T(\mathfrak{A}_0)$.

**Definition 4.3.** A Banach quasi *-algebra $(\mathfrak{X}|| \cdot ||, \mathfrak{A}_0|| \cdot ||_o)$ is said to be strongly regular if $T(\mathfrak{X})$ is sufficient and

$$
\|x\| = \sup_{\Omega \in T(\mathfrak{X})} \Omega(x, x)^{1/2}, \quad \forall x \in \mathfrak{X}.
$$

**Example 4.4.** If $\mathfrak{M}$ is a von Neumann algebra possessing a sufficient family $\mathfrak{F}$ of normal finite traces, then the CQ*-algebra $(\mathfrak{M}_p, \mathfrak{N})$ constructed in Section 3 is strongly regular. This follows from the definition itself of the norm in the completion.

**Example 4.5.** If $\varphi$ is a normal faithful finite trace on $\mathfrak{M}$, then $T(\mathcal{L}^p(\varphi))$, for $p \geq 2$, is sufficient. To see this, we start with defining $\Omega_0$ on $\mathfrak{M} \times \mathfrak{M}$ by

$$
\Omega_0(X, Y) = \varphi(Y^* X), \quad X, Y \in \mathfrak{M}.
$$

Then

$$
|\Omega_0(X, Y)| = |\varphi(Y^* X)| \leq \|X\|_p |Y|_{p'}, \quad \forall X, Y \in \mathfrak{M}.
$$

Since $p \geq 2$, then $\mathcal{L}^p(\varphi)$ is continuously embedded into $\mathcal{L}^p(\varphi)$. Thus, there exists $\gamma > 0$ such that $\|Y\|_{p'} \leq \gamma \|Y\|_p$ for every $Y \in \mathfrak{M}$. Let us define

$$
\tilde{\Omega}(X, Y) = \frac{1}{\gamma} \Omega_0(X, Y), \quad \forall X, Y \in \mathfrak{M}.
$$

Then

$$
|\tilde{\Omega}(X, Y)| \leq \|X\|_p |Y|_{p'}, \quad \forall X, Y \in \mathfrak{M}.
$$

Hence, $\tilde{\Omega}$ has a unique extension, denoted with the same symbol, to $\mathcal{L}^p(\varphi) \times \mathcal{L}^p(\varphi)$. It is easily seen that $\tilde{\Omega} \in T(\mathcal{L}^p(\varphi))$.

Were, for some $X \in \mathcal{L}^p(\varphi)$, $\Omega(X, X) = 0$, for every $\Omega \in T(\mathcal{L}^p(\varphi))$, we would then have $\tilde{\Omega}(X, X) = \|X\|_2^2 = 0$. This, clearly, implies $X = 0$. The equality $\tilde{\Omega}(X, X) = \|X\|_2^2$ also shows that $L^2(\varphi)$ is strongly regular.

Let now $(\mathfrak{X}|| \cdot ||, \mathfrak{A}_0|| \cdot ||_o)$ be a CQ*-algebra with unit $e$ and sufficient $T(\mathfrak{X})$. Let $\pi : \mathfrak{X}_0 \hookrightarrow \mathcal{B}(\mathcal{H})$ be the universal representation of $\mathfrak{A}_0$. Assume that the $C^*$-algebra $\pi(\mathfrak{A}_0) := \mathfrak{M}$ is a von Neumann algebra. In this case, $\mathcal{M}_T(\mathfrak{A}_0) = \mathcal{M}_T(\mathfrak{A}_0)$ and $\mathcal{M}_T(\mathfrak{A}_0)$ is a family of traces satisfying Condition (P). Therefore, by Proposition 3.3, we can construct for $p \geq 1$, the CQ*-algebras $(\mathfrak{M}_p|| \cdot ||_p, \mathfrak{N}_p|| \cdot ||_o)$). Clearly, $\mathfrak{A}_0$ can be identified with $\mathfrak{M}$. It is then natural to pose the question if also $\mathfrak{X}$ can be identified with some $\mathfrak{M}_p$.

**Theorem 4.6.** Let $(\mathfrak{X}|| \cdot ||, \mathfrak{A}_0|| \cdot ||_o)$ be a CQ*-algebra with unit $e$ and and sufficient $T(\mathfrak{X})$.

Then there exist a von Neumann algebra $\mathfrak{M}$ and a monomorphism

$$
\Phi : x \in \mathfrak{X} \rightarrow \Phi(x) := \tilde{X} \in \mathfrak{M}_2
$$

with the following properties:
Proof. Let $\pi$ be the universal representation of $\mathfrak{A}_0$ and assume first that $\pi(\mathfrak{A}_0) : = \mathfrak{M}$ is a von Neumann algebra. By Proposition 4.1, the family of traces $\mathfrak{M}_T(\mathfrak{A}_0)$ is convex and $w^*$-compact. Moreover, for each central positive element $Z$ with $0 \leq Z \leq 1$ and for $\varphi \in \mathfrak{M}_T(\mathfrak{A}_0)$, the trace $\varphi_Z(X) := \varphi(ZX)$ yet belongs to $\mathfrak{M}_T(\mathfrak{A}_0)$. Indeed, starting from the form $\Omega \in T(\mathfrak{X})$ which generates $\varphi$, one can define the sesquilinear form

$$\Omega_Z(x, y) := \Omega(x\pi^{-1}(Z^{1/2}), y\pi^{-1}(Z^{1/2})) \quad \forall x, y \in \mathfrak{X}.$$ 

We check that $\Omega_Z \in T(\mathfrak{X})$.

(i) $\Omega_Z(x, x) = \Omega(x\pi^{-1}(Z^{1/2}), x\pi^{-1}(Z^{1/2})) \geq 0, \quad \forall x \in \mathfrak{X}$

(ii) We have, for every $x \in \mathfrak{X}$ and for every $a, b \in \mathfrak{A}_0$,

$$\Omega_Z(xa, b) = \Omega(xa\pi^{-1}(Z^{1/2}), b\pi^{-1}(Z^{1/2})) = \Omega(a\pi^{-1}(Z^{1/2}), x^*b\pi^{-1}(Z^{1/2})) = \Omega_Z(a, x^*b).$$

(iii) We have, for every $x, y \in \mathfrak{X}$,

$$|\Omega_Z(x, y)| = |\Omega(x\pi^{-1}(Z^{1/2}), y\pi^{-1}(Z^{1/2}))| \leq \|x\pi^{-1}(Z^{1/2})\| \|\pi^{-1}(Z^{1/2})y\| \leq \|x\|\|\pi^{-1}(Z^{1/2})\|_0 \|\pi^{-1}(Z^{1/2})\|_0 \leq \|x\|\|y\|.$$ 

(iv) For every $x \in \mathfrak{X}$,

$$\Omega_Z(x^*, x^*) = \Omega(x^*\pi^{-1}(Z^{1/2}), x^*\pi^{-1}(Z^{1/2})) = \Omega(x\pi^{-1}(Z^{1/2}), x\pi^{-1}(Z^{1/2})) = \Omega_Z(x, x).$$

Moreover, $\Omega_Z$ defines, for every $A = \pi(a) \in \mathfrak{M} = \pi(\mathfrak{A}_0)$, the following trace

$$\varphi_{\Omega_Z}(A) = \Omega_Z(a, e) = \Omega(a\pi^{-1}(Z^{1/2}), \pi^{-1}(Z^{1/2})) = \Omega(a\pi^{-1}(Z), e) = \Omega(\pi^{-1}(AZ), e) = \varphi(\pi(AZ))$$

Then, the family of traces $\mathfrak{M}_T(\mathfrak{A}_0) (= \mathfrak{M}_T(\mathfrak{A}_0))$ satisfies the assumptions of Lemma 3.5; therefore, if $\eta_1, \eta_2 \in \mathfrak{E}_T(\mathfrak{A}_0)$, denoting with $P_1$ and $P_2$ their respective supports, one has $P_1P_2 = 0$.

By the sufficiency of $T(\mathfrak{X})$ we get

$$\|X\|_{2, \mathfrak{M}_T(\mathfrak{A}_0)} := \sup_{\varphi \in \mathfrak{E}_T(\mathfrak{A}_0)} \|X\|_{2, \varphi} = \sup_{\varphi \in \mathfrak{E}_T(\mathfrak{A}_0)} \|X\|_{2, \varphi} \quad \forall X \in \pi(\mathfrak{A}_0).$$

By Proposition 3.3, the Banach space $\mathfrak{M}_2$, completion of $\mathfrak{M}$ with respect to the norm $\| \cdot \|_{2, \mathfrak{M}_T(\mathfrak{A}_0)}$, is a CQ*-algebra. Moreover, since the supports of the extreme traces
satisfy the assumptions of Theorem 3.6, the CQ*-algebra \( \mathfrak{M}_2[\| \cdot \|_{2,\pi_T(\mathfrak{A}_0)}] \), consists of operators affiliated with \( \mathfrak{M} \). We now define the map \( \Phi \).

For every element \( x \in \mathfrak{X} \), there exists a sequence \( \{a_n\} \) of elements of \( \mathfrak{A}_0 \) converging to \( x \) with respect to the norm of \( \mathfrak{X} \). Let \( X_n = \pi(a_n) \), \( n \in \mathbb{N} \). Then,

\[
\| X_n - X_m \|_{2,\pi_T(\mathfrak{A}_0)} := \sup_{\varphi \in \mathfrak{N}_T(\mathfrak{A}_0)} \| \pi(a_n) - \pi(a_m) \|_{2,\varphi} = \sup_{\Omega \in T(\mathfrak{X})} [\Omega((a_n - a_m)^*(a_n - a_m), e)]^{1/2} = \sup_{\Omega \in T(\mathfrak{X})} [\Omega(a_n - a_m, a_n - a_m)]^{1/2} \leq \|a_n - a_m\| \to 0.
\]

Let \( \tilde{X} \) be the \( \| \cdot \|_{2,\pi_T(\mathfrak{A}_0)} \)-limit of the sequence \( (X_n) \) in \( \mathfrak{M}_2 \). We define \( \Phi(x) := \tilde{X} \).

For each \( x \in \mathfrak{X} \), we put

\[
p_T(x)(x) = \sup_{\Omega \in T(\mathfrak{X})} \Omega(x,x)^{1/2}.
\]

Then, owed to the sufficiency of \( T(\mathfrak{X}) \), \( p_T(x) \) is a norm on \( \mathfrak{X} \) weaker than \( \| \cdot \| \). This implies that

\[
\| \tilde{X} \|_{2,\pi_T(\mathfrak{A}_0)}^2 = \lim_{n \to \infty} \sup_{\Omega \in T(\mathfrak{X})} \Omega(a_n,a_n) = \lim_{n \to \infty} p_T(x)(a_n)^2 = p_T(x)(x)^2.
\]

From this equality it follows easily that the linear map \( \Phi \) is well defined and injective. The condition (iii) can be easily proved. If \( (\mathfrak{X}, \mathfrak{A}_0) \) is strongly regular, then, for every \( x \in \mathfrak{X} \), \( p_T(x)(x) = \|x\| \). Thus \( \Phi \) is isometric. Moreover, in this case, \( \Phi \) is surjective; indeed, if \( T \in \mathfrak{M}_2 \), then there exists a sequence \( T_n \) of bounded operators of \( \pi(\mathfrak{A}_0) \) which converges to \( T \) with respect to the norm \( \| \cdot \|_{2,\pi_T(\mathfrak{A}_0)} \). The corresponding sequence \( \{t_n\} \subset \mathfrak{A}_0 \), \( T_n = \Phi(t_n) \), converges to \( t \) with respect to the norm of \( \mathfrak{X} \) and \( \Phi(t) = T \) by definition. Therefore \( \Phi \) is an isometric *-isomorphism.

To complete the proof, it is enough to prove that the given CQ*-algebra \( (\mathfrak{X}, \mathfrak{A}_0) \) can be embedded in a CQ*-algebra \( (\mathfrak{A}, \mathfrak{B}_0) \) where \( \mathfrak{B}_0 \) is a W*-algebra. Of course, we may directly work with \( \pi(\mathfrak{A}_0) \) with \( \pi \) the universal representation of \( \mathfrak{A}_0 \). The family of traces \( \mathfrak{N}_T(\mathfrak{A}_0) \) defined on \( \pi(\mathfrak{A}_0)^\prime\prime \) is not necessarily sufficient. Let \( P_\Omega, \Omega \in T(\mathfrak{X}) \), denote the support of \( \tilde{\omega}_\Omega \) and let

\[
P = \bigvee_{\Omega \in T(\mathfrak{X})} P_\Omega.
\]

Then \( \mathfrak{B}_0 := \pi(\mathfrak{A}_0)^\prime\prime P \) is a von Neumann algebra, that we can complete with respect to the norm

\[
\| X \|_{2,\pi_T(\mathfrak{A}_0)} = \sup_{\Omega \in T(\mathfrak{X})} \tilde{\omega}_\Omega(X^* X), \quad X \in \pi(\mathfrak{A}_0)^\prime\prime P.
\]

We obtain in this way a CQ*-algebra \( (\mathfrak{A}, \mathfrak{B}_0) \) with \( \mathfrak{B}_0 \) a W*-algebra. The faithfulness of \( \pi \) on \( \mathfrak{A}_0 \) implies that

\[
\pi(a) P = \pi(a), \quad \forall a \in \mathfrak{A}_0.
\]

It remains to prove that \( \mathfrak{X} \) can be identified with a subspace of \( \mathfrak{A} \). But this can be shown in the very same way as we did in the first part: for each \( x \in \mathfrak{X} \) there exists
a sequence \( \{a_n\} \subset \mathfrak{A}_0 \) such that \( \|x - a_n\| \to 0 \) as \( n \to \infty \). We now put \( X_n = \pi(a_n) \).

Then, proceeding as before, we determine the element \( \hat{X} \in \mathcal{K} \), where

\[
\hat{X} = \| \cdot \|_{2, \pi_\mathcal{T}(\mathfrak{A}_0)} \lim_{n \to \infty} \pi(a_n) P.
\]

It is easy to see that the map \( x \in \mathcal{X} \to \hat{X} \in \mathcal{K} \) is injective. If \( (\mathcal{X}, \mathfrak{A}_0) \) is regular, but \( \pi(\mathfrak{A}_0) \subset \pi(\mathfrak{A}_0)' \), then \( \Phi \) is an isometry of \( \mathcal{X} \) into \( \mathbb{M}_2 \), but needs not be surjective. \( \square \)

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**References**
