Algebras of unbounded operators and physical applications: a survey

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Abstract

After an historical introduction on the standard algebraic approach to quantum mechanics of large systems we review the basic mathematical aspects of the algebras of unbounded operators. After that we discuss in some details their relevance in physical applications.
I Introduction

During the past 20 years a long series of papers concerning algebras of unbounded operators appeared in the literature, papers which, though being originally motivated by physical arguments, contain essentially no physics at all. On the contrary the mathematical aspects of these algebras have been analyzed in many details and this analysis produced, up to now, the monographs [40] and [2]. Some physics appeared first in [34] and [39], in the attempt to describe systems with a very large ($10^{24}$) number of degrees of freedom, following some general ideas originally proposed in the famous paper of Haag and Kastler, [33].

These authors consider, as widely discussed in the literature, [21], systems with infinite degrees of freedom because, in this way, a simpler approach to, e.g., phase transitions and collective phenomena can be settled up. However, moving from a large but finite to an infinite number of degrees of freedom one has to build up a mathematical apparatus which is rather sophisticated and, as we will see, not yet completely fixed.

More recently other physical applications of algebras of unbounded operators have been proposed by the present author and others, see [4, 5, 8, 9, 11, 12, 13, 14] for instance. In our opinion it is time to review some of these results, trying to connect as much as possible these with the original results based on C*-algebras.

The paper, which is meant to be very pedagogical, is organized as follows: in Section II we give an introduction to non relativistic ordinary quantum mechanics (i.e. quantum mechanics for systems with a finite number of degrees of freedom), useful to fix the notation and some preliminary ideas. Section III is devoted to a longer review to non relativistic quantum mechanics for systems with infinite degrees of freedom, with a particular interest for some physically relevant results and for open problems. In Section IV we introduce some mathematical definitions and results concerning algebras of unbounded operators, while their physical applications are given in Section V. Our conclusions and our future projects are finally contained in Section VI. To keep the paper self-contained we have also added two Appendices. In the first one we give the general construction of the algebraic settings which extends the Haag and Kastler’s construction, while in the second appendix we give a list of information of functional analysis which may be useful to some non particularly mathematically minded.

II Ordinary (non relativistic) quantum mechanics

This and the next sections are heavily based on [41, 42], to which we refer for further details.
The usual description of non relativistic quantum mechanics, as it is taught in many textbooks, is given in some fixed Hilbert $\mathcal{H}$ space as follows:

- Each observable $A$ of the physical system corresponds to a self-adjoint operator $\hat{A}$ in $\mathcal{H}$;
- The pure states of the physical system correspond to normalized vectors of $\mathcal{H}$;
- The expectation values of $A$ correspond to the following mean values: $\langle \psi, \hat{A} \psi \rangle = \rho_\psi(\hat{A}) = \text{tr}(P_\psi \hat{A})$, where we have also introduced a projector operator $P_\psi$ on $\psi$ and $\text{tr}$ is the trace in $\mathcal{H}$;
- The states which are not pure, i.e. the mixed states, correspond to convex linear combinations $\hat{\rho} = \sum_j w_j \rho_{\psi_j}$, with $\sum_j w_j = 1$ and $w_j \geq 0$ for all $j$;

The dynamics (in the Schrödinger representation) is given by a unitary operator $U_t := e^{iHt/\hbar}$, where $H$ is the self-adjoint energy operator, as follows: $\hat{\rho} \rightarrow \hat{\rho}_t = U_t^* \hat{\rho} U_t$. In the Heisenberg representation the states do not evolve in time while the operators do, following the dual rule: $\hat{A} \rightarrow \hat{A}_t = U_t \hat{A} U_t^*$, and the Heisenberg equation of motion is satisfied: $\frac{d}{dt} \hat{A}_t = \frac{i}{\hbar}[H, \hat{A}_t]$. It is very well known that these two different strategies have the same physical content: $\hat{\rho}_t(\hat{A}_t) = \hat{\rho}(\hat{A}_t)$.

A different description of a quantum mechanical system, which is more useful for its extensions to quantum systems with infinite degrees of freedom, is the algebraic description.

In this approach the observables are elements of a C*-algebra $\mathfrak{A}$ (which coincides with $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$). This means, first of all, that $\mathfrak{A}$ is a vector space over $\mathbb{C}$ with a multiplication law such that $\forall A, B \in \mathfrak{A}$, $AB \in \mathfrak{A}$. Also, two such elements can be summed up and the following properties hold: $\forall A, B, C \in \mathfrak{A}$ and $\forall \alpha, \beta \in \mathbb{C}$ we have

$$A(BC) = (AB)C, \quad A(B + C) = AB + AC, \quad (\alpha A)(\beta B) = \alpha \beta (AB).$$

An involution is a map $*: \mathfrak{A} \rightarrow \mathfrak{A}$ such that

$$A^{**} = A, \quad (AB)^* = B^* A^*, \quad (\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$$

A *-algebra $\mathfrak{A}$ is an algebra with an involution *. $\mathfrak{A}$ is a normed algebra if there exists a map, the norm of the algebra, $\| \cdot \| : \mathfrak{A} \rightarrow \mathbb{R}_+$, such that:

$$\| A \| \geq 0, \quad \| A \| = 0 \iff A = 0, \quad \| \alpha A \| = |\alpha| \| A \|, \quad \| A + B \| \leq \| A \| + \| B \|, \quad \| AB \| \leq \| A \| \| B \|.$$ 

If $\mathfrak{A}$ is complete wrt $\| \cdot \|$ then it is called a Banach algebra, or a Banach *-algebra if $\| A^* \| = \| A \|$. Finally, a C*-algebra is a Banach *-algebra with the property $\| A^* A \| = \| A \|^2$. 


Remarks:–(1) Using this description of our physical system we pay more attention on the rules between the elements which describe the system rather than on the way in which these elements concretely act on a given Hilbert space.

(2) All the C*-algebras are isomorphic to a norm-closed, *-closed, algebra of bounded operators on a certain Hilbert space.

(3) All the abelian C*-algebras are isomorphic to the *-algebra of continuous functions, over a locally compact Hausdorff space $X$, which vanish at infinity, $C_0(X)$.

The states are linear, positive and normalized functional on $\mathfrak{A}$, which looks like $\rho(\hat{A}) = \text{tr}(\hat{\rho}A)$, when $\mathfrak{A} = B(\mathcal{H})$, $\hat{\rho}$ is a trace-class operator and $\text{tr}$ is the trace on $\mathcal{H}$. This means in particular that

$$\rho(\alpha_1 A + \alpha_2 B) = \alpha_1 \rho(A) + \alpha_2 \rho(B)$$

and that

$$\rho(A^* A) \geq 0; \quad \rho(1) = 1.$$ 

An immediate consequence of these assumptions, and in particular of the positivity of $\rho$, is that $\rho$ is also continuous, i.e. that $|\rho(A)| \leq \|A\|$ for all $A \in \mathfrak{A}$.

The dynamics in the Heisenberg representation for conservative quantum systems, i.e. for systems which do not interact with the environment, is given by the map

$$\mathfrak{A} \ni A \to \alpha^t(A) = U_t A U_t^* \in \mathfrak{A}, \quad \forall t$$

which defines a 1-parameter group of *-automorphisms of $\mathfrak{A}$ satisfying the following conditions

$$\alpha^t(\lambda A) = \lambda \alpha^t(A), \quad \alpha^t(A + B) = \alpha^t(A) + \alpha^t(B),$$

$$\alpha^t(AB) = \alpha^t(A) \alpha^t(B), \quad \|\alpha^t(A)\| = \|A\|, \text{ and } \alpha^{t+s} = \alpha^t \alpha^s.$$ 

Remark:– in the Schrödinger representation the time evolution is the dual of the one above, i.e. $\hat{\rho} \to \hat{\rho}_t = \alpha^{t*} \hat{\rho}$.

The reason why this algebraic approach to ordinary quantum mechanics is not very much used in the literature follows from the following von Neumann uniqueness theorem: for finite quantum mechanical systems there exists only one irreducible representation (but for unitary equivalence):

let us consider, for instance, two operators $Q$ and $P$ such that $[Q, P] = i\hbar I$. They can be irreducibly represented on $\mathcal{H} = L^2(\mathbb{R})$ as follows: $\hat{q} f(q) = q f(q)$, $\hat{p} f(q) = -i\hbar f'(q)$, $\forall f \in S(\mathbb{R})$, which is dense in $\mathcal{H}$. If now $\hat{q}'$, $\hat{p}'$ is a (different) irreducible representation of $Q$, $P$ on a (different)
Hilbert space $\mathcal{H}$, $[\hat{q}', \hat{p}'] = i\hbar \mathbb{1}$, then there exists an unitary map $V : \mathcal{H} \rightarrow \mathcal{H}'$ such that

$$\hat{q}' = V \hat{q} V^*, \quad \hat{p}' = V \hat{p} V^*.\$$

Notice that a more precise formulation of von Neumann theorem should require the use of the Weyl unitary operators, to avoid domain problems connected with the unboundedness of $Q$ and $P$, [42].

This result can be interpreted as follows: there is no difference in using an abstract C*-algebra or a given Hilbert space when dealing with a quantum system with finite number of degrees of freedom since two different (but irreducible) representations of the same algebra are surely related by a unitary map and, for this reason, they are physically equivalent. As we will discuss in the next section, this is not what happens in quantum mechanics for systems with infinite degrees of freedom, $QM_\infty$ in the following, so that the two descriptions became really different.

### III A short review of (non relativistic) $QM_\infty$

As just stated, when the degrees of freedom of a system increase up to infinity, the uniqueness von Neumann theorem does not need to hold, in the sense that the same physical system may have several inequivalent representations.

A very simple example which exhibits such a feature is an infinite spin chain, the so-called Ising model, whose (formal) hamiltonian is $H = -J \sum_j \sigma_j^3 \sigma_{j+1}^3$. Here $\sigma_j^3$ is the third component of the Pauli matrices localized at the site $j$ of a certain lattice. If $J > 0$ the following vectors both minimize the energy of the system:

$$\psi_0^{(+1)} = \ldots \otimes \uparrow \otimes \uparrow \otimes \uparrow \otimes \ldots$$

and

$$\psi_0^{(-1)} = \ldots \otimes \downarrow \otimes \downarrow \otimes \downarrow \otimes \ldots$$

where $\uparrow$ and $\downarrow$ are eigenstates of $\sigma^3$ with eigenvalues +1 and −1 respectively: $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Furthermore, $\psi_0^{(\pm 1)}$ cannot be mapped into one another by local actions, since for instance $\psi_0^{(+1)}$ can be obtained from $\psi_0^{(-1)}$ only acting with $\sigma^1$ on each site of the infinite lattice! For this reason, there exists no element $Y$ of a local algebra (see below) satisfying the equality $Y \psi_0^{(+1)} = \psi_0^{(-1)}$. As we will show in a while, these two states are related to two different representations of the same abstract C*-algebra, representations which
are labeled by different values of an order parameter, the so called magnetization, \( m \approx< \psi_0^{(\pm1)} : \frac{1}{|V|} \sum_{j \in V} \pi^{(\pm1)}(\sigma_j^3) \psi_0^{(\pm1)} > \rightarrow \pm 1 \). This implies, moreover, that the two representations, called here \( \pi^{(+1)} \) and \( \pi^{(-1)} \), are necessarily \textit{unitarily inequivalent}, since they describe different physics.

It may be worth noticing also that this model exhibits a first example of \textit{spontaneous breaking of a symmetry}: the following symmetry of the \( H, \gamma : \sigma_j^3 \rightarrow -\sigma_j^3 \), is clearly not a symmetry of the ground state, meaning with this that the two vectors \( \psi_0^{(\pm1)} \) are not left invariant by the map \( \gamma \). We will discuss in the following a consequence of this property.

This kind of physical systems can be properly discussed within the framework of C*-algebras, as first proposed by Haag and Kastler, [33], whose construction goes as follows.

**The algebra.** Let \( \Sigma \) be a physical system with infinite degrees of freedom, \( V \subset \mathbb{R}^d \) a finite \( d \)-dimensional region, \( \mathcal{H}_V \) the related Hilbert space (whose construction depends on \( \Sigma \) and will be discussed in a moment), \( \mathcal{A}_V = B(\mathcal{H}_V) \) the associated C*-algebra of bounded operators acting on \( \mathcal{H}_V \) and let finally \( H_V \) be the self-adjoint energy operator for \( \Sigma_V \), the restriction of \( \Sigma \) in \( V \).

The family of algebras \( \{ \mathcal{A}_V \} \) satisfies the following properties:

- \textit{isotony}: if \( V_1 \subset V_2 \) then \( \mathcal{A}_{V_1} \subset \mathcal{A}_{V_2} \). Moreover \( ||.||_2 |_{V_1} = ||.||_1 \) (\( \Rightarrow \mathcal{A}_{V_1}, \mathcal{A}_{V_2} \subset \mathcal{A}_{V_1 \cup V_2} \));

- if \( V_1 \cap V_2 = \emptyset \) then \( [\mathcal{A}_{V_1}, \mathcal{A}_{V_2}] = 0 \).

In particular this last property clearly shows the non relativistic framework we are discussing here, since it simply means that two operators localized in disjoint spatial regions are necessarily independent, i.e. they must commute. Then we define \( \mathfrak{A} = \mathcal{A}_0 = \bigcup \mathcal{A}_V \) and \( \mathfrak{A} \) is the \textit{quasi-local C*-algebra of the bounded observables}.

On this algebra we can introduce the \textit{spatial translations} \( \{ \gamma_x \} \), which is a group of \*-automorphisms of \( \mathfrak{A} \) satisfying the following: \( \gamma_x \mathcal{A}_V = \mathcal{A}_{V+x}, \gamma_{x_1} \gamma_{x_2} = \gamma_{x_1+x_2} \).

It is now worth discussing briefly two examples of this construction.

**Example 1: discrete system**

Let \( X \) be an infinite lattice, \( 0 \in X \), and \( \mathcal{H}_0 \) a finite dimensional Hilbert space (e.g. \( \mathcal{H}_0 = \mathbb{C}^2 \) for Pauli matrices). Let \( \mathcal{H}_x \) a copy of \( \mathcal{H}_0 \) localized in the lattice site \( x \in X \) and \( \mathcal{H}_V = \otimes_{x \in V} \mathcal{H}_x \), which is a \textit{finite dimensional} Hilbert space for each fixed \( V \). Then we consider the C*-algebra \( \mathcal{A}_V = B(\mathcal{H}_V) \) which, if \( \mathcal{H}_0 = \mathbb{C}^2 \), is isomorphic to the set of \( (2|V|) \times (2|V|) \) matrixes with complex entries.
The family of Hilbert spaces and C*-algebras constructed in this way are related to each other in the following easy and natural way: if \( V \subset V' \) then \( H_{V'} = H_V \otimes H_{V\cap V} \) and, \( \forall A \in \mathfrak{A}_V, A \otimes 1_{V\cap V} \in \mathfrak{A}_{V'} \).

Moreover, the map \( \gamma_k(a^{(1)} a^{(2)} \ldots a^{(n)}) = a_{x_1+\mathbf{k}}^{(1)} a_{x_2+\mathbf{k}}^{(2)} \ldots a_{x_n+\mathbf{k}}^{(n)} \) is an automorphism for each \( k \) and it represents the spatial translations.

Finally, the local energy is given by summing up the interactions of all the particles inside \( V \),

\[
H_V = \sum_r \sum_{x_1, \ldots, x_r \in V} V_r(x_1, x_2, \ldots, x_r)
\]

where \( V_r \) is the \( r \)-body interaction.

Example 2: continuous system

The starting point, in this case, are the Fermi or the Bose commutation rules, given in terms of smeared fields \( \Psi[f] = \int_{\mathbb{R}} \Psi(x) f(x) dx \), where we take \( f(x) \in \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \) for technical convenience:

\[
[\Psi[f], \Psi^\dagger[g]]_\pm = \langle \mathcal{g}, f \rangle, \quad [\Psi[f], \Psi^\dagger[g]]_\pm = 0, \quad \forall f, g \in \mathcal{S}(\mathbb{R}).
\]

Let \( \Phi_0 \) be the vacuum of the theory, i.e. a vector such that \( \Psi[f]\Phi_0 = 0, \forall f \in \mathcal{S}(\mathbb{R}) \).

The Hilbert space \( \mathcal{H}_V \) is the norm closure of \( \Psi^\dagger[f_1] \ldots \Psi^\dagger[f_n] \Phi_0 \), where each \( f_j \) is supported in \( V \). Observe that, even if \( V \) is a finite volume, \( \dim(\mathcal{H}_V) = \infty \). This implies that, even for finite systems, unbounded operators may appear in the game. To avoid this unpleasant aspect one usually introduces the following C*-algebra \( \mathfrak{A}_V = \{ X \in \mathcal{B}(\mathcal{H}_V) : [X, N_V] = 0 \} \), where \( N_V = \int_V \Psi^\dagger(x) \Psi(x) dx \) is the number operator. In this way, we will only consider those observables in \( \mathfrak{A}_V \) which are automatically bounded.

One of the relevant operators is the energy, i.e. the local hamiltonian, which for a 2-body interaction, is:

\[
H_V = \frac{\hbar^2}{2m} \int_V dx |\nabla \Psi(x)|^2 + \\
\frac{1}{2} \int_V dx \int_V dx' \Psi^\dagger(x) \Psi^\dagger(x') V(x, x') \Psi(x') \Psi(x),
\]

but we see that, in principle, \( H_V \notin \mathfrak{A}_V \) since \( H_V \) may be unbounded!

The states. Continuing with Haag and Kastler’s construction, we recall that the states of \( \Sigma \) are positive, normalized linear functionals on \( \mathfrak{A} \) which, when restricted to \( V \), reduces to the states over the finite system \( \Sigma_V \) and, therefore, over the finite volume algebra \( \mathfrak{A}_V \). In
other words, they corresponds to a family of density matrices $\rho_V$: $\dot{\rho}(A) = tr_V(\rho_V A)$ for each $A \in \mathcal{A}_V$, here $tr_V$ is the trace in $\mathcal{H}_V$. These states satisfy the following consistency condition: $tr_V(\rho_V A) = tr_V'(\rho_V' A) \forall A \in \mathcal{A}_V$, $V \subset V'$.

They have a physical interpretation which is given by the Ruelle, Dell’Antonio and Doplicher theorem: these states have zero probability to describe an infinite number of particles in a finite region. They are usually called in the literature locally finite states.

Among all the states a particular role is played by the so called pure states: $\rho$ is pure if it is not a convex combination of other states, i.e. if there are no $\rho_1, \rho_2$ and $\lambda \in]0,1[$ such that $\rho = \lambda \rho_1 + (1 - \lambda) \rho_2$. Their relevance is due to the fact that, as we will discuss in the following, they are related to the pure thermodynamical phases of a certain physical system.

Lanford and Ruelle introduced the notion of states with short range correlations (SRCS), which are given as follows: let $B$ be a local bounded observable, $\epsilon$ a positive number. Then $\rho$ is a SRCS if there exists a bounded region $\Lambda$ such that, for all $A$ bounded and localized outside $\Lambda$, then

$$|\rho(AB) - \rho(A)\rho(B)| \leq \epsilon \|A\|.$$  

These states are related to the pure ones. Indeed, in 1969, Ruelle proved that each pure state is automatically a SRCS.

A weaker requirement is the so-called asymptotic abelianess: $\rho$ satisfies the asymptotic abelianess if, for each local $A, B$, we have

$$|\rho(A\gamma_j(B)) - \rho(A)\rho(\gamma_j(B))| \to 0,$$

when $|j| \to \infty$, which means that the quantity $\rho(A\gamma_j(B))$ factorizes whenever the two observables $A$ and $\gamma_j(B)$ are localized in regions of the space which are far away from one another.

**The dynamics.** The next step in our analysis is related to the description of the time evolution of the physical system $\Sigma$. This is obtained from the dynamics of $\Sigma_V$ in Heisemberg representation as follows:

first we define the time evolution of the element $A$ in $\mathfrak{A}_V$ in the volume $V$ as follows:

$\mathfrak{A}_V \ni A \to \alpha^V_t(A) := e^{it\mathcal{H}_V/\hbar}Ae^{-it\mathcal{H}_V/\hbar}$.

secondly we use $\alpha^V_t(A)$ to define $\alpha^t(A)$ as follow $\alpha^t(A) = \tau - \lim_V \alpha^V_t(A)$, where $\tau$ is a reasonable topology of $\mathfrak{A}$, i.e. a topology usually related to $\Sigma$ itself. Possible topologies are the following:

for short range interactions and discrete systems $\tau$ is usually the uniform topology, [32];
for long range interactions it is known that $\alpha^t_V$ is not $\| \cdot \|_-$converging: a possible alternative for $\tau$ is the strong topology (restricted to a relevant family of states). This different topology has been used in many papers, among which [7, 37, 44] and references therein. In this case a state $\rho$ must be chosen in such a way that

$$\rho(\alpha^t_V(A)) \rightarrow \rho(\alpha^t(A)) =: \rho_t(A),$$

and this limit defines the time evolution of the state $\rho$, $\rho_t$, by means of $\rho_t(A) := \rho(\alpha_t(A))$. It is clear that the existence of (sufficiently many) such $\rho$’s has to be checked in each model.

**The symmetry.** It is now possible to introduce the concept of symmetry: an automorphism of $\mathfrak{A}$, $\gamma$, is a symmetry of the system $\Sigma$ if $\alpha^t(\gamma(A)) = \gamma(\alpha^t(A))$ and is a local symmetry if $\gamma : \mathfrak{A}_V \rightarrow \mathfrak{A}_V$ and if $\gamma(H_V) = H_V$. We have already seen an example of a local symmetry at the beginning of this section, when speaking of the Ising model.

Moreover, the automorphism $\gamma$ is a symmetry of the state $\rho$ if $\rho_\gamma(A) := \rho(\gamma(A)) = \rho(A)$, $\forall A \in \mathfrak{A}$.

**Representations and GNS-construction.** A crucial notion, also in view of its physical applications, is that of a *-representation of a *-algebra. This is essentially a map $\pi : \mathfrak{A} \rightarrow B(\mathcal{H})$, for a certain $\mathcal{H}$, which preserves the algebraic structure of $\mathfrak{A}$:

$$\pi(A + B) = \pi(A) + \pi(B), \quad \pi(\lambda A) = \lambda \pi(A),$$

$$\pi(AB) = \pi(A)\pi(B), \quad \pi(A^*) = \pi(A)^*.$$ 

It is clear that $\pi(\mathfrak{A})$ is a *-algebra as well.

It is a well known fact that any state $\rho$ over an abstract C*-algebra $\mathfrak{A}$ produces a unique (but for equivalence) triplet $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$, where $\mathcal{H}_\rho$ is an Hilbert space, $\pi_\rho$ is a representation (in the sense discussed above) and $\Omega_\rho$ is a cyclic vector of $\mathcal{H}_\rho$, i.e. $\pi_\rho(\mathfrak{A})\Omega_\rho$ is dense in $\mathcal{H}_\rho$. Moreover we have, $\forall A \in \mathfrak{A}$,

$$\rho(A) = \langle \Omega_\rho, \pi_\rho(A)\Omega_\rho \rangle.$$ 

Also, $\pi_\rho$ is irreducible if and only if $\rho$ is pure.

We can give here the sketch of the proof: $\mathfrak{A}$ becomes a pre-Hilbert space wrt the following positive semidefinite scalar product: $(A, B) = \rho(A^*B)$. Let $\mathcal{I}_\rho = \{ A \in \mathfrak{A} : \rho(A^*A) = 0 \}$. This is a left ideal of $\mathfrak{A}$ $(A \in \mathcal{I}_\rho, X \in \mathfrak{A}$ then $XA \in \mathcal{I}_\rho)$. We introduce the equivalence classes: $[A] = A + I$, $I \in \mathcal{I}_\rho$ which produce a complex vector space when equipped with the following operations: $[A] + [B] = [A + B]$, $[\lambda A] = \lambda[A]$. In this way the set $\{ [A], A \in \mathfrak{A} \}$ is equipped with
a positive definite scalar product: \( < [A], [B] > = (A, B) = \rho(A^*B) \). If we complete \( \{[A], A \in \mathcal{A}\} \) wrt the norm inherited from \( <, > \) we get our Hilbert space \( \mathcal{H}_\rho \).

The representation \( \pi_\rho \) is defined by \( \pi_\rho(A)[B] := [AB] \), while the cyclic vector \( \Omega_\rho = [I] \). Notice that, incidentally, \( \pi_\rho \) is a bounded representation since \( \|\pi_\rho(A)[B]\| \leq \|A\|\|B\| \), for each \( A, B \in \mathcal{A} \). This means that, \( \forall A \in \mathcal{A} \), then \( \pi_\rho(A) \in B(\mathcal{H}_\rho) \).

**Remarks:**

1. The first obvious remark is that GNS representations generated by different states need not be unitarily equivalent!

2. Each (GNS) representation corresponds to a phase of the physical system. In particular, GNS representations generated by pure states correspond to pure phases [38].

3. States which are only locally different are macroscopically indistinguishable: all the macroscopic observables have the same expectation values. For instance, if we go back to the Ising model, it is clear that

\[
\lim_{V \to \infty} < \Phi_0^{(+1)} \pi^{(+1)}(\sigma^3_V \Phi_0^{(+1)} > = 1 = \lim_{V \to \infty} < \pi^{(+1)}(A) \Phi_0^{(+1)} \pi^{(+1)}(\sigma^3_V) \pi^{(+1)}(A) \Phi_0^{(+1)} > ,
\]

for any strictly localized \( A \in \mathcal{A} \), [41]. Then they produce unitarily equivalent GNS representations. In [41] the author says that two locally different states belong to the same island or to a given folium, see [42].

This has a clear physical interpretation: equal values of the macroscopic observables (the so-called order parameters) label unitarily equivalent representations, which are interpreted as the same phase of the matter. In other words: two different phases of the matter correspond to two representations in which some macroscopic observable assumes different values.

4. An interesting result is the following: even if the algebraic dynamics for \( \Sigma \) cannot be given in an hamiltonian form, nevertheless, under certain assumptions on \( \Sigma \), the dynamics in each representation \( \pi_\rho \) is hamiltonian: there exists a s.a. operator \( \hat{H}_\rho \) such that, \( \forall A \in \mathcal{A} \),

\[
\frac{d}{dt} \alpha_t^\rho(\pi_\rho(A)) = i[\hat{H}_\rho, \alpha_t^\rho(\pi_\rho(A))],
\]

see [41] and references therein. \( \hat{H}_\rho \) is what is often called in literature the effective hamiltonian.

This result has a clear physical interpretation: different phases of \( \Sigma \) may have, and they usually have, different dynamical behaviors, and this is reflected in the different possible expressions for \( H_\rho \).

### III.1 Non-zero temperature

We devote this subsection to some brief considerations on equilibrium states and to the associated phase structure, considering separately the cases of quantum systems with finite and
infinite degrees of freedom.

Let us start considering finite systems. In this case we can prove that the following are equivalent:

(i) $\rho$ is a Gibbs state corresponding to the trace class operator $\hat{\rho} = e^{-\beta H_V} \text{tr}(\text{idem})$, where $\beta^{-1} = kT$;
(ii) $\rho$ minimizes the free energy functional $\hat{F}_V(\rho) = \text{tr}_V(\rho H_V + \beta^{-1} \rho \log(\rho))$;
(iii) $\rho$ is a KMS (i.e. Kubo-Martin-Schwinger) state at the corresponding inverse temperature $\beta$, i.e., roughly speaking, if $A, B \in \mathfrak{A}$, $\rho(AB) = \rho(BA) + i\hbar \beta$.

This equivalence has an immediate consequence: for each temperature there exists an unique equilibrium state, and, therefore, an unique associated GNS representation. This means that, for such a finite system, there exists a single thermodynamical phase of $\Sigma$ at each given temperature.

The situation is completely different for infinite systems. For these systems the role of the thermodynamical limit is crucial, and not only for the existence of the time evolution, as we will discuss in a moment.

In order to keep the analysis simple, it is convenient to make the following assumptions on the finite volume hamiltonian $H_V$:

1. first we require that $H_V$ is such that $H_{V_1 \cup V_2} - H_{V_1} - H_{V_2}$ is a surface effect. This condition holds, for instance, for short range forces;
2. there exists $c > 0$ such that $\|H_V\| \leq c|V|$.

These assumptions imply, first of all, that $\alpha^t(A) = \|A - \lim_{V \to \infty} \alpha_t^V(A)$.

Secondly, let us define the following functionals

$$
\begin{align*}
E_V(\rho_V) &= \text{tr}_V(\rho_V H_V), \\
S_V(\rho_V) &= -k \text{tr}_V(\rho_V \log(\rho_V)), \\
F_V(\rho_V) &= E_V(\rho_V) - T S_V(\rho_V)
\end{align*}
$$

These are called the local energy, the entropy and the free energy functionals. In particular, the entropy functional satisfies the following crucial inequalities, useful to compute the thermodynamical limits of some functional related to $S_V(\rho_V)$: for each $V_1 \cap V_2 = \emptyset$ then

$$
S_{V_1 \cup V_2}(\rho) \leq S_{V_1} + S_{V_2}.
$$

This is the so-called subadditivity property of the entropy. The strong subadditivity property, proved later by Lieb and Ruskai, also holds: for each $V_1, V_2$, the following inequality is satisfied:

$$
S_{V_1 \cup V_2}(\rho) + S_{V_1 \cap V_2}(\rho) \leq S_{V_1} + S_{V_2}.
$$
Remark: In view of the pedagogical nature of this paper, it may be of some interest to comment briefly about the definition of the entropy functional. As a matter of fact this can be seen as a quantum counterpart of a concept arising from information theory, where one consider the entropy as a mean surprise. Let us show how:

let us consider a set of $M$ elementary events $\{E_1, \ldots, E_M\}$ and let $p_j$ be the probability that the event $E_j$ occurs: of course we have $p_j \geq 0$ and $\sum_{j=1}^{M} p_j = 1$. Let further $u_j = -\log(p_j)$ be the surprise related to $E_j$: with this definition it appears clear that, if we are sure that $E_j$ is going to occur ($p_j \approx 1^-$), then there is no surprise at all (and indeed we have $u_j \approx 0$). On the other way, if $E_j$ is extremely rare ($p_j \approx 0^+$), then the surprise is very large (and we find $u_j \approx \infty$).

The mean surprise is defined as

$$MS = \frac{\sum_{i=1}^{M} N_i u_i}{\sum_{i=1}^{M} N_i} = -\sum_{i=1}^{M} p_i \log(p_i),$$

where $p_i = N_i/N$, which is exactly the Shannon expression of the entropy. The generalization from the classical to the quantum entropy gives rise to the definition above.

The assumptions for $H_V$ and the subadditivity of the entropy, imply that the following global density functionals

$$e(\rho) = \lim_{V \nearrow} \frac{E_V(\rho_V)}{|V|}, \quad s(\rho) = \lim_{V \nearrow} \frac{S_V(\rho_V)}{|V|}, \quad f(\rho) = \lim_{V \nearrow} \frac{F_V(\rho_V)}{|V|}$$

exist, as well as the following incremental functionals

$$\begin{align*}
\Delta E(\rho|\rho') &= \lim_{V \nearrow} (E_V(\rho'_V) - E_V(\rho_V)), \\
\Delta S(\rho|\rho') &= \lim_{V \nearrow} (S_V(\rho'_V) - S_V(\rho_V)), \\
\Delta F(\rho|\rho') &= \lim_{V \nearrow} (F_V(\rho'_V) - F_V(\rho_V)),
\end{align*}$$

where $\rho'$ is a local modification of $\rho$, i.e. a state which differs from $\rho$ only on a volume of finite size.

The role of these functionals is crucial in the analysis of the equilibrium states. A state $\tilde{\rho}$ is globally thermodynamically stable (GTS) if it is invariant under translations and if it minimizes $f(\rho)$. It is locally thermodynamically stable (LTS) if $\Delta F(\tilde{\rho}|\rho') \geq 0$ for all $\rho'$, local modification of $\tilde{\rho}$. Then, [41], it is proved that a GTS state is a LTS state, while an LTS state which is invariant under translations is also a GTS state for systems with short range interactions.

Again, this result has a physical interpretation, which can be deduced also from the explicit solution of some easy physical models: a GTS state is an equilibrium state. The LTS states are,
for systems with long range interactions, only metastable states (i.e. states with a long mean life and good thermodynamical properties). Obviously, they are also true equilibrium states under the above assumptions.

Another interesting result relates the LTS and the KMS states: they are exactly the same objects, [41]! This implies, of course, that a KMS state for an infinite system is not an equilibrium state, in general, but only a metastable state. This is different from what happens for finite systems, as we have already pointed out.

It may be worth, at this stage, recalling some general results on the KMS states. Let Σ be an finite system. Then, as we have already said, a state ρ over a C*-algebra $\mathfrak{A}$ is a KMS-state at an inverse temperature $\beta$ (briefly, a $\beta$-KMS state in the following) if, for all observables $A, B$ and for all $t \in \mathbb{R}$,

$$\rho(A_t B) = \rho(B A_{t+i\hbar\beta})$$

For infinite systems this definition does not work, in general, since, e.g., $A_{t+i\hbar\beta}$ may make no sense. Also, and even more substantial, the time evolution may not exist even for real time. Therefore, in reference [32], a different definition was proposed:

$$\rho$$ is a $\beta$-KMS state if for each $A, B \in \mathfrak{A}$ there exists a complex function $F_{AB}(z)$ which is analytical in the strip $\Im(z) \in [0, \hbar\beta]$, continuous on the boundaries, and is such that

$$F_{AB}(t) = \rho(BA_t), \quad F_{AB}(t + i\hbar\beta) = \rho(A_t B)$$

The physical interpretation of KMS-states are well established by some explicitly solvable quantum models: a $\beta$-KMS state, $\rho_\beta$, is nothing but a reservoir at a temperature $T = \frac{1}{k\beta}$. Indeed, given $\Sigma$ described by $\rho_\beta$ and weakly coupled with a finite system $\mathcal{S}$, in the limit $t \to \infty$, one can prove that, independently of the details of the $\Sigma - \mathcal{S}$ interaction and of the initial state of $\mathcal{S}$, this is described by a Gibbs state corresponding to the same inverse temperature $\beta$ of $\Sigma$.

What makes the difference between finite and infinite systems is now the following remark: while for a given temperature the equilibrium state of a finite system is uniquely fixed by any of the three equivalent requirements discussed above, an infinite system $\Sigma$ may possess more than one GTS state at the same temperature. Examples of different GTS states may be constructed as limits of a Gibbs state (for those thermodynamical conditions) corresponding to different boundary conditions. This result has a mathematical interpretation which is quite simple: while the potential $\hat{F}_V(\rho)$ is convex, and therefore admits an unique minimum, the free energy density functional $f(\rho)$ is affine, so that more than a single minimum may be achieved.

If this is the case, $\Sigma$ admits different thermodynamical phases under the same thermodynamical conditions, each corresponding to a different GTS state. We say that the system possesses
macropscopic degeneracy: these different equilibrium states (and the related physical phases)
are labeled by the (different) values of some macropscopic observables (like the magnetization in
the case of ferromagnetic materials), [41].

This fact has several related consequences, which we here list and comment briefly, referring
to specific textbooks for a deeper analysis:

The first consequence of the algebraic approach discussed so far, and the possibility of
having a macropscopic degeneracy for infinitely extended systems, is that it provides a nice
framework to analyze coexisting phases of such systems and, as a consequence, to discuss easily
the occurrence of phase transitions.

A second consequence, which is deeply connected with the first one, is that we can use this
approach to discuss the so called spontaneous breaking of a symmetry:

suppose that Σ has a local symmetry γ and let ∆ = {ρ ∈ A : ρ is GTS}, be the set of GTS
states. Then, since necessarily we have f(ρ) = f(ργ), it follows that for any ρ ∈ ∆ also ργ ∈ ∆:
the symmetry γ maps ∆ into itself.

From that we see that, if ∆ = {ρ1} consists of a single element, γ is necessarily a symmetry
of ρ1: the symmetry is unbroken. In other words, in this case it is clear that ρ1 = (ρ1)γ.

If, on the contrary, ∆ = {ρ1, . . . , ρn}, then, in general, we can only say that (ρi)γ = ρj,
for some i and j not necessarily equal: if this is the case, then the symmetry is spontaneously
broken.

Example (T=0 Ising model): for this model HV is invariant under spin reversal (σj → −σj),
which is therefore a local symmetry, since it also maps A into itself, but the two
(transactionally invariant) ground states associated to the magnetization m = ±1 are, clearly,
no longer invariant: they are mapped into each other by the symmetry. To be more explicit,
the vectors ψ(±1) are the cyclic vectors related, via a standard GNS construction, to the two
states ρ± over the quasi-local C*-algebra of the discrete system, which are characterized by the
values of the following limits:

\[ \lim_{N \to \infty} \rho_\pm \left( \frac{1}{2N+1} \sum_{j=-N}^{N} \sigma_j^3 \right) = \pm 1. \]

In this example it is clear that ∆ = {ρ+, ρ−}.

Whenever a system exhibits a spontaneously symmetry breaking, a related result on the
spectrum of the theory can be deduced by making use of the non relativistic Goldstone’s theo-
rem.
Roughly speaking, this theorem says the following: suppose that the symmetry $\gamma_\lambda$ is generated by a local charge

$$Q_R(t) = \int_{|\vec{x}| \leq R} j_o(\vec{x}, t) \, d^3x,$$

and suppose that $\gamma_\lambda$ commutes with the time translations $\alpha^t$. Then, if $\gamma_\lambda$ is spontaneously broken, i.e. if for some $A \in \mathfrak{A}$, $\lim_{R \to \infty} < Q_R, A >_{\psi_0} \neq 0$, then the energy spectrum cannot have a gap above the ground state.

This theorem has proved to be a very important tool both in condensed matter and in quantum field theory.

We want to end this list of results related to our algebraic approach by mentioning a very interesting relation between KMS states and the Tomita-Takesaki theory, which enriches the list of relevant results which can be easily proven using the C*-algebraic picture of $QM_\infty$.

We start recalling very briefly few facts on this theory:

first let us recall that a von Neumann algebra (VNA) is a selfadjoint (i.e. closed with respect to the adjoint) subset of $B(\mathcal{H})$, $\mathcal{M} \subset B(\mathcal{H})$, such that $\mathcal{M} = \mathcal{M}'$. Here $\mathcal{M}' = \{X \in B(\mathcal{H}), [X, A] = 0 \, \forall A \in \mathcal{M}\}$ is called the commutant of $\mathcal{M}$ and $\mathcal{M}''$, which is constructed in the same way, is its bicommutant. Equivalently, $\mathcal{M} \subset B(\mathcal{H})$ is a VNA if it is weakly closed or strongly closed.

This implies that every VNA is a C*-algebra (indeed, if $\mathcal{M}$ is weakly closed then $\mathcal{M}$ is automatically uniformly closed), while not any C*-algebra is a VNA (e.g. $\mathbb{C}^0(X)$).

Tomita-Takesaki’s theorem is given for $\sigma$–finite VNAs, for which a cyclic and separating vector surely exists. We recall that a vector $\varphi \in \mathcal{H}$ is said separating for $\mathcal{M}$ if, $X \varphi = 0$ for $X \in \mathcal{M}$ is equivalent to $X = 0$. Let $\Omega$ be a cyclic and separating vector, and let $S_0$ and $F_0$ be the densely defined operators

$$S_0 A \Omega = A^* \Omega, \quad F_0 A' \Omega = A'^* \Omega, \quad \forall A \in \mathcal{M}, \forall A' \in \mathcal{M}' .$$

These operators are closable, $S = \overline{S_0}$, $F = \overline{F_0}$, and the polar decomposition of $S$, $S = J \Delta^{1/2}$, produces the modular conjugation $J$ and the modular operator $\Delta$ associated to $(\mathcal{M}, \Omega)$.

Many results have been discussed in the literature concerning $J$ and $\Delta$, but since they have no role here, we will not give further details, referring to [21] for more information. We just want to mention here the following result:
**Tomita-Takesaki theorem:** With the above definitions we have

$$J M J = M', \quad \text{and} \quad \Delta^it M \Delta^{-it} = M,$$

for all $t \in \mathbb{R}$.

It is not very hard to show now that all the KMS states can be used to generate a modular structure in the sense of Tomita-Takesaki:

let $\rho$ be a $-1$-KMS state (i.e. a KMS state corresponding to $\beta = -1$). This is what it is usually called simply a KMS state). This state generates a GNS representation $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$. Let $U_\rho(t)$ the unitary operator which implements $\alpha^t$ in this representation. Then $\Omega_\rho$ is cyclic and, since $\rho$ is KMS, is also separating for $\pi_\rho(\mathfrak{A})''$, i.e. $\pi_\rho(X)\Omega_\rho = 0$ implies that $\pi_\rho(X) = 0$, [21].

Then we are in the assumptions of Tomita-Takesaki’s construction, so that we can introduce a modular conjugation $J_\rho$ and a modular operator $\Delta_\rho$ associated to $(\pi_\rho(\mathfrak{A})'', \Omega_\rho)$. Calling $H_\rho$ the generator of $U_\rho(t)$, we find that

$$\Delta_\rho = e^{H_\rho}.$$  

In other words: given a system $\Sigma$ an effective hamiltonian surely exists in any representation GNS-constructed by a given KMS-state.

It may be worth stressing that these are only few results which can be obtained within an algebraic frameworks. More results on phase transitions, applications to quantum field theory, statistical mechanics etc. can be found in many specialized textbooks, among which we only cite [43, 21, 22, 41, 42].

**III.2 A list of problems**

Instead of giving more results on this canonical scheme, we devote the last part of this section to discuss some limits which are, in our idea, intrinsic with the approach discussed so far, and which suggest the construction of a slightly generalized algebraic framework. These conclusions are based on a simple remark: the main results which have been given in this section are obtained under some requirements, which may not be necessarily satisfied in many relevant conditions. For instance, we have assumed that the norm of the local hamiltonian $H_V$ does not grow faster than $|V|$: $\|H_V\| \leq c|V|$.

However, this is not always true: actually, it is false quite often! For instance, this inequality is violated already by a gas of free bosons, for which $H_V = \sum_{j \in V} a_j^\dagger a_j$, since each creation and annihilation operator is such that $\|a_j\| = \|a_j^\dagger\| = \infty$. 

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This same condition is satisfied, on the contrary, by a gas of free fermions, for which, however, \( \text{dim}(\mathcal{H}_V) < \infty \). This is one of the reasons why, in the analysis of an open system, the free gas of bosons which constitutes the reservoir is frequently replaced by a gas of fermions. We will come back on this point in a moment.

The second assumption considered above is that the interactions are short ranged.

However this is not the case in many situations. For instance, the Coulomb interaction is long ranged, while in the mean field models the real forces are replaced by interactions with an infinite range: this means that, given two particles localized in \( i \) and \( j \), they feel the same strength independently of the difference \( |i - j| \). However, many results can be obtained even under these conditions. In particular we find that:

- \( \alpha_t^V \) is not norm convergent to an algebraic dynamics \( \alpha^t \), but, as we have already sketched before, we can (often) find a different topology which makes of \( \alpha_t^V(A) \) a Cauchy sequence for each (or for many) \( A \in \mathfrak{A} \), see [7, 44, 26] just to cite few authors.

- We have already mentioned that, in general, KMS states are not equilibrium states (i.e. GTS states). Moreover, they are not even limits of Gibbs state; indeed, in [31], it is stated the following result, which we repeat here in a simplified version:

  if \( \alpha_t^V \) is uniformly convergent to \( \alpha^t \) and if, calling \( \omega_\beta^{(V)}(A) = \text{tr}_V(\rho_{\beta,V} A) \), with \( \rho_{\beta,V} = \frac{e^{-\beta H_V}}{\text{tr}(\text{idem})} \), the limit \( \lim_V \omega_\beta^{(V)}(A) = \omega_\beta(A) \) exists for each \( A \in \mathfrak{A}_0 \), then:

  a. \( \omega_\beta \) is a \( \beta \)-KMS state wrt \( \alpha^t \);

  b. \( \omega_\beta \) is associated to a modular operator and a modular conjugation (in the sense of Tomita-Takesaki).

**Remark:** it is clear that we are requiring here the uniform convergence of \( \alpha_t^V \), which, as we have just seen, is violated for long range interactions, and the existence of the limit of \( \omega_\beta^{(V)}(A) \), which is not ensured a priori! Therefore we cannot conclude that limit of Gibbs states are surely KMS states.

- surface effects become volume effects, so that variables at infinity (i.e. completely delocalized operators) appear in the dynamics of strictly localized operators. These are related to the order parameters used to describe different phases, [37] and references therein;

- the density functionals \( e(\rho) \) and \( f(\rho) \) do not necessarily exist, since in the proof of their existence, the assumption that the forces are short ranged is crucial, see [41];

- the Goldstone’s theorem holds only in a modified form [37].

We see that a new world appears whenever the interactions appearing in the physical system modify their range. We also observe that many things can be said but many other aspects are
still to be clarified.

A third assumption which is usually somehow hidden in the C*-algebraic approach to QM\(^\infty\) is related again to the presence on the (almost) unavoidable unbounded operators. Consider, for instance, the position and momentum operators \(\hat{q}\) and \(\hat{p}\). As we have already mentioned, they satisfy the following commutation relations: \([\hat{q}, \hat{p}] = i \mathbb{I}\) (in convenient units) and, as a consequence, it is an easy exercise to check that at least one of them must be unbounded. Actually, it is well known that they are both unbounded. In the literature three possible ways to deal with unbounded operators have been proposed: the first one consists in restricting the action of the operators on some (possibly dense) subset of a given Hilbert space, a sort of common domain of all the operators. A second possibility is to exponentiate these unbounded and self-adjoint operators in order to define unitary (and therefore bounded) operators. The original operators can be recovered by taking suitable derivatives of the unitary maps on certain relevant sets of vectors. A third possibility is the following: we could replace, say, the operator \(\hat{p}\) with a bounded operator \(\hat{p}_N\) whose spectrum coincides with the one of \(\hat{p}\) inside a compact interval \([-N, N]\), and is zero outside this set. It is clear that \(\hat{p}_N\) is bounded and, as \(N \to \infty\), approaches \(\hat{p}\) in some sense. Then one considers only those states \(\omega_N\) on \(\mathfrak{A}\), \(N\)-depending as well, such that \(\omega_N(\hat{p}_N)\) converges in the limit \(N \to \infty\) to some specific quantity, which in some sense represent the mean value of the original operator \(\hat{p}\) on a state which can be interpreted as the limit of the family of states \(\omega_N\). An example of this procedure can be found in [1].

However, quite often this is not enough. As an example, we cite the Lindblad expression for the generator \(L\) of a completely positive semigroup (describing the time evolution of a quantum open system). These structures play a very important role in the analysis of order-disorder transitions out of equilibrium, [41, 42].

More in details, let \(\mathfrak{A}\) and \(\mathfrak{B}\) be C*-algebras. We recall that a map \(f : \mathfrak{A} \to \mathfrak{B}\) is positive if \(f(A) > 0\) for each \(A > 0\). It is completely positive, CP, if, for any finite matrix algebra \(\mathcal{M}\), the mapping \(f \otimes \mathbb{I} : \mathfrak{A} \otimes \mathcal{M} \to \mathfrak{B} \otimes \mathcal{M}\) is positive.

Examples of completely positive maps are the following: (1) the automorphisms of C*-algebras are CP; (2) let \(\mathcal{K} \subset \mathcal{H}\) be both Hilbert spaces and \(P\) the projection operator from \(\mathcal{H}\) into \(\mathcal{K}\). Then \(f(A) = PAP\) is CP.

A quantum dynamical semi-group is a set \(\{T_t : t \geq 0\}\) of completely positive, identity preserving maps of \(\mathfrak{A}\) such that \(T_sT_t = T_{s+t}\) for all \(s, t \geq 0\) and \(T_0 = \mathbb{I}\). If \(T_t\) is normwise continuous in \(t\) for all \(A \in \mathfrak{A}\), then there exists an infinitesimal generator \(L\) defined by the formula

\[
\frac{d}{dt} T_t A = LT_t A = T_t L A, \quad \forall A \in \mathfrak{A}
\]
Lindblad proved that, if $\mathfrak{A} = B(\mathcal{H})$ (for some $\mathcal{H}$), $L$ has necessarily the following expression:

$$LA = i[H, A] + \sum_j \left( V_j^* AV_j - \frac{1}{2} \{V_j^* V_j, A\} \right),$$

where $H$ is self-adjoint and $V_j, \sum_j V_j^* V_j \in \mathfrak{A}$.

Very few results exist for unbounded operators, [29, 24, 15], mainly because of the following technical difficulty: if $T_t$ is a semigroup (and not a group) it follows that $T_t(AB) \neq T_t(A)T_t(B)$. For this reason even if we can introduce a cutoff in the system (so that all the operators we get are bounded), no known general result on perturbation of generators can be used because it will contrast with the final operation of removing the cutoff!

As we have already mentioned, to avoid this kind of difficulties, in many models the boson reservoir is replaced by a fermioncic one, as for instance, in [23] for the open BCS-model, changing a realistic into a, somehow, toy model. This suggests that an alternative procedure should be considered, and this will be the contain of the next sections.

IV Algebras of unbounded operators

In this section we will briefly introduce different examples of what we generically call algebras of unbounded operators, giving only those mathematical results and definitions which are relevant for our purposes. A much deeper analysis of these aspects can be found, for instance, in [2] or in [40].

A possible algebraic framework, which is also the main one we will work with here and in the next section, is the following:

let $\mathfrak{A}$ be a linear space, $\mathfrak{A}_0 \subset \mathfrak{A}$ a $*$-algebra with unit $\mathbb{I}$ (otherwise we can always add it): $\mathfrak{A}$ is a quasi $*$-algebra over $\mathfrak{A}_0$ if

[i] the right and left multiplications of an element of $\mathfrak{A}$ and an element of $\mathfrak{A}_0$ are always defined and linear;

[ii] $x_1(x_2a) = (x_1x_2)a, (ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$, for each $x_1, x_2 \in \mathfrak{A}_0$ and $a \in \mathfrak{A}$;

[iii] an involution $*$ (which extends the involution of $\mathfrak{A}_0$) is defined in $\mathfrak{A}$ with the property $(ab)^* = b^*a^*$ whenever the multiplication is defined.

A quasi $*$-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is locally convex (or topological) if in $\mathfrak{A}$ a locally convex topology $\tau$ is defined such that (a) the involution is continuous and the multiplications are separately
continuous; and (b) $A_0$ is dense in $A[\tau]$.

Let $\{p_\alpha\}$ be a directed set of seminorms which defines $\tau$. The existence of such a directed set can always be assumed. We can further also assume that $A[\tau]$ is complete. Indeed, if this is not so, then the $\tau$-completion $\tilde{A}[\tau]$ is again a topological quasi $*$-algebra over the same $*$-algebra $A_0$.

One may ask why these structures are related to unbounded operators. This can be understood in a simple way just remarking that, if $a$ and $b$ are unbounded operators, then $ab$ and $ba$ do not exist in general. But if $x$ is a third bounded operator, then $xa, ax, bx$ and $xb$ are all well defined, at least if the range of $x$ is contained in the domain of $a$ and $b$. This is reflected by the fact that $A$ is not a $*$-algebra, while $A_0$ is, but only a quasi $*$-algebra: not all its elements can be mutually multiplied, but we can safely multiply an element of $A$ with an element of $A_0$.

This abstract argument can be made more explicit by the next example, which shows explicitly that some concrete realization of $(A, A_0)$ contains unbounded operators.

**Example:** Let $H$ be a separable Hilbert space and $N$ an unbounded, densely defined, self-adjoint operator. Let $D(N^k)$ be the domain of the operator $N^k$, $k \in \mathcal{N}$, and $D$ the domain of all the powers of $N$: $D \equiv D^\infty(N) = \cap_{k \geq 0}D(N^k)$. This set is dense in $H$. Let us now introduce $L^\dagger(D)$, the $*$-algebra of all the closable operators defined on $D$ which, together with their adjoints, map $D$ into itself. Here the adjoint of $X \in L^\dagger(D)$ is $X^\dagger = X^\ast \mid_D$. \footnote{We need to introduce a map which, given an element $X \in L^\dagger(D)$, produces another element $X^\dagger \in L^\dagger(D)$. The most natural choice, which is clearly $X^\dagger \equiv X^*$, is only compatible with $L^\dagger(D) = B(H)$, i.e. with $N$ bounded, which is not what we want. Recalling that $D(X^*) \supseteq D$, it is clear that $X^\ast \mid_D$ is well defined. Further one can prove that $\dagger$ has the properties of an involution and maps $L^\dagger(D)$ into itself.}

In $D$ the topology is defined by the following $N$-depending seminorms: $\phi \in D \to \|\phi\|_n \equiv \|N^n\phi\|, n \in \mathbb{N}_0$, while the topology $\tau_0$ in $L^\dagger(D)$ is introduced by the seminorms

$$X \in L^\dagger(D) \to \|X\|^f_k \equiv \max \left\{ \|f(N)XN^k\|, \|N^kXf(N)\| \right\},$$

where $k \in \mathbb{N}_0$ and $f \in \mathcal{C}$, the set of all the positive, bounded and continuous functions on $\mathbb{R}_+$, which are decreasing faster than any inverse power of $x$: $L^\dagger(D)[\tau_0]$ is a complete $*$-algebra.

It is clear that $L^\dagger(D)$ contains unbounded operators. Indeed, just to consider the easiest example, it contains all the positive powers of $N$. Moreover, if $N$ is the closure of $N_o = a^\dagger a$, with $[a, a^\dagger] = I$, $L^\dagger(D)$ also contains all positive powers of $a$ and $a^\dagger$.

Let further $L(D, D')$ be the set of all continuous maps from $D$ into $D'$, with their topologies (in $D'$ this is the strong dual topology, see Appendix 2), and let $\tau$ denotes the topology defined
by the seminorms

\[ X \in \mathcal{L}(\mathcal{D}, \mathcal{D}') \rightarrow \|X\| = \|f(N)Xf(N)\|, \]

\( f \in \mathcal{C} \). Then \( \mathcal{L}(\mathcal{D}, \mathcal{D}')[\tau] \) is a complete vector space.

In this case \( \mathcal{L}^\dagger(\mathcal{D}) \subset \mathcal{L}(\mathcal{D}, \mathcal{D}') \) and the pair

\( (\mathcal{L}(\mathcal{D}, \mathcal{D}')[\tau], \mathcal{L}^\dagger(\mathcal{D})[\tau_0]) \)

is a concrete realization of a locally convex quasi *-algebra.

**Remark:** let us now suppose that \( \mathcal{D} \equiv \mathcal{S}(\mathbb{R}) \), the set of the test functions, and \( \mathcal{D}' = \mathcal{S}'(\mathbb{R}) \). Since \( \mathcal{S}(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}) \), it is easy to check that \( \mathcal{L}^\dagger(\mathcal{S}) \subset \mathcal{L}(\mathcal{S}, \mathcal{S}') \). Let \( \Psi(x) \in \mathcal{S}'(\mathbb{R}) \). We define the map \( Z_\Psi \) as follows: \( (Z_\Psi f)(x) = \Psi(x)f(x), \forall f(x) \in \mathcal{S}(\mathbb{R}) \). Since \( \Psi(x)f(x) \in \mathcal{S}'(\mathbb{R}) \), and since \( Z_\Psi \) is continuous, we conclude that \( Z_\Psi \in \mathcal{L}(\mathcal{S}, \mathcal{S}') \). It is clear that, for instance, \( Z_2^\dagger \), does not exist for generic \( \Psi \), and this reflects the fact that \( \mathcal{L}(\mathcal{S}, \mathcal{S}') \) is not an algebra.

Using a quasi *-algebra is not the only possibility to include unbounded operators in a reasonable algebraic framework. For completeness we briefly mention now two other possibilities which, however, will play no role in the physical applications considered in the next section.

We begin recalling that a partial *-algebra [3] is a complex vector space \( \mathfrak{A} \) with involution * (with the usual properties) and a subset \( \Gamma \subset (\mathfrak{A}, \mathfrak{A}) \) such that

\( (x, y) \in \Gamma \) iff \( (y^*, x^*) \in \Gamma \);

if \( (x, y), (x, z) \in \Gamma \) then \( (x, \lambda y + \mu z) \in \Gamma \) for all \( \lambda, \mu \in \mathbb{C} \);

if \( (x, y) \in \Gamma \) then there exists an element \( x \cdot y \in \mathfrak{A} \). This multiplication satisfies the following properties:

\[ x \cdot (y + \lambda z) = x \cdot y + \lambda x \cdot z \text{ and } (x \cdot y)^* = y^* \cdot x^*, \forall (x, y), (x, z) \in \Gamma. \]

Such a structure is a generalization of a quasi *-algebra, meaning with that that each quasi *-algebra is also a partial *-algebra. However, from our point of view, this appear to be too general to be used in concrete applications, and for this reason is not particularly relevant in our scheme.

Other examples of algebras of unbounded operators are the so-called CQ*-algebras [10], which can be seen as particular cases of topological quasi *-algebras. We do not give here the general definition but only its simplest version with some examples, referring to [10, 16, 17, 18, 19] for more details.

A (proper) CQ*-algebra is a quasi *-algebra such that: \( \mathfrak{A}_0 \|[\cdot, \cdot]_0 \) is a C*-algebra; \( \mathfrak{A}[\|\cdot\|] \) is a Banach space in which \( \mathfrak{A}_0 \) is dense; the two norms are related as follows:

\[ \|x\|_0 = \max \left\{ \sup_{\|a\| \leq 1} \|ax\|, \sup_{\|a\| \leq 1} \|xa\| \right\}, \forall x \in \mathfrak{A}_0. \]
This is a natural generalization of C*-algebras: indeed the completion of any C*-algebra $(\mathfrak{A}_0, \|\|_0)$ with respect to a weaker norm $\|\|_s$ satisfying: (i) $\|A^*\|_s = \|A\|_s$, $\forall A \in \mathfrak{A}_0$ and (ii) $\|AB\|_s \leq \|A\|_s \|B\|_0$, $\forall A, B \in \mathfrak{A}_0$, is a CQ*-algebra.

Let us now consider few examples of CQ*-algebras:

**Examples (commutative cases):**

1. We begin with $(L^p(X, \mu), C_0(X))$, where $X$ is a compact space and $C_0(X)$ is the set of continuous functions on $X$ [16];

2. The second abelian example is $(L^p(X, \mu), L^\infty(X, \mu))$, where $(X, \mu)$ is a measure space with $\mu$ a Borel measure on the locally compact Hausdorff space $X$ [16].

**Examples (non commutative cases):**

3. We first mention the non commutative $L^p$ spaces, [17].

4. A second example can be constructed as follows: let $\mathcal{H}$ be a Hilbert space with scalar product $(.,.)$ and $S$ an unbounded selfadjoint operator, with $S \geq I$, with dense domain $D(S)$. The subspace $D(S)$ becomes a Hilbert space, denoted by $\mathcal{H}_{+1}$, with the scalar product $(f,g)_{+1} = (Sf,Sg)$, and let $\mathcal{H}_{-1}$ denote the conjugate dual of $\mathcal{H}_{+1}$. Then $\mathcal{H}_{-1}$ itself is a Hilbert space. Given further

$$\mathfrak{A} = \{X \in \mathcal{B}(\mathcal{H}_{+1}, \mathcal{H}_{-1}) : X \text{ is compact from } \mathcal{H}_{+1} \text{ into } \mathcal{H}_{-1}\},$$

$$\mathfrak{A}_b = \{X \in \mathcal{B}(\mathcal{H}_{+1}) : X \text{ is compact in } \mathcal{H}_{+1}\},$$

then $(\mathfrak{A}[\|\|,], *, \mathfrak{A}_b[\|\|, b])$ is a (non proper) CQ*-algebra of operators, whose definition can be found in [18].

**Remark:** We also want to stress that these structures have been used in relation with Tomita-Takesaki’s theory, [19].

It is not surprising that, analogously to what happens for C*-algebras, a crucial role also from the point of view of physical application is played by the *-representations of a quasi *-algebra.

Let now $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra, $\mathcal{D}_\pi$ a dense domain in a certain Hilbert space $\mathcal{H}_\pi$, and $\pi$ a linear map from $\mathfrak{A}$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$, where

$$\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi) = \{X \text{ closable in } \mathcal{H}_\pi : D(X) = \mathcal{D}_\pi \text{ and } D(X^*) \supseteq \mathcal{D}_\pi\}. $$

This is a partial *-algebra with the usual operations $X + Y$, $\lambda X$, the involution $X^\dagger = X^*|_{\mathcal{D}_\pi}$ and the weak product $X \Box Y \equiv X^\dagger Y$ (defined whenever $Y\mathcal{D}_\pi \subset D(X^*)$ and $X^\dagger \mathcal{D}_\pi \subset D(Y^*)$).
Notice that these conditions produce the definition of the set $\Gamma \subset (\mathcal{L}(\mathcal{D}_\pi, \mathcal{H}_\pi), \mathcal{L}(\mathcal{D}_\pi, \mathcal{H}_\pi)))$.

Let furthermore

$$\mathcal{L}(\mathcal{D}_\pi) = \{ A \in \mathcal{L}(\mathcal{D}_\pi, \mathcal{H}_\pi) : A, A^\dagger \in \mathcal{D}_\pi \to \mathcal{D}_\pi \}.$$  

$\mathcal{L}(\mathcal{D}_\pi)$ is a $^*$-algebra and the weak multiplication $\Box$ reduces to the ordinary multiplication of operators.

In our context a $^*$-representation of $\mathfrak{A}$ is a linear map from $\mathfrak{A}$ into $\mathcal{L}(\mathcal{D}_\pi, \mathcal{H}_\pi)$ such that:

(i) $\pi(a^*) = \pi(a)^\dagger$, $\forall a \in \mathfrak{A}$;

(ii) if $a \in \mathfrak{A}$, $x \in \mathfrak{A}_0$, then $\pi(a) \Box \pi(x)$ is well defined and $\pi(ax) = \pi(a) \Box \pi(x)$.

Moreover, if

(iii) $\pi(\mathfrak{A}_0) \subset \mathcal{L}(\mathcal{D}_\pi)$,

then $\pi$ is said to be a $^*$-representation of the quasi $^*$-algebra $(\mathfrak{A}, \mathfrak{A}_0)$.

As for C*-algebras, we see here that a $^*$-representation preserves the algebraic structure of the abstract quasi $^*$-algebra $(\mathfrak{A}, \mathfrak{A}_0)$.

**Remark:**– It may be worth noticing that it might appear more natural to represent $(\mathfrak{A}, \mathfrak{A}_0)$ in another quasi $^*$-algebra $(\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}_\pi'), \mathcal{L}(\mathcal{D}_\pi'))$, analogous to the one constructed above with $\mathcal{D}_\pi$ instead of $\mathcal{D} = D^\infty(N)$. We will return on this quasi $^*$-algebra in the following. Nevertheless, it is usually more convenient to use $\mathcal{L}(\mathcal{D}_\pi, \mathcal{H}_\pi)$ for the following reasons:

1. if $a \in \mathfrak{A}$ then $\pi(a) \in \mathcal{L}(\mathcal{D}_\pi, \mathcal{H}_\pi)$. Therefore $\forall \varphi \in \mathcal{D}_\pi \pi(a)\varphi \in \mathcal{H}_\pi$. Of course, if we decide to represent $a$ as an element of $\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}_\pi')$, we go out of the Hilbert space ($\pi(a)\varphi \notin \mathcal{H}_\pi$, in general)! This is not exactly what one expects from a representation of a $^*$-algebra, since the abstract elements of the algebra are usually represented acting and living on some Hilbert space;

2. we also have a technical reason to use $\mathcal{L}(\mathcal{D}_\pi, \mathcal{H}_\pi)$, which will appear clear in a moment: in the theorem on the derivations given in the next section the topology $\tau_\pi$ plays a role, and this can be defined on $\mathcal{L}(\mathcal{D}_\pi, \mathcal{H}_\pi)$ but not on $\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}_\pi')$;

3. finally we can observe that any partial $^*$-algebra is a quasi $^*$-algebra: therefore the choice of representing a quasi $^*$-algebra into a partial $^*$-algebra of operators is consistent.

The $^*$-representation $\pi$ is called **ultra-cyclic** if there exists $\xi_0 \in \mathcal{D}_\pi$ such that $\pi(\mathfrak{A}_0)\xi_0 = \mathcal{D}_\pi$. $\pi$ is **faithful** if $\pi(x) = 0$ implies $x = 0$.

As we have anticipated, we use $\pi$ to introduce a certain topology on $\pi(\mathfrak{A})$: let $\pi$ be a $^*$-representation of $\mathfrak{A}$. The strong topology $\tau_\pi$ on $\pi(\mathfrak{A})$ is defined by the seminorms: $\{p_\xi(\cdot) ; \xi \in \mathcal{D}_\pi \}$.
\(p_\varepsilon(\pi(a)) \equiv \|\pi(a)\|, \ a \in \mathcal{A}, \ \xi \in \mathcal{D}_\varepsilon\). This will be used in the next section. It may be worth noticing that \(\|\pi(a)\|\) would make no sense in general if \(\pi(a)\) was an element of \(L^1(\mathcal{D}, \mathcal{D}')\).

As for ordinary C*-algebras, even now it is possible to give a GNS-like construction. As a matter of fact, several possible extensions of this construction exist, but we will here mention only one, \[45\].

Let us assume here that the topology \(\tau\) is given by a norm \(||.||\). Therefore \(\mathcal{A}\) is a Banach space and we suppose, for simplicity, that \((\mathcal{A}, \mathcal{A}_0)\) has a unit \(I\). Let \(\varphi\) a sesquilinear form on \(\mathcal{A} \times \mathcal{A}\) such that

1. \(\varphi(x, x) \geq 0, \ \forall x \in \mathcal{A}\);
2. \(\varphi(ax, y) = \varphi(x, a^*y), \ \forall a \in \mathcal{A}, \ x, y \in \mathcal{A}_0\);
3. there exists \(\gamma > 0\) such that \(|\varphi(x, y)| \leq \gamma \|x\|\|y\|, \ \forall x, y \in \mathcal{A}_0\).

These conditions imply that

\[N_\varphi = \{a \in \mathcal{A} : \varphi(a, a) = 0\} = \{a \in \mathcal{A} : \varphi(a, b) = 0, \ \forall b \in \mathcal{A}\}.

Let \(\lambda_\varphi(\mathcal{A}) = \mathcal{A}/N_\varphi\) and let us introduce a scalar product on this vector space as follows:

\[<\lambda_\varphi(a), \lambda_\varphi(b)> = \varphi(a, b).

Let further \(\mathcal{H}_\varphi\) be the completion of \(\lambda_\varphi(\mathcal{A})\) wrt the norm inherited by this scalar product. One can check that \(\lambda_\varphi(\mathcal{A}_0)\) is dense in \(\mathcal{H}_\varphi\).

Let us finally define a map \(\pi^\varphi\):

\[\pi^\varphi_\varphi(a)\lambda_\varphi(x) = \lambda_\varphi(ax), \ a \in \mathcal{A}, \ x \in \mathcal{A}_0:\]

Then \(\pi^\varphi\) is a *-representation of \(\mathcal{A}\) in \(L^1(\lambda_\varphi(\mathcal{A}_0), \mathcal{H}_\varphi)\). Moreover,

1. \(\lambda_\varphi(\mathcal{A}_0) = \pi^\varphi_\varphi(\mathcal{A}_0)\lambda_\varphi(I)\) (i.e. \(\lambda_\varphi(I)\) is ultra-cyclic);
2. \(\varphi(a, b) = <\pi^\varphi_\varphi(a)\lambda_\varphi(I), \pi^\varphi_\varphi(b)\lambda_\varphi(I)>, \ \forall a, b \in \mathcal{A}\).

Remarks:— (1) this approach uses sesquilinear forms instead of linear functionals: indeed, while \(\omega(a^*b)\) is not well defined for generic \(a, b \in \mathcal{A}\), independently of the choice of the linear functional \(\omega\), \(\varphi(a, b)\) surely makes sense if \(\varphi\) is a generic sesquilinear form on \(\mathcal{A} \times \mathcal{A}\). This allows to define a scalar product on \(\mathcal{A} \times \mathcal{A}\), as shown above;

(2) the physical interpretation is analogous to that discussed in the previous section: different sesquilinear forms produce different representations which can still be interpreted as different phases of the matter.
V Physical applications

In this section we will show how the algebraic framework discussed so far can be of some usefulness in the rigorous treatment of some physical systems.

V.1 Existence of an effective hamiltonian

We begin with reviewing some recent results obtained in collaboration with A. Inoue and C. Trapani, [13, 14], and concerning the possibility of introducing, under certain conditions on Σ, an effective hamiltonian.

Definition: Let \((\mathcal{A}, A_0)\) be a quasi \(*\)-algebra. A \(*\)-derivation of \(A_0\) is a linear map \(\delta : A_0 \to \mathcal{A}\) with the following properties:

(i) \(\delta(x^*) = \delta(x)^*\), \(\forall x \in A_0\);
(ii) \(\delta(xy) = x\delta(y) + \delta(x)y\), \(\forall x, y \in A_0\).

As we see, a \(*\)-derivation of \(A_0\) does exactly what one expects from a similar object: it is a linear map, it preserves the adjoint, and it satisfies the Leibnitz rule, of course only for those elements for which this can be defined. From a physical point of view we have already seen that it is quite hard for a derivation to be implemented by an hamiltonian at an algebraic level: even if a local hamiltonian does exist (i.e. the energy for finite \(V\)), usually this sequence of operators do not converge to a self-adjoint operator \(H\) in most topologies, even if the sequence \(e^{iHv}Xe^{-iHv}\) converges for each observable \(X\). This means that the dynamics is, in general, hamiltonian only at a local level. However, as we have already discussed in Section III, under some conditions of \(\Sigma\) an effective hamiltonian exists in \(B(H_\rho)\), i.e. in the \(C^*\)-algebra obtained, via the GNS-representation, from some state \(\rho\). The role of the representation appears evident now, and we will recover the relevance of certain representations even in our settings. In particular, we will restrict ourselves to those \(*\)-representations \(\pi\) of \((\mathcal{A}, A_0)\) such that, whenever \(x \in A_0\) satisfies \(\pi(x) = 0\), then \(\pi(\delta(x)) = 0\). Under this rather natural assumption, the linear map

\[
\delta_\pi(\pi(x)) = \pi(\delta(x)), \quad x \in A_0,
\]

is well-defined on \(\pi(\mathcal{A}_0)\) with values in \(\pi(\mathcal{A})\) and it is a \(*\)-derivation of \(\pi(\mathcal{A}_0)\). We call \(\delta_\pi\) the \(*\)-derivation induced by \(\pi\).

Given such a representation \(\pi\) and its dense domain \(D_\pi \subset \mathcal{H}_\pi\), we consider the graph topology \(t_\pi\) generated by the seminorms

\[
\xi \in D_\pi \to \|A\xi\|, \quad A \in L^\dagger(D_\pi).
\]
Let $D'_\pi$ be the conjugate dual space of $D_\pi$ and $t'_\dagger$ the strong dual topology of $D'_\pi$, i.e., see our second appendix, the topology generated by the seminorms

$$D'_\pi \ni z \mapsto \rho_\pi(z) := \sup_{x \in \mathcal{E}} |\langle x, z \rangle|,$$

where $\langle, \rangle$ is the form which puts in duality $D_\pi$ and $D'_\pi$ and $\mathcal{E}$ is a bounded set in $D_\pi$. Then we get the usual rigged Hilbert space

$$D_\pi[t_\dagger] \subset \mathcal{H}_\pi \subset D'_\pi[t'_\dagger].$$

Let $\mathcal{L}(D_\pi, D'_\pi)$ denote the space of all continuous linear maps from $D_\pi[t_\dagger]$ into $D'_\pi[t'_\dagger]$. Then one has

$$\mathcal{L}(D_\pi) \subset \mathcal{L}(D_\pi, D'_\pi).$$

Each operator $A \in \mathcal{L}(D_\pi)$ can be extended to an operator $\hat{A}$ on the whole $D'_\pi$ in the following way:

$$\langle \hat{A}\xi', \eta \rangle = \langle \xi', A^\dagger \eta \rangle, \quad \forall \xi' \in D'_\pi, \, \eta \in D_\pi.$$ 

Therefore the left and right multiplication of $X \in \mathcal{L}(D_\pi, D'_\pi)$ and $A \in \mathcal{L}(D_\pi)$ can always be defined:

$$(X \circ A)\xi = X(A\xi), \quad \text{and} \quad (A \circ X)\xi = \hat{A}(X\xi), \quad \forall \xi \in D_\pi,$$

and for that we can conclude, as already anticipated before, that $(\mathcal{L}(D_\pi, D'_\pi), \mathcal{L}(D_\pi))$ is a quasi *-algebra.

Let $\delta$ be a *-derivation of $\mathfrak{A}_0$ and $\pi$ an ultra-cyclic *-representation of $(\mathfrak{A}, \mathfrak{A}_0)$ with ultracyclic vector $\xi_0$. Then $\pi(\mathfrak{A}_0) \subset \mathcal{L}(D_\pi)$. We say that the *-derivation $\delta_\pi$ induced by $\pi$ is spatial if there exists $H_\pi = H_\pi^\dagger \in \mathcal{L}(D_\pi, D'_\pi)$ such that $H_\pi \xi_0 \in \mathcal{H}_\pi$ and

$$\delta_\pi(\pi(x)) = i\{H_\pi \circ \pi(x) - \pi(x) \circ H_\pi\}, \quad \forall x \in \mathfrak{A}_0.$$ 

The meaning of this definition is clear: a derivation produces in a representation $\pi$ a spatial induced derivation if this can be implemented by a symmetric operator $H_\pi$: the way in which $\delta_\pi$ is, in a certain obvious sense, described by $H_\pi$ is via a generalized commutator, i.e. a commutator where we may need to consider some adequate extensions of the the operators involved. To be more explicit, if $x \in \mathfrak{A}_0$, then $\pi(x) \in \mathcal{L}(D_\pi)$, so that, since by definition $H_\pi \in \mathcal{L}(D_\pi, D'_\pi)$, it is clear that $H_\pi \circ \pi(x)\psi = H_\pi \pi(x)\psi \in D'_\pi$ for each $\psi \in D_\pi$. Viceversa,
in general we have \( \pi(x) \circ H_\pi \psi = \hat{\pi}(x)H_\pi \psi \in D'_\pi \), since \( H_\pi \psi \in D'_\pi \) for each \( \psi \in D_\pi \), so that \( \pi(x)(H_\pi \Psi) \) is not well defined. This kind of difficulties, however, is not a big surprise here since they are almost everywhere whenever one deals with unbounded operators, and it is easily overcome here by means of the \( \circ \) multiplication.

The main result concerning spatial derivations is contained in the following theorem, [13], which extends and analogous result for C*-algebras which can be found, for instance, in [21].

**Theorem 1** Let \((A[\tau], A_0)\) be a locally convex quasi *-algebra with identity and \(\delta\) be a *-derivation of \(A_0\).

Then the following statements are equivalent:

(i) There exists a \((\tau - \tau_0)\)-continuous, ultra-cyclic *-representation \(\pi\) of \(A\), with ultra-cyclic vector \(\xi_0\), such that the *-derivation \(\delta_\pi\) induced by \(\pi\) is spatial.

(ii) There exists a positive linear functional \(f\) on \(A_0\) such that \(f(x^*x) \leq p(x)^2, \forall x \in A_0\), for some continuous seminorm \(p\) of \(\tau\) and, denoting with \(\hat{f}\) the continuous extension of \(f\) to \(A\), the following inequality holds:

\[
|\hat{f}(\delta(x))| \leq C \left( \sqrt{f(x^*x)} + \sqrt{f(xx^*)} \right), \quad \forall x \in A_0,
\]

for some positive constant \(C\).

(iii) There exists a positive sesquilinear form \(\varphi\) on \(A \times A\) such that:

- \(\varphi\) is invariant, i.e. \(\varphi(ax, y) = \varphi(x, a^*y)\), for all \(a \in A\) and \(x, y \in A_0\);
- \(\varphi\) is \(\tau\)-continuous, i.e. \(|\varphi(a, b)| \leq p(a)p(b)\), for all \(a, b \in A\), for some continuous seminorm \(p\) of \(\tau\);

\(\varphi\) satisfies the following inequality:

\[
|\varphi(\delta(x), 1)| \leq C \left( \sqrt{\varphi(x, x)} + \sqrt{\varphi(x^*, x^*)} \right), \quad \forall x \in A_0,
\]

for some positive constant \(C\).

**Remarks:**

1. even if \(\delta\) cannot be written as \(\delta(x) = i[H, x]\), for any \(H \in A\), if the above theorem can be applied, then \(\delta_\pi\) is, essentially, the commutator with a certain symmetric operator, \(H_\pi\). Again, as we expect for physical reasons, the dynamics depends on the representation \(\pi\) and, as a consequence, on the phase of the matter.

2. The above theorem can be used to answer to the following question: suppose we add to a spatial *-derivation \(\delta_0\) a *perturbation* \(\delta_p\) such that \(\delta = \delta_0 + \delta_p\) is again a *-derivation.
Under which conditions is $\delta$ still spatial? A sufficient condition for this to be true, [13], is that $|\tilde{f}(\delta_p(x))| \leq |\tilde{f}(\delta_0(x))|$, for all $x \in \mathfrak{A}_0$, which is exactly what we expect since this means simply that $\delta_p$ is smaller than $\delta_0$. If we call $H_\pi, H_{\pi,0}$ and $H_{\pi,p}$ the operators which implement $\delta, \delta_0$ and $\delta_p$, we can also prove that $i[H_\pi, A]\psi = i[H_{\pi,0} + H_{\pi,p}, A]\psi$, for all $A \in \mathcal{L}^1(\mathcal{D}_\pi)$ and $\psi \in \mathcal{D}_\pi$.

This theorem is the starting point to consider the problem of the removal of the cutoff. This means that we are assuming, as usual, that the dynamical behavior of the infinite system is obtained as a suitable limit of its restriction to a finite volume $V_L$. At the infinitesimal level, this means that we have a family of inner derivations $\delta_L$, with $h_L$ their associated energies, (i.e. $\delta_L(x) = i[h_L, x]$ for all $x \in \mathfrak{A}_0$ and for all $L$) but we don’t now if the limit of these derivations is still inner or, at least, if the induced limiting derivation is spatial in some particular representations. Let us now be more precise.

Let $\mathcal{S} = \{ (\mathfrak{A}, \mathfrak{A}_0), \Sigma, \alpha^t \}$ be a physical system, where, extending Sewell’s notation, [42], $(\mathfrak{A}, \mathfrak{A}_0)$ is a quasi $*$-algebra, $\Sigma$ the set of states over $(\mathfrak{A}, \mathfrak{A}_0)$ and $\alpha^t$ the time evolution. Let further $\{ \mathcal{S}_L = (\mathfrak{A}_L \subset \mathfrak{A}_0, \Sigma, \alpha^t_L), L \in \Lambda \}$ be a family of regularized systems, i.e. $\mathcal{S}_L$ is the restriction of $\mathcal{S}$ to some finite volume $V_L$. Here $\Lambda$ is a set of indexes, used to label the finite volume systems $S_L$. We suppose here that, for each fixed $L$, the dynamics $\alpha^t_L$ is generated by a $*$-derivation $\delta^t_L$: $\alpha^t_L(x) = \tau_0 - \sum_{k=0}^{\infty} \frac{t^k}{k!} \delta^t_L(x) = e^{iht}xe^{-iht}, \ \forall x \in \mathfrak{A}_L$.

Here $\tau_0$ is a topology on $\mathfrak{A}_0$. Actually, this assumption is not necessary even if this is really what we have in mind, and what actually happens for ordinary C*-algebras at least for discrete systems, where each $h_L$ is a bounded operator.

**Definition 2** The family $\{ \mathcal{S}_L, L \in \Lambda \}$ is said to be c-representable if there exists a $*$-representation $\pi$ of $(\mathfrak{A}, \mathfrak{A}_0)$ into some $(\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}_\pi'), \mathcal{L}^1(\mathcal{D}_\pi))$ such that:

(i) $\pi$ is $(\tau - \tau_\pi)$-continuous;

(ii) $\pi$ is ultra-cyclic with ultra-cyclic vector $\xi_0$;

(iii) when $\pi(x) = 0$, then $\pi(\delta_L(x)) = 0, \ \forall L \in \Lambda$.

Any such representation $\pi$ is a c-representation.

Making use of this definition we can prove the following Proposition, [14]:

**Proposition 3** Let $\{ \mathcal{S}_L, L \in \Lambda \}$ be a c-representable family and $\pi$ a c-representation. Let $h_L = h^+_L \in \mathfrak{A}_L$ be the element which implements $\delta_L$: $\delta_L(x) = i[h_L, x], \ \forall x \in \mathfrak{A}_0, \ \forall L \in \Lambda$. Suppose that $\delta_L(x)$ is $\tau$-Cauchy $\forall x \in \mathfrak{A}_0$ and that $\sup L \| \pi(h_L)\xi_0\| < \infty$. 28
Then, one has

(a) $\delta(x) = \tau - \lim_L \delta_L(x)$ exists in $\mathfrak{A}$ and is a $*$-derivation of $\mathfrak{A}_0$;

(b) $\delta_x$, the $*$-derivation induced by $\pi$, is well defined and spatial.

Remarks:-- (1) It is clear that if the sequence $\{h_L\}$ is $\tau$-convergent, then $\delta_L(x)$ is automatically $\tau$-Cauchy $\forall x \in \mathfrak{A}_0$.

(2) This Proposition implies that any physical system $\mathcal{S}$ with a $c$-representable regularized family $\{\mathcal{S}_L, L \in \Lambda\}$ admits an effective hamiltonian in the sense of [11, 44, 41, 34].

(3) This is our version of Sewell’s result on the existence of different effective hamiltonians in different (GNS-like) representations: a physical system $\Sigma$ exhibits different dynamics in its different thermodynamical phases.

(4) We want to cite here an open problem which, in our opinion, deserves a deeper analysis: what happens if we consider a representation $\pi'$ globally equivalent but locally different from a given $c$-representation $\pi$? Is $\pi'$ still a $c$-representation? Do the related effective hamiltonians coincide? This is indeed what one expects in connection with what has been discussed in the $C^*$-algebraic approach, even if no explicit proof of this claim exists at this stage.

V.2 The time evolution $\alpha^t$

In this subsection we will consider the problem of the existence of the algebraic dynamics for a system with infinite degrees of freedom at the level of automorphisms of a certain quasi $*$-algebra, instead of considering only its infinitesimal behavior. More explicitly, the problem is the following: suppose that we have been able to prove that the derivation $\delta$ exists. Nevertheless, in general we have no information about $\delta^2, \delta^3, \ldots$. Moreover, even if all these maps do exist, this does not mean that the series $\sum_{k=0}^{\infty} \frac{t^k}{k!} \delta^k(x)$, which defines $\alpha^t(x)$ when $x$ and $H$ are both bounded, exists as well, for a generic $x \in \mathfrak{A}_0$. In other words, the existence of $\delta(x)$ does not imply, by no means, the existence of $\alpha^t(x)$ for $x \in \mathfrak{A}$ or even for $x \in \mathfrak{A}_0$.

Furthermore, the effective hamiltonian $H_{\pi}$, whose existence has been proved in the previous subsection, is symmetric but not self-adjoint, and therefore the spectral theorem cannot be used to define $e^{iH_{\pi}t}$ and, as a consequence, to conclude that $\pi(\alpha^t(x)) = e^{iH_{\pi}t} \pi(x) e^{-iH_{\pi}t}$! So the following crucial problem arises: how to define a time evolution in this case?

This is a rather hard problem already in a standard setting, when there is no problem in multiplying elements of the algebra. Here this is quite a dangerous operation and the difficulties are even more than before. We will devote the rest of this subsection to some partial results
which can be used to analyze certain classes of physical systems. It may be stressed that no
general result really exists at this stage.

V.2.1 From $\delta$ to $\alpha$: the first way

We begin by considering a class of models which are suggested by the mean field spin models,
reviewing some results first obtained in [14].

Let us assume that the finite volume hamiltonians $h_L$ can be written in terms of some s.a.
elements $s^\alpha_L$, $\alpha = 1, 2, ..., N$, which are $\tau$-converging to some elements $s^\alpha \in \mathfrak{A}$, commuting with
all the elements of $\mathfrak{A}_0$:

$$
s^\alpha = \tau - \lim_L s^\alpha_L, \quad [s^\alpha, x] = 0, \forall x \in \mathfrak{A}_0.
$$

For mean field spin models $s^\alpha$ is the magnetization and $\tau$ is the strong topology restricted to
a relevant family of states, [7], or, alternatively, the so-called physical topology, [9, 11, 12, 34],
adopting the Lassner’s terminology.

We say that $\{s^\alpha_L\}$ is uniformly $\tau$-continuous if, for each continuous seminorm $p$ of $\tau$ and
for all $\alpha = 1, 2, ..., N$, there exists another continuous seminorm $q$ of $\tau$ and a positive constant $c_{p,q,\alpha}$ such that

$$
p(s^\alpha_L a) \leq c_{p,q,\alpha} q(a), \forall a \in \mathfrak{A}, \forall L \in \Lambda.
$$

From this definition it also follows that $p(as^\alpha_L) \leq c_{p,q,\alpha} q(a), \forall a \in \mathfrak{A}$, and that the same
inequalities can be extended to $s^\alpha$. Then we have

**Lemma 4** If $\{s^\alpha_L\}$ is a uniformly $\tau$-continuous sequence and if $\tau - \lim_L s^\alpha_L = s^\alpha$, $\forall \alpha$, then

$$
\tau - \lim_L (s^\alpha_L)^k = (s^\alpha)^k, \forall \alpha \text{ and for } k = 1, 2, ....
$$

**Proposition 5** Suppose that (1) $\forall x \in \mathfrak{A}_0 \ [h_L, x]$ depends on $L$ only through $s^\alpha_L$ and (2) $s^\alpha_L \underset{\tau}{\longrightarrow} s^\alpha$ and $\{s^\alpha_L\}$ is a uniformly $\tau$-continuous sequence.

Then, for each $k \in \mathbb{N}$, the following limit exists

$$
\tau - \lim_L \delta^k_L [h_L, x]_k = \tau - \lim_L \delta^k_L (x), \forall x \in \mathfrak{A}_0,
$$

and defines an element of $\mathfrak{A}$ which we call $\delta(k)(x)$.

**Remark:** The reason why we prefer to use $\delta(k)(x)$ instead of $\delta^k(x)$ is just to stress in this
way that it is not possible to write $\delta^k(x) = i^k[h, x]_k$, since first of all no global $h$ does exist and,
secondly, even if it does, $[h, x]_k$ is not well defined in general because of domain difficulties.

Once we have obtained conditions for all the multiple commutators to exist in some reason-
able sense, we still need to find conditions for which the infinite series which defines $\alpha^l(x)$ do
converge. For that it is convenient to introduce here the following definition:
Definition 6 We say that \( x \in A_0 \) is a generalized analytic element of \( \delta \) if, for all \( t \), the series \( \sum_{k=0}^{\infty} \frac{t^k}{k!} \pi(\delta(k)(x)) \) is \( \tau_s \)-convergent. The set of all generalized analytic elements is denoted with \( G \).

Therefore we have,[14],

Proposition 7 Let \( x_\gamma \) be a net of elements of \( A_0 \) and suppose that, whenever \( \pi(x_\gamma) \xrightarrow{\tau_s} \pi(x) \) then \( x_\gamma \xrightarrow{\tau} x \). Then, \( \forall x \in G \) and \( \forall t \in \mathbb{R} \), the series \( \sum_{k=0}^{\infty} \frac{t^k}{k!} \delta(k)(x) \) converges in the \( \tau \)-topology to an element of \( A \) which we call \( \alpha^t(x) \).

Moreover, \( \alpha^t \) can be extended to the \( \tau \)-closure \( \overline{G} \) of \( G \).

It may be worth noticing that, even if the assumptions are rather strong, they are satisfied, for instance, by mean field spin models!

V.2.2 From \( \delta \) to \( \alpha^t \): the second way

Let \( \pi \) be a faithful *-representation of the quasi *-algebra \( (A, A_0) \) and \( \delta \) a *-derivation of \( A_0 \) such that \( \delta_\pi \), is well-defined on \( \pi(A_0) \) with values in \( \pi(A) \).

We define the following subset of \( A_0 \) (a domain of regularity of \( \delta \))

\[
A_0(\delta) := \{ x \in A_0 : \delta^k(x) \in A_0, \ \forall k \in \mathbb{N}_0 \}.
\]

Whenever \( \delta \) is regular the set \( A_0(\delta) \) is large. For instance, if \( \delta \) is inner in \( A_0 \) with an implementing element \( h \in A_0 \), then \( A_0(\delta) = A_0 \). For general \( \delta \), \( A_0(\delta) \) contains, at least, all the multiples of the identity \( 1 \) of \( A_0 \).

\( A_0(\delta) \) is a *-algebra which is mapped into itself by \( \delta \). Moreover it is easy to check that \( \pi(\delta^k(x)) = \delta^k_\pi(\pi(x)) \), \( \forall x \in A_0(\delta) \) and \( \forall k \in \mathbb{N}_0 \). Therefore it follows that \( \delta^k_\pi(\pi(x)) \in \pi(A_0) \).

Let \( \sigma_s \) be the topology on \( A \) defined via \( \tau_s \) in the following way:

\[
A \ni a \rightarrow q_\xi(a) = p_\xi(\pi(a)) = \| \pi(a) \xi \|, \quad \xi \in D_\pi.
\]

Then we have the following theorem, [14]:

Theorem 8 Let \( (A, A_0) \) be a quasi *-algebra with identity, \( \delta \) a *-derivation on \( A_0 \) and \( \pi \) a faithful *-representation of \( (A, A_0) \) such that the induced derivation \( \delta_\pi \) is well defined. Then, we have:
(1) if the following inequality holds

$$\forall \eta \in D, \exists c_\eta > 0 : p_\eta(\delta_\pi(\pi(x))) \leq c_\eta p_\eta(\pi(x)), \ \forall x \in A_0(\delta),$$

then $$\sum_{k=0}^{\infty} \frac{\delta^k(x)}{k!} \delta_\pi(\pi(x))$$ converges for all $$t$$ in the topology $$\sigma_\pi$$ to an element of $$A_0(\delta)^{\sigma_\pi}$$ which we call $$\alpha^t(x)$$; $$\alpha^t$$ can be extended to $$A_0(\delta)^{\sigma_\pi}$$ and

$$\alpha^{t+\tau}(x) = \alpha^t(\alpha^\tau(x)), \ \forall t, \tau, \forall x \in A_0(\delta);$$

(2) Suppose that the following inequality holds

$$\exists c > 0 : \forall \eta_1 \in D, \exists A_{\eta_1} > 0, n \in \mathbb{N} \text{ and } \eta_2 \in D, p_{\eta_1}(\delta_\pi(\pi(x))) \leq A_{\eta_1} c^k k! k^n p_{\eta_2}(\pi(x)), \ \forall x \in A_0(\delta), \forall k \in \mathbb{N}_0,$$

then $$\sum_{k=0}^{\infty} \frac{\delta^k(x)}{k!} \delta_\pi(\pi(x))$$ converges, for $$t < \frac{1}{c}$$ in the topology $$\sigma_\pi$$ to an element of $$A_0(\delta)^{\sigma_\pi}$$ which we call $$\alpha^t(x)$$; $$\alpha^t$$ can be extended to $$A_0(\delta)^{\sigma_\pi}$$ and, $$\forall x \in A_0(\delta),$$

$$\alpha^{t+\tau}(x) = \alpha^t(\alpha^\tau(x)), \ \forall t, \tau, \text{ with } t + \tau < \frac{1}{c}.$$ 

Remarks:- (1) As we see, this theorem gives sufficient conditions for $$\alpha^t$$ to be defined (as a converging series) at least on a certain subset of $$A_0.$$  

(2) Here and in the previous approach the spatiality of the derivation is not required. It is obvious that, when $$H_\pi$$ exists as a self-adjoint operator mapping $$D_\pi$$ into $$H_\pi,$$ we could use the spectral theorem to define $$\pi(\alpha^t(x)) = e^{iH_\pi t} \pi(x) e^{-iH_\pi t};$$

V.2.3 A different point of view

In a recent paper, [12], we have considered the problem of the existence of $$\alpha^t$$ from a slightly different point of view, which is maybe more suitable for systems with a finite number of degrees of freedom. This is because we have assumed that the energy operator of our quantum system does exist as a self-adjoint, unbounded and densely defined operator $$H_0 \geq 1.$$ Then, it is known that the operator $$e^{iH_0 t},$$ and therefore the time evolution of an observable $$X,$$ can be defined via the spectral theorem. However, but for finite dimensional Hilbert spaces, our claim is that the natural algebraic framework to discuss the dynamical behavior of the system is $$\mathcal{L}(D)[\tau_0],$$ where $$D = D^\infty(H_0),$$ rather than $$B(H).$$ Indeed, if $$\text{dim}(H) = \infty,$$ it is clear that in
general $H_0 \notin B(\mathcal{H})$ and that $\delta$ does not map $B(\mathcal{H})$ into itself. On the other hand, it is evident that $H_0 \in \mathcal{L}^1(\mathcal{D})[\tau_0]$ and that $\delta : \mathcal{L}^1(\mathcal{D})[\tau_0] \to \mathcal{L}^1(\mathcal{D})[\tau_0]$.

These claims are based on the following natural procedure:
l et $H_0 = \int_1^{\infty} \lambda dE(\lambda)$ be the spectral decomposition of $H_0$, see Appendix 2. We define, for $L \geq 1$, the projectors $Q^0_L = \int_1^{L} dE(\lambda)$ and we introduce the regularized hamiltonian $H_L = Q^0_L H_0 Q^0_L$.

For each $L$, we see that $Q^0_L, H_L \in B(H) \cap \mathcal{L}^1(\mathcal{D})$ and that $[Q^0_L, H_0] = [H_0, H_L] = 0$.

If $\tau_0$ is the topology on $\mathcal{L}^1(\mathcal{D})$ generated by the seminorms

$$\mathcal{L}^1(\mathcal{D}) \ni A \mapsto \|A\|^{f,k} = \max\{\|H_0^k Af(H_0)\|, \|f(H_0)AH^k\|\},$$

then we have:

(i) $H_L \to H_0$ with respect to the topology $\tau_0$;

(ii) $\{e^{itH_L}\}$ is $\tau_0$-Cauchy in $\mathcal{L}^1(\mathcal{D})$ and converges to $e^{itH_0}$

(iii) $\forall A \in \mathcal{L}^1(\mathcal{D})$, the sequence $\{e^{itH_L} Ae^{-itH_L}\}$ is $\tau_0$-Cauchy in $\mathcal{L}^1(\mathcal{D})$ and converges to $e^{itH_0} Ae^{-itH_0}$.

We can therefore conclude that $H_0, e^{itH_0},$ and $\alpha^t(A) := e^{itH_0} Ae^{-itH_0}$ all belong to $\mathcal{L}^1(\mathcal{D})$, $\forall A \in \mathcal{L}^1(\mathcal{D})$. Moreover we can also show that

$$\alpha^t(A) = \tau_0 - \lim_L e^{itH_L} Ae^{-itH_L} = \left(\tau_0 - \lim_L e^{itH_L}\right) A \left(\tau_0 - \lim_L e^{-itH_L}\right).$$

This suggests the use of $\mathcal{L}^1(\mathcal{D})[\tau_0]$ as a natural algebraic and topological framework for the analysis of the time evolution of, at least, finite quantum systems. Of course, a similar construction can be repeated also for $QM_\infty$ systems, at least for those systems for which an unbounded, self-adjoint and densely defined operator $M$ exists such that $[M, H_L] = 0$ (on a dense domain), [34].

In the same paper we have considered the role of a perturbation in this approach: let $H = H_0 + B$, and suppose that the spectral decomposition of the free hamiltonian $H_0$ is explicitly known while the spectral decomposition of the perturbed hamiltonian $H$ cannot be exactly found, which is exactly what usually happens in concrete situations. We have shown that the convenient algebraic structure is again $\mathcal{L}^1(\mathcal{D})$, with $\mathcal{D} = D^\infty(H_0)$, (since, if $H_0$ has discrete spectrum, we know an o.n. set in $\mathcal{D}$ and, as a consequence, we know $\mathcal{D}$) but the technically convenient topology, $\tau$, is that given by the seminorms

$$\mathcal{L}^1(\mathcal{D}) \ni A \mapsto \|A\|^{f,k} = \max\{\|H^k Af(H)\|, \|f(H)AH^k\|\},$$

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because with this choice some of the above convergence results can be established. Moreover
this apparent difference between the algebraic and the topological frameworks, can be easily
controlled. Indeed we have proven in [12] that, if (1) \( D(H_0) \subseteq D(B) \) and if \( H = H_0 + B \) is
self-adjoint on \( D(H_0) \), and (2) if \( D^\infty(H_0) = D^\infty(H) \) (hypothesis for which we gave necessary
and sufficient conditions), then \( \tau_0 \equiv \tau \).

Under these assumptions we can therefore undertake a deeper analysis of the existence of
the algebraic dynamics for a perturbed hamiltonian. We refer to [12] for a detailed analysis,
which eventually produces a rigorous definition of the Schrödinger dynamics.

V.3 Fixed point results

This is an alternative procedure which again produces a rigorous definition of the dynamics of
a (closed) physical system, [5], and which is based on a generalization of well known fixed point
theorems.

Let \( \mathfrak{B} \) be a \( \tau \)-complete subspace of \( \mathcal{L}^1(D) \) and \( T \) a map from \( \mathfrak{B} \) into \( \mathfrak{B} \). We say that \( T \) is
a weak \( \tau \) strict contraction over \( \mathfrak{B} \), briefly a \( w\tau sc(\mathfrak{B}) \), if there exists a constant \( c \in ]0,1[ \) such
that, for all \( (h,k) \in C_N := (C,N_0) \), \( N_0 = \mathbb{N} \cup \{0\} \), there exists a pair \( (h',k') \in C_N \) satisfying
\[
\|Tx - Ty\|_{h,k} \leq c\|x - y\|_{h',k'} \quad \forall x, y \in \mathfrak{B}.
\] (5.1)

In what follows we will consider equations of the form \( Tx = x \), \( T \) being a \( w\tau sc(\mathfrak{B}) \). The
first step consists in introducing the following subset of \( \mathfrak{B} \):
\[
\mathfrak{B}_L \equiv \left\{ x \in \mathfrak{B} : \sup_{(h,k) \in C_N} \|Tx - x\|_{h,k} \leq L \right\},
\] (5.2)

\( L \) being a fixed positive real number.

**Lemma 9** Let \( T \) be a \( w\tau sc(\mathfrak{B}) \). Then

(a) if \( T0 = 0 \) then any \( x \in \mathfrak{B} \) such that \( \sup_{(h,k) \in C_N} \|x\|_{h,k} \leq L_1 \) belongs to \( \mathfrak{B}_L \) for \( L \geq L_1(1 + c) \);

(b) if \( \|T0\|_{h,k} \leq L_2 \) for all \( (h,k) \in C_N \), then any \( x \in \mathfrak{B} \) such that \( \sup_{(h,k) \in C_N} \|x\|_{h,k} \leq L_1 \)
belongs to \( \mathfrak{B}_L \) for \( L \geq L_1(1 + c) + L_2 \);

(c) if \( x \in \mathfrak{B}_L \) then \( T^n x \in \mathfrak{B}_L \), for all \( n \in \mathcal{N} \);

(d) \( \mathfrak{B}_L \) is \( \tau \)-complete;

(e) if \( \mathfrak{B}_L \) is not empty, then \( T \) is a \( w\tau sc(\mathfrak{B}_L) \).
\( \mathcal{B}_L \) is non empty, see [5]. The existence of a fixed point is ensured by the following Proposition:

**Proposition 10** Let \( T \) be a \( \tau \)-sc\( (\mathcal{B}) \). Then

(a) \( \forall x_0 \in \mathcal{B}_L \) the sequence \( \{x_n \equiv T^n x_0\}_{n \geq 0} \) is \( \tau \)-Cauchy in \( \mathcal{B}_L \). Its \( \tau \)-limit, \( x \in \mathcal{B}_L \), is a fixed point of \( T \);

(b) if \( x_0, y_0 \in \mathcal{B}_L \) satisfy the condition \( \sup_{(h,k) \in C} \|x_0 - y_0\|^{h,k} < \infty \), then \( \tau - \lim_n T^n x_0 = \tau - \lim_n T^n y_0 \).

For physical applications we need to consider the case in which these maps depend on an external parameter:

let \( I \subset \mathbb{R} \) be a set such that 0 is one of its accumulation points. A family of weak \( \tau \) strict contractions \( \{T_\alpha\}_{\alpha \in I} \) is said to be uniform if

1) \( T_\alpha : \mathcal{B} \to \mathcal{B} \ \forall \alpha \in I, \mathcal{B} \) being a \( \tau \)-complete subspace of \( \mathcal{L}^+(\mathcal{D}) \);

2) \( \forall (h,k) \in C_N \) and \( \forall \alpha \in I \) there exist \( (h',k') \in C_N \), independent of \( \alpha \), and \( c_\alpha \in [0,1[ \), independent of \( (h,k) \), such that

\[
\|T_\alpha x - T_\alpha y\|^{h,k} \leq c_\alpha \|x - y\|^{h',k'}, \quad \forall x, y \in \mathcal{B}; \tag{5.3}
\]

3) \( c_- \equiv \lim_{\alpha \to 0} c_\alpha \in [0,1[ \).

We further say that the family \( \{T_\alpha\}_{\alpha \in I} \) is \( \tau \)-strong Cauchy if, for all \( (h,k) \in C_N \) and \( \forall y \in \mathcal{B} \),

\[
\|T_\alpha y - T_\beta y\|^{h,k} \to 0. \tag{5.4}
\]

We call \( \mathcal{B}_L^{(\alpha)} \) the following set \( \mathcal{B}_L^{(\alpha)} \equiv \{ x \in \mathcal{B} : \sup_{(h,k) \in C} \|T_\alpha x - x\|^{h,k} \leq L \} \).

**Proposition 11** Let \( \{T_\alpha\}_{\alpha \in I} \) be a \( \tau \)-strong Cauchy uniform family of \( \tau \)-sc\( \mathcal{B} \). Then

(1) There exists a \( \tau \)-sc\( \mathcal{B} \), \( T \), which satisfies the following relations:

\[
\|Ty - T_\alpha y\|^{h,k} \to 0 \quad \forall y \in \mathcal{B}, \forall (h,k) \in C_N
\]

and

\[
\|Ty - Tz\|^{h,k} \leq c_- \|y - z\|^{h',k'} \quad \forall y,z \in \mathcal{B},
\]

where \( (h',k') \) are those of inequality (5.3).

(2) let \( \{x_\alpha\}_{\alpha \in I} \) be a family of fixed points of the net \( \{T_\alpha\}_{\alpha \in I} : T_\alpha x_\alpha = x_\alpha, \forall \alpha \in I. \) If \( \{x_\alpha\}_{\alpha \in I} \) is a \( \tau \)-Cauchy net then, calling \( x \) its \( \tau \)-limit in \( \mathcal{B} \), \( x \) is a fixed point of \( T \).
(3) If the set $\cap_{\alpha \in I} \mathcal{B}_L^{(\alpha)}$ is not empty and if the following commutation rule holds

$$T_\alpha(T_\beta y) = T_\beta(T_\alpha y), \quad \forall \alpha, \beta \in I \text{ and } \forall y \in \mathcal{B},$$

then, calling

$$x_\alpha = \tau - \lim_{n \to \infty} T_n^0 x^0, \text{ where } x^0 \in \cap_{\alpha \in I} \mathcal{B}_L^{(\alpha)},$$

each $x_\alpha$ is a fixed point of $T_\alpha$, $T_\alpha x_\alpha = x_\alpha$ and $\{x_\alpha\}_{\alpha \in I}$ is a $\tau$-Cauchy net. Moreover $\tau - \lim_{\alpha \to 0} x_\alpha$ is a fixed point of $T$.

As an application we have proven in [5] that, under certain technical assumptions, the time evolution of a given operator $x$,

$$x_\alpha(t) = x + i \int_0^t ds [H_\alpha, x_\alpha(s)],$$

is associated with a uniform family of $\text{w}_\tau \text{sc}(\mathcal{L}^+), \{U_\alpha\}$, which is also $\tau$-strong Cauchy. This implies that, because of the Proposition above, the dynamics for the physical system can be obtained as a $\tau$-limit of the regularized dynamics $x_\alpha(t)$, which is a fixed point of $U := \lim_\alpha U_\alpha$.

V.4 Explicit estimates

We end this excursus of (class of) models for which the time evolution is under control, by considering the so-called almost mean field Ising model, defined by the following finite volume hamiltonian

$$H_V = \frac{J}{|V|} \sum_{i,j \in V} \sigma_i^3 \sigma_j^3, \quad (5.5)$$

with $0 < \gamma \leq 1$, [8]. Particularly relevant in the mathematical description of this model is the almost magnetization operator $S^3_V := \frac{1}{|V|} \sum_{p \in V} \sigma_p^3$. In fact, if $A$ is a local observable, its regularized time evolution $\alpha_\tau(A) := e^{iH_V t} A e^{-iH_V t}$ in general depends on $t$, $A$ and $S^3_V$.

In the first appendix it is discussed in some details how to construct the relevant Hilbert space for the model, $\mathcal{H}_{\{n\}}$, the dense domain $\mathcal{D}_{\{n\}}$, the $O^*$-algebra $\mathcal{L}(\mathcal{D}_{\{n\}})$, a $*$-representation of the C*-spin algebra $\mathfrak{A}_s$ and the physical topology $\tau_0$, following [34]. Here we introduce also a different topology $\tau$ on $\mathfrak{A}_s$, which has proved to be of some usefulness, as follows:

$$\tau : \quad ||A||_{\{n\}} := ||f(M_{\{n\}} \pi_{\{n\}}(A) f(M_{\{n\}})||,$$

where $f$ belongs to $\mathcal{C}$. With these definitions, calling $\mathfrak{A}_0$ the $\tau_0$-completion of $\mathfrak{A}_s$ and $\mathfrak{A}$ the $\tau$-completion of $\mathfrak{A}_s$, we proved in [8] that:
• $(\mathcal{A}[\tau], \mathcal{A}_0[\tau_0])$ is a topological quasi $\ast$-algebra;

• all the powers of the almost magnetization $S^V_3$ are $\tau_0$-converging in $\mathcal{A}$;

• the finite volume dynamics $\alpha^t$ $\tau_0$-converges to a one-parameter group of automorphisms $\alpha^t$ of $\mathcal{A}_0$;

• $\alpha^t$ solves the $\tau_0$-limit of the finite volume Heisenberg equation of motion.

Another spin model which can be analyzed within the same algebraic framework is the almost mean field Heisenberg model,

$$H_V = \frac{J}{|V|^2} \sum_{i,j \in V} \sum_{\alpha=1}^3 \sigma_i^\alpha \sigma_j^\alpha,$$

with $\frac{1}{2} < \gamma \leq 1$, see [9], which differs from the Ising model because it is intrinsically three-dimensional.

A different class of models that we have considered using the same approach involves free and interacting bosons, [4]. The formal Hamiltonian $H$ for the one mode free bosons is simply the number operator $N = a^\dagger a$, $a$ and $a^\dagger$ being the annihilation and creation operators for the bosons. They satisfy the canonical commutation relation $[a, a^\dagger] = I$. (More properly, $N$ is the unique self-adjoint extension of the symmetric operator $a^\dagger a$.)

The construction of the topological quasi $\ast$-algebra is the usual one. Let $\mathcal{D} := D_\infty(N) = \cap_{k \geq 0} D(N^k)$. This set is dense in the Fock-Hilbert space $\mathcal{H}$ constructed in the standard way. Starting from $\mathcal{D}$ we can define the $\ast$-algebra $L^+(\mathcal{D})$. It is clear that all powers of $a$ and $a^\dagger$ belong to this set. The topology in $L^+(\mathcal{D})$ is, using Lassner’s terminology in [34], the usual quasi-uniform topology:

$$X \in L^+(\mathcal{D}) \rightarrow \|X\|_f^k := \max \{\|f(N)X N^k\|, \|N^k X f(N)\|\},$$

where $f \in C$ and $k \geq 0$. We have already discussed several times along this paper that $L^+(\mathcal{D})[\tau_0]$ is a complete locally convex topological $\ast$-algebra.

Let $\mathcal{E}_l$ be the subspace of $\mathcal{H}$ generated by all the vectors which are proportional to $(a^\dagger)^l \Phi_0$. Let also $\mathcal{F}_L$ be the direct sum $\mathcal{F}_L := \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_L$. Finally, let $N = \sum_{l=0}^\infty \Pi_l$ be the spectral decomposition of the number operator $N$. The operators $\Pi_l$ are projection operators, as well as the operators $Q_L = \sum_{l=0}^L \Pi_l$. The following properties are obvious:

$$\Pi_k \Pi_l = \delta_{kl} \Pi_l, \quad \Pi_l^\dagger = \Pi_l; \quad Q_L Q_M = Q_L, \quad \text{if} \ L \leq M, \quad Q_L^\dagger = Q_L.$$
It is clear that $\Pi_k : \mathcal{H} \to \mathcal{E}_k$, and $Q_L : \mathcal{H} \to \mathcal{F}_L$. The operator $Q_L$ is used to cut-off the hamiltonian, by replacing $a$ with $a_L := Q_L a Q_L$. The regularized hamiltonian is simply $H_L = Q_L N Q_L = N Q_L$ and the related time evolution is $\alpha^t_L(X) = e^{iH_L t} X e^{-iH_L t}$. This occupation number cut-off produces a self adjoint bounded operator $H_L$ and we have shown in [4] that the limits of $\alpha^t_L(a^n)$ and $\alpha^t_L((a^\dagger)^n)$ exist in $\mathcal{L}^+(\mathcal{D})[\tau_0]$ for all $n \in \mathbb{N}$. We have already seen that this result has been generalized by those in [12].

The same algebraic framework turns out to be useful also in the analysis of the thermodynamical limit of the interacting model described by the following formal hamiltonian:

$$H_V = \frac{J}{|V|} \sum_{i,j \in V} \sigma^3_i \sigma^3_j + a^\dagger a + \gamma (a + a^\dagger) \sigma^3_V,$$

where $\sigma^3_V = \frac{1}{|V|} \sum_{i \in V} \sigma^3_i$. Here the algebra $\mathcal{L}^+(\mathcal{D})$ must be replaced by $\mathfrak{A} = B(\mathcal{H}_{spin}) \otimes \mathcal{L}^+(\mathcal{D})$. The topology on $\mathfrak{A}$, $\tau_{comp}$, is generated by the following seminorms: $\| X A \|_{f,k,\Psi} \equiv \| X \|_{f,k} \| A \Psi \|$, $X \in \mathcal{L}^+(\mathcal{D})$ and $A \in \mathcal{B}(\mathcal{H}_{spin})$. It is worthwhile to remind also that $\Psi$ cannot be a generic vector in $\mathcal{H}_{spin}$, but must belong to the set

$$\mathcal{F} = \left\{ \Psi \in \mathcal{H}_{spin} : \lim_{|V|, \infty} \frac{1}{|V|} \sum_{p \in V} \sigma^3_p \Psi = \sigma^3_\infty \Psi, \| \sigma^3_\infty \| \leq 1 \right\}.$$

As before, the regularized hamiltonian is obtained by replacing $a$ with $a_L := Q_L a Q_L$, so that the new hamiltonian $H_{V,L}$ depends on two, in principle, unrelated cutoffs. The existence of the limit of $\alpha^{t,L}_V(X) = e^{iH_{V,L} t} X e^{-iH_{V,L} t}$ is ensured by the following result, [4]: the limit of $\alpha^{t,L}_V(a)$ for $|V|$ and $L$ both diverging exists in $\mathfrak{A}[\tau_0]$. Moreover, if the two cutoffs satisfy the relation $|V| = L^r$, for a certain integer $r > 1$, the same holds true also for $\alpha^{t,L}_V(\sigma^a_\infty)$.

### V.5 Few words on other results

In this paper we have only discussed in some details results related to those research lines I am more involved which, as already mentioned, are mainly related to quasi $^*$-algebras. We dedicate this short section to a very brief list of different lines of research, starting with the analysis of one-parameter groups of $^*$-automorphisms in the context of a particular class of partial $^*$-algebras, the so-called partial $O^*$-algebras. This analysis is important because both time evolution and physical symmetries are examples of $^*$-automorphisms. Some results on the existence of the time evolution for a given physical system, its continuity and the spatiality of the related derivation can be found in [2] and in references therein.
Another application of algebras of unbounded operators originates from the analysis of point-like quantum fields as discussed in [30, 36]. Here the field \( A(x) \) is represented as a sesquilinear form on a certain domain \( D \subseteq H \). One of the basic Wightman axioms is that the smeared field \( A(f) = \int_{\mathbb{R}^4} A(x) f(x) dx \) exists as a well defined operator in \( D \) for any given \( f \in C_0^\infty(\mathbb{R}^4) \). However this may not be true and different possible definitions of point-like field have been proposed in the literature. A detailed analysis on this subject can be found, for instance, in [28].

We end this short subsection mentioning a last application of quasi \(*\)-algebras in the analysis of the dynamics of a free Bose system confined in a segment of length \( l \). A contradiction arising from the analysis of this system, which originates from the use of the Bogoliubov inequality, disappear when one constructs the CCR quasi \(*\)-algebra as in [35] or, alternatively, adopting the point of view of [20] where the authors generalize the notion of states on unbounded operator algebras.

Other physical applications of algebras of unbounded operators can be found in [25].

VI Work in progress and future projects

We want to discuss here some preliminary results concerning a situation in which the algebraic framework is somehow fixed and no global hamiltonian exists, but only a family of finite volume energy operators. This is essentially what happens in the standard formulation of QM\(_\infty\). More in details, let \( S \) be a self-adjoint, unbounded, densely defined operator on a Hilbert space \( H \). For simplicity we assume that its spectrum is discrete, even if most of the results do not depend on this aspect: \( S = \sum_{l=0}^\infty s_l P_l \). Let \( D = D^\infty(S) \), and \( \mathcal{L}^\dagger(D) \) and \( \tau \) constructed as usual. Let further \( H_L = \sum_{l=0}^L h_l P_l \) be our regular hamiltonian: \( H_L \in B(H), \forall L \). It is worth stressing that we are assuming, for the time being, that the spectral projections of \( H_L \) are the same as those of \( S \).

In some of our previous attempt, and in particular in what we have discussed in Subsection V.2.3, we had \( \tau - \lim H_L \in \mathcal{L}^\dagger(D) \). This implies, in particular, that \( H \) exists and \( H \in \mathcal{L}^\dagger(D) \). In this case we have seen that there is absolutely no problem in defining a Shrödinger or an Heisenberg dynamics. In [6] we have proven that this is not necessary. More in details, we have proven that for each sequence \( \{h_l\} \), if \( \{s_l^{-1}\} \) is in \( l^2(\mathbb{N}_0) \), then

1. \( e^{iH_L t} \) \( \tau \)-converges to an element \( T_t \in \mathcal{L}^\dagger(D) \);
2. \( \forall X \in \mathcal{L}^\dagger(D) \) the sequence \( e^{iH_L t} X e^{-iH_L t} \) \( \tau \)-converges to an element \( \alpha^t(X) \in \mathcal{L}^\dagger(D) \);

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3. \( \forall X \in \mathcal{L}^\dagger(D) \) we have \( \alpha^t(X) = T_t X T_{-t} \);

4. if \( Q_M = \sum_{l=0}^{M} X \in \mathcal{L}^\dagger(D) \), \( X_M = Q_M X Q_M \) and \( \delta_L(X_M) = i[H_L, X_M] \) then

\[
\alpha^t(X) = \tau - \lim_{L,M,N} \sum_{j=0}^{N} \frac{t^j}{j!} \delta_L(X_M).
\]

Remark:– We see, therefore, that the time evolution of each element of \( \mathcal{L}^\dagger(D) \) can be defined (in three different ways!) even if \( H_L \) does not define any hamiltonian of the system \( \Sigma \), i.e. even if \( H_L \) does not converge in any natural topology. This is relevant for us since it is exactly what happens in the most general physical situations, as we have widely discussed in Section II.

However, here we are assuming that \( S \) and \( H_L \) admit the same spectral projections. It is natural to ask what happens if this is no longer true. In this case we have the following partial results, [6]:

suppose that \( S = \sum_{l=0}^{\infty} s_l P_l \) and \( H_M = \sum_{l=0}^{M} h_l E_l \), with \( E_j \neq P_j \). Then something can be said also in this case. In particular

- if \([E_l, P_j] = 0\) for all \( l, j \), or if \( E_l \neq P_l \) only for a finite number of \( l \)'s, the above results still can be proved;

- let \( \{\psi_l\} \) and \( \{\phi_l\} \) be two different orthonormal bases of \( \mathcal{H} \) and suppose that \( P_l = |\phi_l><\phi_l| \) and \( E_l = |\psi_l><\psi_l| \). It is clear that \([E_l, P_j] \neq 0\) in general. Nevertheless, if \( \psi_l \) is a finite linear combination of the \( \phi_j \)'s, then again the above results still can be proved.

The last result of [6] which we want to cite here concerns the role of the Gibbs and the KMS states for this situation: let \( \rho_L := \frac{e^{-\beta H_L}}{\text{tr}_L(e^{-\beta H_L})} \) be the density matrix of a Gibbs state at the inverse temperature \( \beta \). Then it is easy to check that \( \tau - \lim_L \rho_L \) exists in \( \mathcal{L}^\dagger(D) \). But it is still to be investigated is whether this limit is a KMS state (in some sense).

As it is clear even if many results have been obtained within this context, many others are still to be obtained. In particular, the following research lines are already opened:

1. we need a deeper analysis of the previous results when \( S \) and \( H_L \) are essentially different and, in particular, if \([H_{L_1}, H_{L_2}] \neq 0\).

2. What can be said about Goldstone’s theorem when \( \alpha^t_{\nu} \) does not converge uniformly (or \( \mathcal{F} \)-strongly) to \( \alpha^t \)? What does this theorem become in a quasi *-algebraic framework?
3. Can we define a KMS state when \( \alpha_t^{V} \) does not converge uniformly (or \( \mathcal{F} \)-strongly) to \( \alpha_t^{0} \)?

4. Is there any relation between these KMS-like states and the phase structure of the physical system?

5. Is there any relation between these KMS-like states and the Tomita-Takesaki theory? (something is discussed in [2])

6. What about local modifications? Do two states \( \rho \) and \( \chi \) which are only locally different generate unitarily equivalent representations? And what can be said about the related effective hamiltonians? (Some results are already discussed in [11])

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Appendix 1: the algebras for $\Sigma$

Let $\Sigma$ be a system with infinite degrees of freedom. We recall that the Haag Kastler’s construction of the C*-algebra associated to $\Sigma$, can be schematized as follows:

\[ V \xrightarrow{\cdot} \mathcal{H}_V \xrightarrow{\mathfrak{A}_V := B(\mathcal{H}_V)} \mathfrak{A} = \mathfrak{A}_0 \upharpoonright \mathfrak{A}_V \quad \mathfrak{A}_0 = \bigcup \mathfrak{A}_V \]

which means that to each volume $V$ it is associated an Hilbert space $\mathcal{H}_V$ and a C*-algebra $B(\mathcal{H}_V)$, whose union produce $\mathfrak{A}_0$. Taking the completion of $\mathfrak{A}_0$ wrt the C*-norm, we get the C*-algebra of the quasi-local operators. We refer to Section II for more details on the construction $\mathcal{H}_V$, and, in particular, to what concerns the states on $\mathfrak{A}$ and the dynamics.

The construction of the topological quasi*-algebra for a spin system goes as follows, [34]:

1. let $\mathcal{H} = \mathbb{C}^2$ and, $\vec{n} \in \mathbb{R}^3$, $|\vec{n}| = 1$, and $|\vec{n}> \in \mathcal{H}$ fixed (but for a phase) by requiring that $(\vec{\sigma} \cdot \vec{n})|\vec{n}> = |\vec{n}>$ and $|\vec{n}>$ is normalized. For further extensions, it may be worth remarking that this is just a way to extract a certain vector $|\vec{n}>$ out of $\mathcal{H}$.

2. let $\{\vec{n}_p\}$ be a sequence of normalized vectors in $\mathbb{R}^3$ and $\{|\vec{n}_p>\}$ the related normalized vectors in $\mathcal{H}_p$, all copies of $\mathcal{H}$, constructed as in 1. We put $\{|n>\} = \otimes_{p=1}^\infty |\vec{n}_p>$. Of course $\{|n>\} \in \mathcal{H}_\infty := \otimes_{p=1}^\infty \mathcal{H}_p$, which is a non-separable Hilbert space, [43]. Also, because it is defined via an infinite tensor product, the scalar product must be defined with a certain care. We don’t want to discuss these mathematical details here, since they do not play a major role here, and again we refer to [43].

3. Let $\pi$ be a natural realization of $\mathfrak{A}_0$: $\pi(\sigma^\alpha_n)|\{n\}> = (\otimes_{p\neq j}|\vec{n}_p> \otimes (\sigma^\alpha_j |\vec{n}_j>)$, and $\mathcal{H}_{\{n\}}$ be the closure in $\mathcal{H}_\infty$ of the space $\pi(\mathfrak{A}_0)|\{n\}>$. This is a separable Hilbert space.

4. An o.n. basis of $\mathcal{H}_{\{n\}}$ is given by the set $\{|\{m\}, \{n\}>\} = \otimes_p |m_p, \vec{n}_p>$, where $m_p = 0, 1$ for each $p$ and $\sum_p m_p < \infty$. Here we have defined each vector $|m, \vec{n}> := (\vec{\sigma} \cdot \vec{n}^-)^m |\vec{n}>$, 42
m = 0, 1, where \( \vec{n}^- = \frac{1}{2}(\vec{n}^1 - i \vec{n}^2) \), \( \vec{n}^1, \vec{n}^2 \) and \( \vec{n} \) being an o.n. set in \( \mathbb{R}^2 \).

5. The operator \( M_{\{m\}} \{\{m\}\}, \{\{n\}\} \rangle \rangle = (1 + \sum_p m_p) \{\{m\}\}, \{\{n\}\} \rangle \rangle \) is unbounded, self-adjoint and greater than \( \mathbb{I} \). We use this to define a dense subset of \( \mathcal{H}_{\{n\}} \), \( \mathcal{D}_{\{n\}} = D^\infty(M_{\{n\}}) \), and \( \mathcal{D}_{\{n\}} \) to define the \( \mathcal{O}^\ast \)-algebra \( \mathcal{L}^\dagger(\mathcal{D}_{\{n\}}) \).

6. We find that \( \pi(\mathfrak{A}_0) \subset \mathcal{L}^\dagger(\mathcal{D}_{\{n\}}) \).

7. We can introduce a topology \( \tau_0 \) on \( \mathfrak{A}_0 \) as follows: \( \forall X \in \mathfrak{A}_0 \) we put, as usual,

\[
\|X\|_{(f,k)}^{(n)} := \max\left\{ \|f(M_{\{n\}})\pi(X)M_{\{n\}}^k\|, \|M_{\{n\}}^k\pi(X)f(M_{\{n\}})\| \right\}.
\]

As we see, these seminorms are labeled by \( (f,k) \) and by \( \{n\} \).

8. Taking the completion \( \mathfrak{A} \) of \( \mathfrak{A}_0 \) wrt the topology \( \tau_0 \) we get a topological \( \ast \)-algebra. The realization of \( \mathfrak{A}_0 \) can be extended to \( \mathfrak{A} \) and we find that \( \hat{\pi}(\mathfrak{A}) \subset \mathcal{L}^\dagger(\mathcal{D}_{\{n\}}) \).

9. In the analysis of concrete models like the BCS model of superconductivity, \([34]\), it is necessary to introduce a different topology, and some physically relevant limits have to been searched in \( \mathcal{L}(\mathcal{D}_{\{n\}}, \mathcal{D}'_{\{n\}}) \), which is constructed as we have discussed in Section IV.

The same construction can be repeated for other infinite discrete quantum system obtained as infinite tensor product of finite dimensional Hilbert spaces. For instance, if \( \text{dim}(\mathcal{H}) = N \) and if \( \psi_j, j = 0, 1, \ldots, N - 1 \) is an o.n. basis of \( \mathcal{H} \), we can still take a vector \( \Psi_0 = \otimes_{p \in \mathbb{Z}} \psi_{0,p} \) which belongs to the non separable Hilbert space \( \mathcal{H}_\infty \) constructed as before. Starting from this vector we can introduce a separable Hilbert space as the closure in \( \mathcal{H}_\infty \) of \( \pi(\mathfrak{A}_0)\Psi_0 \), \( \pi \) being the natural realization of the algebra of the matrices \( N \times N \) – and, finally, a number-like operator \( \hat{N}_\Psi \) which is unbounded, self-adjoint and densely defined (and play the role of \( M_{\{n\}} \) above).

The rest of the construction can be easily repeated, and a topological quasi \( \ast \)-algebra associated to the physical system can be finally constructed.
Appendix 2: general facts in functional analysis

This appendix is devoted to list few well known facts and results in functional analysis which are used throughout this paper, and that we have decided to give here to keep the paper self contained.

1. Operators in a Hilbert space $\mathcal{H}$: $A$ is defined on a domain $D(A)$ which, if $A$ is bounded, can be taken to be all of $\mathcal{H}$. If $A$ is unbounded (i.e. if $\sup_{\varphi \in D(A)} \|A\varphi\| = \infty$), then $D(A)$ is a proper subspace of $\mathcal{H}$. (e.g. $D(\hat{x}), D(\hat{p}) \subset L^2(\mathbb{R})$, since $xf(x) \notin L^2(\mathbb{R})$ for each $f(x) \in L^2(\mathbb{R})$).

2. Closed operator: An operator $A$ is closed iff, for each sequence $\{\varphi_n\} \subset D(A)$ converging to $\varphi$ and such that $A\varphi_n$ converges to $\Psi$, then $\Psi = A\varphi$.

3. Extension and closure of an operator: Given two operators $A_1$ and $A_2$ on $\mathcal{H}$ we say that $A_1$ is an extension of $A_2$, and we write $A_1 \supset A_2$, if $D(A_1) \supset D(A_2)$ and $A_1\varphi = A_2\varphi$ for each $\varphi \in D(A_2)$. An operator $A$ is said to be closable if it has a closed extension. Every closable operator has a smallest closed extension, called its closure: $\overline{A}$. $A$ is closable iff for each sequence $\{\varphi_n\} \subset D(A)$ converging to 0 and such that $A\varphi_n$ converges to $\Psi$, then $\Psi = 0$.

4. Adjoint of an operator, bounded case: in this case $A^*$ is defined as $\langle f, A^*g \rangle = \langle Af, g \rangle$, for each $f, g \in \mathcal{H}$. If $A = A^*$ then $A$ is self-adjoint.

5. Adjoint of an operator, unbounded case: Again we put $\langle f, A^*g \rangle = \langle Af, g \rangle$, for each $f \in D(A)$ and $g \in D(A^*)$, where $D(A^*) = \{g \in \mathcal{H} : \exists g_A \in \mathcal{H}$ such that $\langle f, g_A \rangle = \langle Af, g \rangle\}$. Obviously we have $g_A =: A^*g$.

6. Symmetric operator: let $A$ be densely defined in $\mathcal{H}$. $A$ is symmetric if $A \subset A^*$, that is, if $D(A) \subset D(A^*)$ and $A\varphi = A^*\varphi$ for each $\varphi \in D(A)$. Equivalently, $A$ is symmetric if $\langle Af, g \rangle = \langle f, Ag \rangle$, for each $f, g \in D(A)$. $A$ is self-adjoint if $A$ is symmetric and if $D(A) = D(A^*)$. A symmetric operator $A$ is called essentially self-adjoint if its closure $\overline{A}$ is self-adjoint. In this case there exists only one self-adjoint extension of $A$.

7. Density matrices and traces: A density matrix, $\rho$, is an operator on $\mathcal{H}$ defined as $\rho = \sum_{n=1}^{\infty} w_n P_{\psi_n}$, where $P_{\psi_n}$ are orthogonal projectors on the o.n. set $\{\psi_n\}$ and $w_n \geq 0$ with $\sum_{n=1}^{\infty} w_n = 1$. Therefore $\rho$ is bounded and positive. Clearly $tr(\rho) = \sum_{n=1}^{\infty} \langle \psi_n, \rho \psi_n \rangle = \sum_{n=1}^{\infty} w_n = 1$. Remind that $tr$ does not depend on the choice of o.n. basis.
8. **Spectral analysis**: if $A = A^*$ has a discrete spectrum then it can be written as $A = \sum_{n=1}^{\infty} \lambda_n P_{\psi_n}$, where $\{\lambda_n\}$ and $\{\psi_n\}$ are the eigenvalues and the eigenvectors of $A$. If $A$ has not discrete spectrum then we have $A = \int \lambda \, dE(\lambda)$, in a weak sense, where $\{E(\lambda)\}$ is a family of mutually commuting operators, such that $E(-\infty) = 0$, $E(\infty) = I$, $E(\lambda) \leq E(\lambda')$ if $\lambda \leq \lambda'$, and $E(\lambda) \to E(\lambda')$ if $\lambda \to \lambda'$ from above. (if $A$ has discrete spectrum then $E(\lambda) = \sum_{\lambda_n < \lambda} P_{\psi_n}$)

9. **One-parameter unitary groups**: Stone’s theorem: A one parameter group of unitary transformations of $\mathcal{H}$ is a family $U_t$ of unitary operators in $\mathcal{H}$, $t \in \mathbb{R}$, such that $U_t U_s = U_{t+s}$ and $U_0 = I$. This is strong-continuous if $U_t \to I$ strongly when $t \to 0$. In this case Stone’s theorem states that there exists an unique self-adjoint operator $K$ in $\mathcal{H}$ such that, $\forall f \in D(K)$, $\frac{d}{dt} U_t f = iKU_t f = iU_t Kf$. Then we can simply write $U_t = e^{iKt}$. $K$ is the infinitesimal generator of the group.

More precisely we have the following: let $U_t$ be a strongly continuous 1-parameter group of unitary operators. The vectors $\psi \in \mathcal{H}$ for which $\lim_{t,0} (-i) \frac{U_t - I}{t} \psi$ exists form a dense set $\mathcal{D}$ in $\mathcal{H}$. This limit defines a self-adjoint operator $K$ which is the infinitesimal generator of the 1-parameter group.

A related result is the following: let $K$ be a self-adjoint operator with spectral resolution $E_\alpha$. Then the operators $U_t = \int_{\mathbb{R}} e^{i\alpha t} dE_\alpha$ form a 1-parameter group of unitary operators with $K$ as infinitesimal generator.

10. **Tensor products**: $\mathcal{H}_1 \otimes \mathcal{H}_2 = \text{linear span}\{f_1 \otimes f_2, f_j \in \mathcal{H}_j, j = 1, 2\}$, with scalar product $\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle = \langle f_1, g_1 \rangle_1 + \langle f_2, g_2 \rangle_2$. The operators $A_j \in B(\mathcal{H}_j)$, $j = 1, 2$, define a bounded operator $A_1 \otimes A_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ as $(A_1 \otimes A_2)(f_1 \otimes f_2) = A_1 f_1 \otimes A_2 f_2$. The extension to $\otimes_{p \in \mathbb{Z}} \mathcal{H}_p$ is rather subtle, and it is discussed in details in [43].

11. **Strong dual topology**: Let $E$ be a locally convex space and $F$ its dual, i.e. the set of the bounded linear functionals on $E$. The strong dual topology is a topology on $F$, $\beta(F, E)$, defined by the following seminorms:

$$F \ni f \to \rho_A(f) := \sup_{x \in A} |f(x)|,$$

which are labeled by the bounded subset of $E$, $A \subset E$. 45
For $\mathcal{D}$ and $\mathcal{D}'$ this becomes

$$\mathcal{D}' \ni z \mapsto \rho_{\mathcal{E}}(z) := \sup_{x \in \mathcal{E}} | < x, z > |,$$

where $<,>$ is the form which puts in duality $\mathcal{D}$ and $\mathcal{D}'$ and $\mathcal{E}$ is a bounded set in $\mathcal{D}$. 
References


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