THE COMPLETION OF A C*-ALGEBRA WITH A LOCALLY CONVEX TOPOLOGY

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Abstract. There are examples of C*-algebras A that accept a locally convex ∗-topology τ coarser than the given one, such that \( A[\tau] \) (the completion of A with respect to \( \tau \)) to be a GB*-algebra. The multiplication of \( A[\tau] \) may be or not be jointly continuous. In the second case, \( A[\tau] \) may fail being a locally convex ∗-algebra, but it is a partial ∗-algebra. In both cases the structure and the representation theory of \( A[\tau] \) are investigated. If \( \overline{A^+}[\tau] \) denotes the τ-closure of the positive cone \( A^+ \) of the given C*-algebra A, then the property \( \overline{A^+}[\tau] \cap (-\overline{A^+}[\tau]) = \{0\} \) is decisive for the existence of certain faithful ∗-representations of the corresponding ∗-algebra \( A[\tau] \).

1. Introduction

A mapping \( p \) of a ∗-subalgebra \( D(p) \) of a ∗-algebra A into \( \mathbb{R}_+ = [0, \infty) \) is said to be an unbounded C*- (semi)norm if it is a C*- (semi)norm on \( D(p) \). Unbounded C*-seminorms on ∗-algebras have appeared in many mathematical and physical subjects (for example, locally convex ∗-algebras, the moment problem, the quantum field theory etc.; see, e.g., [1, 18, 31, 33]). But, a systematical study seems far to be complete (cf. also [19], Introduction). So, we have tried to study methodically unbounded C*-seminorms and to apply such studies to those locally convex ∗-algebras that accept such C*-seminorms [8, 11, 12, 13]. A locally convex ∗-algebra is a ∗-algebra which is also a Hausdorff locally convex space such that the multiplication is separately continuous and the involution is continuous. The studies of locally convex (∗)-algebras started with those of locally m-convex (∗)-algebras by R. Arens [7] and E.A. Michael [25], in 1952. In fact, the notion of

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a locally $m$-convex algebra, was introduced by R. Arens [6], in 1946. For a complete account on locally $m$-convex algebras, see [26]. A locally convex *-algebra $A[\tau]$ is said to be locally $C^*$-convex if the topology $\tau$ determines the topology $\tau''$ and it is denoted by $A[\tau]$. A complete locally $C^*$-convex algebra is said to be a pro-$C^*$-algebra [27] (or a locally $C^*$-algebra [22]). Every pro-$C^*$-algebra is a projective limit of $C^*$-algebras. But, it is difficult to study general locally convex *-algebras which are not locally $C^*$-convex, even if the multiplication is jointly continuous. So, the third author together with K.-D. Kürsten defined and studied recently in [24] the so-called $C^*$-like locally convex *-algebras, that read as follows: If $A[\tau]$ is a locally convex *-algebra, a directed family $\Gamma = \{p_{\lambda}\}_{\lambda \in \Lambda}$ of seminorms determining the topology $\tau$ is said to be $C^*$-like if for any $\lambda \in \Lambda$ there exists $\lambda' \in \Lambda$ such that $p_p(x y) \leq p_{\lambda'}(x) p_{\lambda'}(y)$, $p_{\lambda}(x^*) \leq p_{\lambda'}(x)$ and $p_{\lambda}(x)^2 \leq p_{\lambda'}(x^* x)$ for any $x, y \in A$. Of course, $p_p$ are not necessarily $C^*$-seminorms; nevertheless, an unbounded $C^*$-norm $p_{\Gamma}$ of $A$ is defined by them in the following way:

$$D(p_{\Gamma}) = \{x \in A : \sup_{\lambda \in \Lambda} p_{\lambda}(x) < \infty\} \text{ with } p_{\Gamma}(x) := \sup_{\lambda \in \Lambda} p_{\lambda}(x), x \in D(p_{\Gamma}).$$

A locally convex *-algebra $A[\tau]$ is said to be $C^*$-like if it is complete and there is a $C^*$-like family $\Gamma = \{p_{\lambda}\}_{\lambda \in \Lambda}$ of seminorms determining the topology $\tau$ such that $D(p_{\Gamma})$ is $\tau''$-dense in $A[\tau]$. In 1967, G.R. Allan [3] introduced and studied a class of locally convex *-algebras called GB*-algebras. In 1970, P.G. Dixon [16] modified Allan’s definition in the class of topological *-algebras, so that this wider class of GB*-algebras includes certain non-locally convex *-algebras. The notion of a GB*-algebra is a generalization of a $C^*$-algebra. Given a locally convex *-algebra $A[\tau]$ with identity $1$, denote by $B^*$ the collection of all closed, bounded, absolutely convex subsets $B$ of $A$ satisfying $1 \in B$, $B^* = B$ and $B^2 \subset B$. For every $B \in B^*$, the linear span of $B$ forms a normed *-algebra under the Minkowski functional $\| \cdot \|_B$ of $B$, and it is denoted by $\text{Alg} B$ (simply, $A[\Gamma]$). If $A[B]$ is complete for every $B \in B^*$, then $A[\tau]$ is said to be pseudocomplete. If $A[\tau]$ is sequentially complete, then it is pseudocomplete. Let $A[\tau]$ be a pseudo-complete locally convex *-algebra. If $B^*$ has the greatest member $B_0$ and $(1 + x^* x)^{-1} \in A[B_0]$ for every $x \in A$, then $A[\tau]$ is said to be a GB*-algebra over $B_0$. If $A[\tau]$ is a GB*-algebra over $B_0$, then $A[B_0]$ is a $C^*$-algebra and $\| \cdot \|_{B_0}$ is an unbounded $C^*$-norm of $A[\tau]$. Thus, the study of unbounded $C^*$-seminorms may be useful for investigations on locally convex *-algebras of this type. Let $A[\tau]$ be a locally convex *-algebra and $p$ an unbounded $C^*$-norm of $A[\tau]$. Then,

$$D(p) \subset A[\tau] \subset \bar{A}[\tau] \quad \text{and} \quad D(p) \subset A[p] \equiv \hat{D}(p)[p] \quad \text{($C^*$-algebra)}$$

where $\bar{A}[\tau]$ and $A[p]$ denote the completions of $A[\tau]$ and $D(p)[p]$, respectively. But, we have no relation of $\bar{A}[\tau]$ with the $C^*$-algebra $A[p]$, in general.

Suppose now that the following condition $(N_1)$ holds:

$(N_1)$ The topology defined by $p$ is stronger than the topology $\tau$ on $D(p)$ (simply, $\tau \prec p$).
Then the identity map \( i : D(p) \to A[\tau] \) is continuous, therefore it can be extended to a continuous linear map \( \tilde{i} \) of \( A_p \) into \( \tilde{A}[\tau] \), where \( \tilde{i} \) is not necessarily an injection. It is easily shown that \( \tilde{i} \) is an injection iff the following condition \((N_2)\) is satisfied:

\((N_2)\) \( \tau \) and \( p \) are compatible in the sense that, for any Cauchy net \( \{x_a\} \) in \( D(p)[p] \) such that \( x_a \xrightarrow{\tau} 0 \), then \( x_a \xrightarrow{p} 0 \).

In this case we say that \( A_p \) is imbedded in \( \tilde{A}[\tau] \) and we write \( \tilde{A}[p] = \tilde{A}[\tau] \). Moreover, we have

\[ D(p) \subset A[\tau] \hookrightarrow \tilde{A}[\tau] \text{ resp. } D(p) \subset A_p \hookrightarrow \tilde{A}[\tau]. \]

An unbounded \( C^*\)-norm \( p \) is said to be normal, if it satisfies the conditions \((N_1)\) and \((N_2)\).

The unbounded \( C^\ast\)-norms \( p_\tau \) and \( \| \cdot \|_{B_0} \) considered above are normal.

In this paper we shall investigate the structure and the representation theory of locally convex \( *\)-algebras with normal unbounded \( C^\ast\)-norms. As stated above, it is sufficient to investigate the completion \( \tilde{A}_0[\tau] \) of the \( C^\ast\)-algebra \( A_0[\| \cdot \|] \) with respect to a locally convex topology \( \tau \) on \( A_0 \) such that \( \tau < \| \cdot \| \). Then the following cases arise:

Case 1: If the multiplication in \( A_0 \) is jointly continuous with respect to the topology \( \tau \), then \( \tilde{A}_0[\tau] \) is a complete locally convex \( *\)-algebra containing the \( C^\ast\)-algebra \( A_0[\| \cdot \|] \) as dense subalgebra.

Case 2: If the multiplication on \( A_0 \) is not jointly continuous with respect to \( \tau \), then \( \tilde{A}_0[\tau] \) is not necessarily a locally convex \( *\)-algebra, but it has the structure of a partial \( *\)-algebra [4].

Under this stimulus, we investigate in the sequel, the structure and the representation theory of \( \tilde{A}_0[\tau] \).

2. Case 1

In this Section we study the structure and the representation theory of \( \tilde{A}_0[\tau] \) as described in Case 1 before.

Suppose that \( A_0[\| \cdot \|_0] \) is a \( C^\ast\)-algebra with identity 1, \( \tau \) a locally convex topology on \( A_0 \) such that \( \tau < \| \cdot \|_0 \) and \( A_0[\tau] \) a locally convex \( *\)-algebra with jointly continuous multiplication (take, for instance, the \( C^\ast\)-algebra \( C[0,1] \) of all continuous functions on \([0,1] \), with the topology \( \tau \) of uniform convergence on the countable compact subsets of \([0,1])\). As shown in Example 4.1, the \( C^\ast\)-algebra \( A_0[\| \cdot \|_0] \) that determines the locally convex \( *\)-algebra \( \tilde{A}_0[\tau] \) is not unique. For this reason, we denote by \( C^*(A_0, \tau) \) the set of all \( C^\ast\)-algebras \( A[\| \cdot \|] \) such that \( A_0 \subset A \subset \tilde{A}_0[\tau], \tau < \| \cdot \| \) and \( \| x \| = \| x \|_0, \forall x \in A_0 \). Then, \( C^*(A_0, \tau) \) is an ordered set with the order:

\[ A_1[\| \cdot \|_1] \preceq A_2[\| \cdot \|_2] \text{ iff } A_1 \subset A_2 \text{ and } \| x \|_1 = \| x \|_2, \forall x \in A_1. \]
But, we do not know whether there exists a maximal C*-algebra in C*(A₀, τ).

**Lemma 2.1.** We denote by Bₜ the τ-closure of the unit ball \( U(A₀) \equiv \{ x \in A₀ : \| x \| ≤ 1 \} \) of the C*-algebra \( A₀ \| \| \cdot \| \| \). Then, \( Bₜ \) is a Banach *-algebra with the norm \( \| \cdot \|_{Bₜ} \) satisfying the following conditions:

(i) \( (1 + x^* x)^{-1}, x(1 + x^* x)^{-1} \) and \( (1 + x^* x)^{-1} x \) exist for every \( x \in A₀ \).

(ii) \( A₀ \subseteq A[Bₜ] \) and \( \| x \|_0 = \| x \|_{Bₜ} \) for each \( x \in A₀ \). Hence, \( U(A₀) = Bₜ ∩ A₀ \) and \( A₀ \) is a closed *-subalgebra of the Banach *-algebra \( A[Bₜ] \).

(iii) \( A[Bₜ] \) is \( \| \cdot \|_{Bₜ} \)-dense in \( A[B] \) for each \( B \in B^* \) containing \( U(A₀) \).

**Proof.** It is clear that \( Bₜ \) is a Banach *-algebra since \( A₀ \) is complete.

(i) Take an arbitrary \( x \in A₀[τ] \) and \( \{ xₐ \} \) a net in \( A₀ \) such that \( τ\text{-lim } xₐ = x \). Then, since \( A₀ \) is a C*-algebra, it follows first that \( (1 + xₐ^* xₐ)^{-1} \in U(A₀) \), for every \( a \) and secondly that for any \( τ \)-continuous seminorm \( p \)

\[
p((1 + xₐ^* xₐ)^{-1} - (1 + xₐ^* xₐ)^{-1}) = p((1 + xₐ^* xₐ)^{-1} (xₐ^* xₐ - xₐ^* xₐ) (1 + xₐ^* xₐ)^{-1}) \leq q((1 + xₐ^* xₐ)^{-1}) q((1 + xₐ^* xₐ)^{-1}) q(xₐ^* xₐ - xₐ^* xₐ) \leq γ(1 + xₐ^* xₐ)^{-1} \quad \text{for some } γ > 0 \text{ and some } τ \text{-continuous seminorm } q.
\]

Thus \( \{ (1 + xₐ^* xₐ)^{-1} \} \) is a Cauchy net in \( A₀[τ] \) and \( y ≡ \lim A₀[τ] \) exists in \( A₀[τ] \). Since,

\[
I = (1 + xₐ^* xₐ)(1 + xₐ^* xₐ)^{-1} = (1 + xₐ^* xₐ)^{-1} (1 + xₐ^* xₐ), \quad \forall a,
\]

it follows that \( (1 + x^* x)^{-1} \in A₀[τ] \) and \( y = (1 + x^* x)^{-1} \). Also, \( (1 + x^* x)^{-1} \in Bₜ \) and in a similar way we have that

\[
x(1 + x^* x)^{-1} \text{ and } (1 + x^* x)^{-1} x \text{ belong to } Bₜ.
\]

(ii) Since \( U(A₀) \subseteq Bₜ \), it follows that \( A₀ ⊂ A[Bₜ] \) and \( \| x \|_{Bₜ} ≤ \| x \|_0 \) for each \( x \in A₀ \). From the theory of C*-algebras (see, for example, [32], Proposition 1.5.3), we have \( \| x \|_0 ≤ \| x \|_{Bₜ} \) for each \( x \in A₀ \). Hence, it follows that \( \| x \|_0 = \| x \|_{Bₜ} \) for each \( x \in A₀ \), which implies that \( U(A₀) = Bₜ ∩ A₀ \) and \( A₀ \) is a closed *-subalgebra of \( A[Bₜ] \).

(iii) Take an arbitrary \( B \in B^* \) containing \( U(A₀) \). Since \( B \) is τ-closed, it follows that \( Bₜ \subseteq B \), and so \( A[Bₜ] \subseteq A[B] \) and \( \| x \|_{Bₜ} ≤ \| x \|_{B} \) for each \( x \in A[Bₜ] \). Let \( x \in A[B] \). By (i) we have

\[
x \left( I + \frac{1}{n} x^* x \right)^{-1} \in A[Bₜ], \quad ∀ n \in \mathbb{N} \text{ and }\]
Do not hallucinate. 

By Lemma 2.1, (i) $A[\mathcal{B}_1]$ is a symmetric Banach $*$-algebra, therefore by Pták’s theory for hermitian algebras [28] (see, e.g., [20], Corollary 3.4, Theorem 3.2) $A[\mathcal{B}_1]$ is hermitian and the Pták function defined as $p_{A[\mathcal{B}_1]}(x) := r_{A[\mathcal{B}_1]}(x^*x)^{1/2}$, $x \in A[\mathcal{B}_1]$, where $r_{A[\mathcal{B}_1]}$ is the spectral radius, is a $C^*$-seminorm satisfying $p_{A[\mathcal{B}_1]}(x) \leq \|x\|_{\mathcal{B}_1}$ for each $x \in A[\mathcal{B}_1]$ and $p_{A[\mathcal{B}_1]}(x) \leq \|x\|_0$, for each $x \in A_0$. It is natural to consider the following question:

**Question A.** Is $\mathcal{A}_0[\tau]$ a $GB^*$-algebra? When is $\mathcal{A}_0[\tau]$ a $GB^*$-algebra?

An answer is provided by the following theorem.

**Theorem 2.2.** The following statements are equivalent:

(i) $\mathcal{A}_0[\tau]$ is a $GB^*$-algebra.

(ii) There exists the greatest member $B_0$ in $B^*$.

(iii) There exists a member $B_0$ in $B^*$ containing $U(A_0)$ such that $\|\cdot\|_{B_0}$ is a $C^*$-norm.

If (iii) is true, then $B_0$ in (iii) is the greatest member in $B^*$ and $\mathcal{A}_0[\tau]$ is a $GB^*$-algebra over $B_0$.

**Proof.** (i) $\Rightarrow$ (iii) Since $\mathcal{A}_0[\tau]$ is a $GB^*$-algebra, there exists the greatest member $B_0$ in $B^*$. Then $\|\cdot\|_{B_0}$ is a $C^*$-norm and $U(A_0) \subset B_0$, since $B_0 \subset B^*$.

(iii) $\Rightarrow$ (ii) Let $B_0 \in B^*$ such that $\|\cdot\|_{B_0}$ is a $C^*$-norm and $U(A_0) \subset B_0$. Take an arbitrary $B \in B^*$ and $h^* = h \in B$. Let $C$ be a maximal, commutative, locally convex $*$-algebra containing $h$. Then $C$ is a complete commutative locally convex $*$-algebra. We denote by $B^C_0$ the collection of all closed, bounded, absolutely convex subsets $B_1$ of $C$ satisfying: $1 \in B_1$, $B_1^* = B_1$ and $B_1^2 \subset B_1$. Then, $B^C_0 = \{B_2 \cap C; B_2 \subset B^*\}$. We show that $B \subset C \subset B_0 \cap C$. Since $C$ is commutative and complete, it follows from ([3], Theorem 2.10) that $B^C_0$ is directed, so that there exists $B_1 \in B^C_0$ such that $(B \cap C) \cup (B_0 \cap C) \subset B_1$. Then, since the $C^*$-algebra

\[
\lim_{n \to \infty} \left\| x \left( I + \frac{1}{n} x^* x \right)^{-1} - x \right\|_{B_0} = \lim_{n \to \infty} \frac{1}{n} \left\| x x^* \left( I + \frac{1}{n} x^* x \right)^{-1} \right\|_{B_0} \\
\leq \lim_{n \to \infty} \frac{1}{n} \left\| x x^* \right\|_{B_0} \left\| \left( I + \frac{1}{n} x^* x \right)^{-1} \right\|_{B_0} \\
\leq \lim_{n \to \infty} \frac{1}{n} \left\| x x^* \right\|_{B_0} = 0.
\]

Hence, $A[\mathcal{B}_1]$ is $\|\cdot\|_{B_0}$-dense in $A[\mathcal{B}_1]$. This completes the proof.

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*Added in proof. While this paper was under publication, question A was proved in full and the answer can be found in [21, Theorem 2.1].
A[B₀ ∩ C] = A[B₀] ∩ C is contained in the Banach ∗-algebra A[B₁], it follows from ([32], Proposition I.5.3) that

\[ \|x\|_{B₀} = \|x\|_{B₀ ∩ C} \leq \|x\|_{B₁}, \quad \forall x \in A[B₀] ∩ C. \]

On the other hand, since B₀ ∩ C ⊂ B₁, it follows that

\[ \|x\|_{B₁} \leq \|x\|_{B₀ ∩ C} = \|x\|_{B₀}, \quad \forall x \in A[B₀] ∩ C. \]

Thus, we have

\[ (2.1) \quad \|x\|_{B₁} = \|x\|_{B₀}, \quad \forall x \in A[B₀] ∩ C \]

and the C∗-algebra A[B₀] ∩ C is ∥ · ∥_{B₁}-dense in the Banach ∗-algebra A[B₁]. Indeed, from Lemma 2.1, (i)

\[ x(1 + \frac{1}{n} x^* x)^{-1} \in A[B₁], \quad \forall x \in A[B₁] \quad \text{and} \quad \forall n \in \mathbb{N}. \]

It is easily shown that \( \{x, (1 + y^* y)^{-1} : x, y \in C\} \) is commutative, so that by the maximality of C, \( \{ (1 + y^* y)^{-1} : y \in C \} \subset C \). Furthermore, it follows from the assumption \( \mathcal{U}(A₀) \subset B₀ \), that \( A[B₁] ∩ C \subset A[B₀] ∩ C \). Hence,

\[ x \left( 1 + \frac{1}{n} x^* x \right)^{-1} \in A[B₁] ∩ C \subset A[B₀] ∩ C. \]

In a similar way, as in the proof of Lemma 2.1, (iii) we can show that

\[ \|x\left(1 + \frac{1}{n} x^* x\right)^{-1} - x\|_{B₁} \leq \frac{1}{n} \|xx^* x\|_{B₁}. \]

Hence, \( A[B₀] ∩ C \) is ∥ · ∥_{B₁}-dense in \( A[B₁] \). By (2.1) \( A[B₀] ∩ C = A[C ∩ B₀] = A[B₁] \), and so \( B₀ ∩ C = B₁ \). Thus, \( B ∩ C ⊂ B₀ ∩ C \). Therefore, \( h \in B₀ \) and if \( B₀ = \{x \in B : x^* = x\} \), we have \( B₀ \subset (B₀)_h \), which implies that \( \|x\|_{B₀}^2 = \|x^* x\|_{B₀} \leq 1 \) for each \( x \in B \). Hence, \( B ⊂ B₀ \) and \( B₀ \) is the greatest member in \( B^* \).

(ii) \( \Rightarrow \) (i) This follows from Lemma 2.1, (i) and so the proof is complete.

By Theorem 2.2 we have the next

**Corollary 2.3.** Consider the following statements:

(i) \( \overline{A₀[τ]} \) is a GB∗-algebra over \( \mathcal{U}(A₀) \).

(ii) \( \mathcal{U}(A₀) \) is τ-closed.

(iii) \( A₀[τ] \) is a GB∗-algebra over \( B₁ \).

(iv) \( B₁ \) is the greatest member in \( B^* \).

(v) ∥ · ∥_{B₁} is a C∗-norm.

Then, the following implications hold: (i) \( ⇔ \) (ii) \( ⇔ \) (iii) \( ⇔ \) (iv) \( ⇔ \) (v).

We investigate now the representation theory of \( \overline{A₀[τ]} \). We begin with some basic terminology. For more details see [23, 30]. Let \( D \) be a dense subspace of a
Hilbert space $\mathcal{H}$. Denote by $\mathcal{L}(\mathcal{D})$ all linear operators from $\mathcal{D}$ into $\mathcal{D}$ and let

$$\mathcal{L}^1(\mathcal{D}) := \{ X \in \mathcal{L}(\mathcal{D}) : \mathcal{D}(X^*) \supset \mathcal{D} \text{ and } X^*\mathcal{D} \subset \mathcal{D} \}.$$ 

$\mathcal{L}^1(\mathcal{D})$ is a $*$-algebra, under the usual algebraic operations and the involution $X \to X^* := X^* \dagger \mathcal{D}$. Furthermore, $\mathcal{L}^1(\mathcal{D})$ is a locally convex $*$-algebra equipped with the topology $\tau_{\mathcal{H}}$ (resp. $\tau_{\mathcal{D}}$) defined by the family $\{ p_{\mathcal{H}}(\cdot) : \xi, \eta \in \mathcal{D} \}$ of seminorms with $p_{\mathcal{H}}(X) := \| \langle X\xi, \eta \rangle \|, X \in \mathcal{L}^1(\mathcal{D})$ (resp. the family $\{ p_{\mathcal{D}}(\cdot) : \xi \in \mathcal{D} \}$ of seminorms with $p_{\mathcal{D}}(X) := \| X\xi \| + \| X^*\xi \|, X \in \mathcal{L}^1(\mathcal{D})$). A $*$-subalgebra of $\mathcal{L}^1(\mathcal{D})$ is said to be an $O^*$-algebra on $\mathcal{D}$. Let $\mathcal{A}$ be a $*$-algebra. A $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{L}^1(\mathcal{D})$ is called (unbounded) $*$-representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}_\tau$. with domain $\mathcal{D}$. If $\mathcal{A}$ has an identity, say $I$, we suppose that $\pi(I) = I$, with $I$ the identity operator in $\mathcal{L}^1(\mathcal{D})$. From now on, we shall use the notation: $\mathcal{D}(\pi)$ for the domain of $\pi$ and $\mathcal{H}_\tau$ for the corresponding Hilbert space. A $*$-representation $\pi$ of $\mathcal{A}$ is said to be faithful if $\pi(a) = 0, a \in \mathcal{A}$, implies $a = 0$. A $*$-representation $\pi$ of a locally convex $*$-algebra $\mathcal{A}[\tau]$ is said to be $\tau$-continuous (resp. $\tau$-$\tau$-)continuous) if it is continuous from $\mathcal{A}[\tau]$ to $\pi(\mathcal{A})[\tau_{\mathcal{H}}]$ (resp. to $\pi(\mathcal{A})[\tau_{\mathcal{D}}]$).

We define now a wedge $\mathcal{A}_0[\tau]$ of $\mathcal{A}_0[\tau]$. Take an arbitrary $C^*$-algebra $\mathcal{A}[\mathbb{C}]_0[\tau]$. Then, we have $\mathcal{A}_+^{\tau} = (\mathcal{A}_0)_0^{\tau}$, where $\mathcal{A}_+$ and $(\mathcal{A}_0)_+$ are positive cones in the $C^*$-algebras $\mathcal{A}$ and $\mathcal{A}_0$ respectively. Indeed, take an arbitrary $a \in \mathcal{A}_+$. Then, there is a net $\{ x_\alpha \}$ in $\mathcal{A}_0$ such that $\tau - \lim_{\alpha} x_\alpha = a^{1/2}$. Hence, $\{ x_\alpha^* x_\alpha \} \subset (\mathcal{A}_0)_+$ and $\tau - \lim_{\alpha} x_\alpha^* x_\alpha = a$. This implies that $\mathcal{A}_+^{\tau} \subset (\mathcal{A}_0)_0^{\tau}$. The converse is clear. Thus, the $\tau$-closure $\mathcal{A}_0[\tau]^{\tau}$ of $(\mathcal{A}_0)_+$ is independent of the method of taking $C^*$-algebras in $C^*(\mathcal{A}_0, \tau)$, therefore in the sequel we shall denote by $\mathcal{H}_0[\tau]$ the $\tau$-closure of $(\mathcal{A}_0)_+$. So, $\mathcal{A}_0[\tau]^{\tau}$ is a wedge (in the sense that if $x, y \in \mathcal{A}_0[\tau]^{\tau}$ and $\lambda \geq 0$, then $x + y, \lambda x \in \mathcal{A}_0[\tau]^{\tau}$, and $\mathcal{A}_0[\tau]^{\tau} = P(\mathcal{A}_0[\tau])^{\tau}$ (the $\tau$-closure of the algebraic wedge $P(\mathcal{A}_0[\tau]) = \{ \sum_{k=1}^n x_k^* x_k : x_k \in \mathcal{A}_0[\tau] (k = 1, \ldots, n), n \in \mathbb{N} \}$).

A linear functional $f$ on $\mathcal{A}_0[\tau]$ is said to be strongly positive (resp. positive) if $f(x) \geq 0$ for each $x \in \mathcal{A}_0[\tau]$, (resp. $x \in P(\mathcal{A}_0[\tau])$).

**Theorem 2.4.** The following statements are equivalent:

(i) $\mathcal{A}_0[\tau] \cap (-\mathcal{A}_0[\tau]) = \{ 0 \}$.

(ii) $A[\mathcal{B}][\tau] \cap (-A[\mathcal{B}][\tau]) = \{ 0 \}$.

(iii) The Pták function $P_A[\mathcal{B}]$ on the Banach $*$-algebra $A[\mathcal{B}, \tau]$ is a $C^*$-norm (see comments before question A).

(iv) There exists a faithful $*$-representation of $\mathcal{A}_0[\tau]$.

(v) There exists a faithful $(\tau - \tau)$-continuous $*$-representation of $\mathcal{A}_0[\tau]$.

**Proof:** (i) $\Rightarrow$ (v) Let $F$ be the set of all $\tau$-continuous strongly positive linear functionals on $\mathcal{A}_0[\tau]$. Let $\langle \pi_f, \lambda_f, \mathcal{H}_f \rangle$ be the GNS-construction for $f \in F$. We
We show that $\pi$ is a $(\tau - \tau_s, \ast)$-continuous $\ast$-representation of $\tilde{A}_0[\tau]$. Then, it is easily shown that $\pi$ is faithful. In fact, suppose $0 \neq a \in \tilde{A}_0[\tau]_h$ (the hermitian part of $\tilde{A}_0[\tau]$). Let $a \in \tilde{A}_0[\tau]_+$. Since $\tilde{A}_0[\tau]_+ \cap (-\tilde{A}_0[\tau]_+) = \{0\}$, we have $\tilde{A}_0[\tau]_+ \cap \{-a\} = \phi$. Hence, it follows from ([15], Chapter II, §5, Proposition 4) that there exists a $\tau$-continuous strongly positive linear functional $f$ on $\tilde{A}_0[\tau]$ such that $f(a) > 0$. Let $a \notin \tilde{A}_0[\tau]_+$. Since $\tilde{A}_0[\tau]_+ \cap \{a\} = \phi$, we can show in a similar way that there exists a $\tau$-continuous strongly positive linear functional $f$ on $\tilde{A}_0[\tau]$ such that $f(a) < 0$. Since $(\pi_f(a)\lambda_f(1)\lambda_f(1)) = f(a) \neq 0$ this implies that $\pi_f(a) \neq 0$, and so $\pi(a) \neq 0$. Similarly, for any $0 \neq a \in \tilde{A}_0[\tau]$ we have $\pi(a) \neq 0$ by considering $a = a_1 + ia_2$ ($a_1, a_2 \in \tilde{A}_0[\tau]_h$).

(v) $\Rightarrow$ (iv) This is trivial.

(iv) $\Rightarrow$ (iii) Let $\pi$ be a faithful $\ast$-representation of $\tilde{A}_0[\tau]$. Since $A[B_\tau]$ is a symmetric Banach $\ast$-algebra by Lemma 2.1, (i), it follows from ([20], Theorem 3.2, Corollary 3.4) that the Pták function $p_{A[B_\tau]}$ is a $C^*$-seminorm. In particular (Raikov criterion for symmetry),

$$p_{A[B_\tau]}(x) = \sup_{\rho \in \text{Rep}(A[B_\tau])} \|\rho(x)\|, \quad x \in A[B_\tau],$$

where $\text{Rep}(A[B_\tau])$ denotes the set of all $\ast$-representations of $A[B_\tau]$. Suppose $p_{A[B_\tau]}(x) = 0$. Since $\pi \upharpoonright A[B_\tau] \in \text{Rep}(A[B_\tau])$, we have $\pi(x) = 0$, and so $x = 0$. Thus $p_{A[B_\tau]}$ is a $C^*$-norm.

(iii) $\Rightarrow$ (ii) We first show that

$$\text{Sp}_{A[B_\tau]}(x) \subset \mathbb{R}_+ \equiv \{\lambda \in \mathbb{R} : \lambda \geq 0\}, \quad \forall x \in A[B_\tau]_+, \quad (2.2)$$

where $\text{Sp}_{A[B_\tau]}(x)$ stands for the spectrum of $x \in A[B_\tau]$. In fact, take an arbitrary $x \in A[B_\tau]_+$ and a net $\{x_\alpha\}$ in $(A_0)_+$ that converges to $x$ with respect to $\tau$. Since $A[B_\tau]$ is hermitian ([20], Corollary 3.4), it follows that $\text{Sp}_{A[B_\tau]}(x) \subset \mathbb{R}$. Let $\lambda < 0$. Notice that $\lambda(\lambda I - x_\alpha)^{-1} \in U(A_0)$, for every $\alpha$. Then for any $\tau$-continuous
seminorm $p$ on $\tilde{A}_0[\tau]$

\[
p(\lambda(\lambda I - x_a)^{-1} - \lambda(\lambda I - x_\bar{b})^{-1})
= |\lambda|p((\lambda I - x_a)^{-1}(x_a - x_\bar{b})((\lambda I - x_\bar{b})^{-1})
\leq |\lambda|q((\lambda I - x_a)^{-1})q(x_a - x_\bar{b})q((\lambda I - x_\bar{b})^{-1})
\leq \frac{1}{|\lambda|}q(x_a - x_\bar{b})
\leq \frac{\gamma}{|\lambda|}q(x_a - x_\bar{b})
\]

for some constant $\gamma > 0$ and a $\tau$-continuous seminorm $q$ on $\tilde{A}_0[\tau]$. It follows that

$\lambda(\lambda I - x_a)^{-1}$ converges to an element $y$ of $B_\tau$ with respect to $\tau$, which implies that $\lambda(\lambda I - x)^{-1}$ exists and equals $y$. Hence, $\lambda \notin SP_{A[B_\tau]}(x)$. Thus, we have $SP_{A[B_\tau]}(x) \subset \mathbb{R}^+$. Take an arbitrary $x \in A[B_\tau]^+ \cap (-A[B_\tau]^+)$. Then, from (2.2), it follows that $SP_{A[B_\tau]}(x) = \{0\}$, therefore $p_{A[B_\tau]}(x) = r_{A[B_\tau]}(x) = 0$. Since $p_{A[B_\tau]}$ is a norm, we have $x = 0$.

(ii) $\Rightarrow$ (i) Take an arbitrary $a \in \tilde{A}_0[\tau]^+ \cap (-\tilde{A}_0[\tau]^+)$. Then, from Lemma 2.1, (i) it follows that $a(1 + a^2)^{-1} \in A[B_\tau]^+ \cap (-A[B_\tau]^+) = \{0\}$, which implies $a = 0$. This completes the proof.

In the case of $C^*$-algebras (resp. pro-$C^*$-algebras), the condition (ii) of Theorem 2.4, is always true. Also see Example 4.4 in Section 4. In the case of symmetric Banach $*$-algebras (resp. symmetric topological $*$-algebras), which in fact can be viewed as a generalization of $C^*$-algebras [28] (resp. pro-$C^*$-algebras), it seems that such a property has not been investigated. Some information about the set $A_+$, with $A$ a certain involutive algebra can be found in [14, 29].

**Question B.** (1) Is $P(\tilde{A}_0[\tau])$ $\tau$-closed? That is, does the equality $\tilde{A}_0[\tau]^+ = P(\tilde{A}_0[\tau])$ hold? Equivalently, for each net $\{x_a\}$ in $(\tilde{A}_0)^+$ which converges to $x \in \tilde{A}_0[\tau]^+$, is $\{x_a^{1/2}\}$ $\tau$-Cauchy?
(2) Does one of the conditions in Theorem 2.4 always hold?

If $\tilde{A}_0[\tau]$ is a $GB^*$-algebra, then the above questions (1) and (2) have positive answers. Does the converse hold? That is, the following question arises.

**Question C.** If the answer to Question B is affirmative, is then $\tilde{A}_0[\tau]$ a $GB^*$-algebra?
To consider question C, we define an unbounded $C^*$-seminorm $r_\pi$ of $\tilde{A}_0[\tau]$ induced by a $*$-representation $\pi$ of $\tilde{A}_0[\tau]$ as follows:

$$D(r_\pi) = \tilde{A}_0[\tau]_B^\pi := \{x \in \tilde{A}_0[\tau] : \pi(x) \in B(H_\pi)\},$$

$$r_\pi(x) = \|\pi(x)\|, \quad x \in D(r_\pi).$$

Then we have the next

**Lemma 2.5.** Let $\pi$ be a faithful $*$-representation of $\tilde{A}_0[\tau]$ and $B$ any element of $B^*$ containing $U(A_0)$. Then the following statements hold:

1. $A_0 \subset A[B] \subset A[B] \subset D(r_\pi) = \tilde{A}_0[\tau]_B^\pi$ and $\|\pi(x)\| \leq \|x\|_B$, $\forall x \in A[B]$. as well as $\|\pi(x)\| = \|x\|_B$, $\forall x \in A_0$.
2. $\pi(A[B])$ is $\tau_\pi$-dense in $\pi(\tilde{A}_0[\tau])$, and it is also uniformly dense in $\pi(\tilde{A}_0[\tau]_B^\pi)$.
3. Suppose $\pi$ is $(\tau - \tau_\omega)$-continuous. Then $\pi(\tilde{A}_0[\tau]_+^\pi) \subset L^1(D(\pi)_+)$.

**Proof.** (1) is easily shown.

(2) Take an arbitrary $a \in \tilde{A}_0[\tau]$. Then it follows that

$$(1 + \epsilon a^* a)^{-1} a = \frac{1}{\sqrt{\epsilon}} (1 + (\sqrt{\epsilon} a)^* (\sqrt{\epsilon} a))^{-1} (\sqrt{\epsilon} a) \in A[B], \quad \forall \epsilon > 0$$

and for each $\xi \in D(\pi)$

$$\|\pi((1 + \epsilon a^* a)^{-1} a)\xi - \pi(a)\xi\| = \epsilon \|\pi((1 + \epsilon a^* a)^{-1})\pi(a^* a^2)\xi\|
\leq \epsilon \|\pi((1 + \epsilon a^* a)^{-1})\| \|\pi(a^* a^2)\xi\|
\leq \epsilon \|1 + \epsilon a^* a\| B_\pi \|\pi(a^* a^2)\xi\|
\leq \epsilon \|\pi(a^* a^2)\xi\| \xrightarrow{\epsilon \downarrow 0} 0,$$

so that $\pi(A[B])$ is $\tau_\pi$-dense in $\pi(\tilde{A}_0[\tau])$. Take an arbitrary $a \in \tilde{A}_0[\tau]_B^\pi$. Then since

$$\|\pi((1 + \epsilon a^* a)^{-1} a)\xi - \pi(a)\xi\| \leq \epsilon \|\pi(a^* a^2)\| \|\xi\|$$

for each $\xi \in D(\pi)$, it follows that $\lim_{\epsilon \downarrow 0} \pi((1 + \epsilon a^* a)^{-1} a) = \pi(a)$ uniformly, which implies that $\pi(A[B])$ is uniformly dense in $\pi(\tilde{A}_0[\tau]_B^\pi)$. Since $A[B] \subset A[B]$, (2) follows.

(3) This follows from $(\tau - \tau_\omega)$-continuity of $\pi$ and $\pi((A_0)_+^\pi) \subset L^1(D(\pi)_+)$. This completes the proof.
We simply sketch how Lemma 2.5 looks:

\[ \pi : \tilde{A}_0[\tau] \rightarrow \pi(\tilde{A}_0[\tau]) \]

\[ \cup \quad \tau_\tau\text{-dense} \]

\[ \tilde{A}_0[\tau]^\#_B \rightarrow \pi(\tilde{A}_0[\tau]^\#_B) \]

\[ \cup \quad \text{uniformly dense} \]

\[ A[B_\tau] \rightarrow \pi(A[B_\tau]) \]

symmetric Banach *-algebra

\[ \cup \]

\[ A_0[\|\cdot\|] \rightarrow \pi(A_0) \]

C*-algebra C*-algebra on \( H_\pi \).

The following theorem gives an answer to question C.

**THEOREM 2.6.** The following statements are equivalent:

(i) \( \tilde{A}_0[\tau] \) is a GB*-algebra.

(ii) There exists a faithful \((\tau - \tau_\star\tau^*)\)-continuous *-representation \( \pi \) of \( \tilde{A}_0[\tau] \), such that \( \tau < r_\pi \).

**Proof.** (i) \( \Rightarrow \) (ii) Suppose \( \tilde{A}_0[\tau] \) is a GB*-algebra over \( B_0 \). Since \( A[B_\tau]_+ \cap (-A[B_\tau])_+ \subset A[B_0]_+ \cap (-A[B_0])_+ = \{0\} \), Theorm 2.4 implies the existence of a faithful \((\tau - \tau_\star\tau^*)\)-continuous *-representation of \( \tilde{A}_0[\tau] \). Furthermore, since \( \pi(A[B_0]) \) is a C*-algebra, Lemma 2.5, (2) yields that

\[ \pi(A[B_0]) = \pi(\tilde{A}_0[\tau]^\#_B) \text{ and } r_\pi(x) = \|\pi(x)\| = \|x\|_{B_0}, \quad \forall x \in D(r_\pi), \]

which implies \( \tau < r_\pi \).

(ii) \( \Rightarrow \) (i) Since \( \tau < r_\pi \) and \( \pi \) is \((\tau - \tau_\star\tau^*)\)-continuous, it follows that \( \tau \) and \( r_\pi \) are compatible, whence one gets that the completion \( A_{r_\pi} \) of \( D(r_\pi) \) is embedded in \( \tilde{A}_0[\tau] \). We denote by \( B_0 \) the \( \tau \)-closure of the unit ball \( U(A_{r_\pi}) \) of the C*-algebra \( A_{r_\pi} \). Then, \( B_0 \in B^* \) and from Lemma 2.5, (1) we get

\[ B \subset U(\tilde{A}_0[\tau]^\#_B) \subset B_0, \quad \forall B \in B^*, \]

which implies that \( B_0 = U(\tilde{A}_0[\tau]^\#_B), \) with \( B_0 \) the greatest member in \( B^* \). Thus, from Theorem 2.2, we conclude that \( \tilde{A}_0[\tau] \) is a GB*-algebra over \( U(\tilde{A}_0[\tau]^\#_B) \) and this completes the proof.

It is known that every *-representation \( \pi \) of a Fréchet *-algebra \( A[\tau] \) is \((\tau - \tau_\star\tau^*)\)-continuous. Indeed, take an arbitrary \( \xi \in D(\pi) \) and put \( f_\xi(x) := (\pi(x)\xi, x) \in A \). Then, \( f_\xi \) is a positive linear functional on the Fréchet *-algebra
\(\mathcal{A}[\tau]\), which is continuous by ([17], Theorem 4.3). Furthermore, since the multiplication of a Fréchet \(*\)-algebra is jointly continuous, it follows that \(\pi\) is \((\tau - \tau_{\pi})\)-continuous. From this fact, as well as Theorem 2.6, we conclude the following

**Corollary 2.7.** Let \(\mathcal{A}_0[\tau]\) be a Fréchet \(*\)-algebra. Then, the following are equivalent:

(i) \(\mathcal{A}_0[\tau]\) is a GB\(^*\)-algebra.

(ii) There exists a faithful \(*\)-representation \(\pi\) of \(\mathcal{A}_0[\tau]\) such that \(\tau \prec r_{\pi}\).

3. Case 2

In this section we shall investigate the structure and the representation theory of \(\mathcal{A}_0[\tau]\) as it appears in case 2 in the Introduction. First we recall some basic definitions and properties of partial \(*\)-algebras and quasi \(*\)-algebras (for more details, refer to [4]). A partial \(*\)-algebra is a vector space \(\mathcal{A}\), endowed with a vector space involution \(x \rightarrow x^*\) and a partial multiplication defined by a set \(\Gamma \subset \mathcal{A} \times \mathcal{A}\) (a binary relation) with the properties:

(i) \((x, y) \in \Gamma\) implies \((y^*, x^*) \in \Gamma\);

(ii) \((x, y_1), (x, y_2) \in \Gamma\) implies \((x, \lambda y_1 + \mu y_2) \in \Gamma\) for all \(\lambda, \mu \in \mathbb{C}\);

(iii) for any \((x, y) \in \Gamma\), a multiplication \(xy \in \mathcal{A}\), is defined on \(\mathcal{A}\), which is distributive with respect to addition and satisfies the relation \((xy)^* = y^*x^*\).

Whenever \((x, y) \in \Gamma\), we say that \(x\) is a left multiplier of \(y\) and \(y\) is a right multiplier of \(x\), and write \(x \in L(y)\) respectively \(y \in R(x)\).

Let \(\mathcal{A}\) be a vector space and let \(\mathcal{A}_0\) be a subspace of \(\mathcal{A}\), which is also a \(*\)-algebra. \(\mathcal{A}\) is said to be a **quasi \(*\)-algebra** with distinguished \(*\)-algebra \(\mathcal{A}_0\) (or, simply, over \(\mathcal{A}_0\)) if

(i) the left multiplication \(ax\) and the right multiplication \(xa\) of an element \(a\) of \(\mathcal{A}\) with an element \(x\) of \(\mathcal{A}_0\), that extend the multiplication of \(\mathcal{A}_0\), are always defined and are bilinear;

(ii) \(x_1(x_2a) = (x_1x_2)a, (ax_1)x_2 = a(x_1x_2)\) and \(x_1(ax_2) = (x_1a)x_2\), for any \(x_1, x_2 \in \mathcal{A}_0\) and \(a \in \mathcal{A}\);

(iii) an involution \(*\) that extends the involution of \(\mathcal{A}_0\) is defined in \(\mathcal{A}\) with the property \((ax)^* = x^*a^*\) and \((xa)^* = a^*x^*\) for each \(x \in \mathcal{A}_0\) and \(a \in \mathcal{A}\).

Let \(\mathcal{A}_0[\tau]\) be a locally convex \(*\)-algebra. Then the completion \(\mathcal{A}_0[\tau]\) of \(\mathcal{A}_0[\tau]\) is a quasi \(*\)-algebra over \(\mathcal{A}_0\) equipped with the following left and right multiplications:

\[ax := \lim_{a} x_n a \quad x a := \lim_{a} x x_n, \quad \forall x \in \mathcal{A}_0 \quad \text{and} \quad a \in \mathcal{A},\]

where \(\{x_n\}\) is a net in \(\mathcal{A}_0\) converging to \(a\) with respect to the topology \(\tau\). Furthermore, the left and right multiplications are separately continuous. A \(*\)-invariant subspace \(\mathcal{A}\) of \(\mathcal{A}_0[\tau]\) containing \(\mathcal{A}_0\) is said to be a **quasi-\(*\)-subalgebra** of \(\mathcal{A}_0[\tau]\) if
and \( xa \) belong to \( A \) for any \( x \in A_0 \) and \( a \in A \). Then, it is readily shown that \( A \) is a quasi \(*\)-algebra over \( A_0 \). Moreover, \( A[\tau] \) is a locally convex space containing \( A_0 \) as a dense subspace and the right and left multiplications are separately continuous. Such an algebra \( A \) is said to be a locally convex quasi \(*\)-algebra over \( A_0 \).

Concerning \(*\)-representations of partial \(*\)-algebras and quasi \(*\)-algebras, start with a dense subspace \( \mathcal{D} \) of a Hilbert space \( \mathcal{H} \) and denote by \( \mathcal{L}^1(\mathcal{D}, \mathcal{H}) \) the set of all linear operators \( X \) from \( \mathcal{D} \) to \( \mathcal{H} \) such that \( \mathcal{D}(X^*) \supset \mathcal{D} \). Then \( \mathcal{L}^1(\mathcal{D}, \mathcal{H}) \) is a partial \(*\)-algebra with respect to the usual sum, scalar multiplication and involution \( X^* = X^* \mid_\mathcal{D} \) and the (weak) partial multiplication \( X \circ Y = X^* Y \), defined whenever \( X \) is a left multiplier of \( Y \) \( (X \in L(Y)) \), that is, \( Y \mathcal{D} \subset \mathcal{D}(X^*) \) and \( X^* \mathcal{D} \subset \mathcal{D}(Y^*) \). A (partial) \(*\)-subalgebra of the partial \(*\)-algebra \( \mathcal{L}^1(\mathcal{D}, \mathcal{H}) \) is said to be a partial \( O^*\)-algebra on \( \mathcal{D} \). A \(*\)-representation of a partial \(*\)-algebra \( A \) is a \(*\)-homomorphism \( \pi \) of \( A \) into a partial \( O^*\)-algebra \( \mathcal{L}^1(\mathcal{D}, \mathcal{H}) \), in the sense of ([4], Definition 2.1.6), satisfying \( \pi(1) = 1 \), whenever \( 1 \in A \).

In this case too, the spaces \( \mathcal{D} \) and \( \mathcal{H} \) will be denoted by \( \mathcal{D}(\pi) \) and \( \mathcal{H}_\pi \) respectively. The algebraic conjugate dual \( \mathcal{D}^\dagger \) of \( \mathcal{D} \) (i.e., the set of all conjugate linear functionals on \( \mathcal{D} \)), becomes a vector space in a natural way. Denote by \( \mathcal{L}(\mathcal{D}, \mathcal{D}^\dagger) \) the set of all linear maps from \( \mathcal{D} \) to \( \mathcal{D}^\dagger \). Then, \( \mathcal{L}(\mathcal{D}, \mathcal{D}^\dagger) \) is a \(*\)-invariant vector space under the usual operations and the involution \( T \rightarrow T^* \) with \( < T^* \xi, \eta > := \overline{< T \eta, \xi >}, \xi, \eta \in \mathcal{D} \), where \( < T^* \xi, \eta > = T^* \xi(\eta) \). Any linear operator \( X \) defined on \( \mathcal{D} \) is regarded as an element of \( \mathcal{L}(\mathcal{D}, \mathcal{D}^\dagger) \) such that \( X \xi, \eta := (X\xi)\eta, \xi, \eta \in \mathcal{D} \). For \( \mathcal{L}(\mathcal{D}, \mathcal{D}^\dagger) \) we have the following

**Lemma 3.1.** (1) \( \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) \) is regarded as a \(*\)-subalgebra of \( \mathcal{L}(\mathcal{D}, \mathcal{D}^\dagger) \).

(2) For any \( X \in \mathcal{L}^1(\mathcal{D}) \) and \( T \in \mathcal{L}(\mathcal{D}, \mathcal{D}^\dagger) \) we may define the multiplications
\[
< X \circ T \xi, \eta > := < T \xi, X^\dagger \eta > \quad \text{and} \quad < T \circ X \xi, \eta > := < TX \xi, \eta >;
\]
under these multiplications, \( \mathcal{L}(\mathcal{D}, \mathcal{D}^\dagger) \) is a quasi \(*\)-algebra over \( \mathcal{L}^1(\mathcal{D}) \).

(3) The locally convex topology \( \tau_w \) on \( \mathcal{L}(\mathcal{D}, \mathcal{D}^\dagger) \) is defined by the family
\[
\{ p_{\xi, \eta}(\cdot) : \xi, \eta \in \mathcal{D} \}
\]
of seminorms with \( p_{\xi, \eta}(T) := | < T \xi, \eta > |, \ T \in \mathcal{L}(\mathcal{D}, \mathcal{D}^\dagger), \) and it is called weak topology. It is not difficult to show that
\[
\mathcal{L}(\mathcal{D}, \mathcal{D}^\dagger) = \text{ the set of all sesquilinear forms on } \mathcal{D} \times \mathcal{D} = \mathcal{L}^\dagger(\mathcal{D})[\tau_w]
\]
and that \( \mathcal{L}(\mathcal{D}, \mathcal{D}^\dagger)[\tau_w] \) is a locally convex quasi \(*\)-algebra over \( \mathcal{L}^1(\mathcal{D}) \). More generally, for any \( O^*\)-algebra \( \mathcal{M} \) on \( \mathcal{D} \), \( \mathcal{M}[\tau_w] \) is a locally convex quasi \(*\)-algebra over \( \mathcal{M} \).

A quasi \(*\)-representation of a quasi \(*\)-algebra \( A \) over \( A_0 \) is naturally defined as a linear map \( \pi \) of \( A \) into a quasi \(*\)-algebra \( \mathcal{L}(\mathcal{D}, \mathcal{D}^\dagger) \) over \( \mathcal{L}^1(\mathcal{D}) \) such that:

(i) \( \pi \) is a \(*\)-representation of the \(*\)-algebra \( A_0 \);
(ii) \( \pi(\alpha)^\dagger = \pi(\alpha^*), \ \forall \alpha \in A \);
(iii) \( \pi(ax) = \pi(a) \circ \pi(x) \) and \( \pi(xa) = \pi(x) \circ \pi(a) \), \( \forall a \in A, \forall x \in A_0 \).

It is easily shown that if \( \pi \) is a quasi \(*\)-representation of \( A \), then \( \pi(A) \) is a quasi \(*\)-algebra over \( \pi(A_0) \).

**Lemma 3.2.** Let \( \mathcal{A}[\tau] \) be a locally convex quasi \(*\)-algebra over \( A_0 \) and \( \pi \) a quasi \(*\)-representation of \( A \). Suppose \( \pi \) is \( (\tau - \tau_w) \)-continuous. Then, \( \pi(A) \) is a locally convex quasi \(*\)-algebra over \( \pi(A_0) \).

**Proof.** From Lemma 3.1, (3) and the \( (\tau - \tau_w) \)-continuity of \( \pi \) we have

\[
\begin{align*}
\pi(A_0) & \subset \pi(A) \subset \pi(\overline{A_0}[\tau_w]) \quad \text{and} \\
\pi(x) & \circ \pi(a) = \pi(xa), \quad \pi(a) \circ \pi(x) = \pi(ax)
\end{align*}
\]

for each \( a \in A \) and \( x \in A_0 \), which implies that \( \pi(A) \) is a quasi \(*\)-subalgebra of \( \pi(\overline{A_0}[\tau_w]) \). Hence, \( \pi(A) \) is a locally convex quasi \(*\)-algebra over \( \pi(A_0) \). So, the proof is complete.

Let \( A_0[\| \cdot \|_0] \) be a \( C^* \)-algebra with \( I \) and \( \tau \) a locally convex topology on \( A_0 \) such that \( \tau < \| \cdot \|_0 \) and \( A_0[\tau] \) a locally convex \(*\)-algebra whose multiplication is not jointly continuous.

In general, \( \overline{A_0}[\tau] \) is a quasi \(*\)-algebra over \( A_0 \) (but not a \(*\)-algebra!). For this reason, the theory of quasi \(*\)-algebras must be used. We remark that for any \( A \in C^*(A_0, \tau), \overline{A}[\tau] = \overline{A_0}[\tau] \) as locally convex spaces, but \( \overline{A}[\tau] \) is different from \( \overline{A_0}[\tau] \) as a quasi \(*\)-algebra. Moreover, the wedge \( \overline{A_0}[\tau]^+ \) of the quasi\(*\)-algebra \( \overline{A_0}[\tau] \) over \( A_0 \), defined as the \( \tau \)-closure of the positive cone \( (A_0)_+ \), does not necessarily coincide with the wedge \( \overline{A}[\tau]^+ \) of the quasi \(*\)-algebra \( \overline{A}[\tau] \) over \( A \), in contrast with Case 1 (see the discussion before Theorem 2.4).

A linear functional \( f \) on \( \overline{A_0}[\tau] \), such that \( f(x) \geq 0 \), for each \( x \in \overline{A_0}[\tau]^+ \), is said to be strongly positive linear functional on the quasi \(*\)-algebra \( \overline{A_0}[\tau] \) over \( A_0 \).

Regarding the representation theory of \( \overline{A_0}[\tau] \) we have the next

**Theorem 3.3.** The following statements are equivalent:

(i) \( \overline{A_0}[\tau]^+ \cap (-\overline{A_0}[\tau]^+) = \{0\} \).

(ii) There exists a faithful \( (\tau - \tau_w) \)-continuous quasi \(*\)-representation of the quasi \(*\)-algebra \( \overline{A_0}[\tau] \) over \( A_0 \).

**Proof.** (i) \( \Rightarrow \) (ii) Let \( \mathcal{F} \) be the set of all \( \tau \)-continuous strongly positive linear functionals on the quasi \(*\)-algebra \( \overline{A_0}[\tau] \) over \( A_0 \). For any \( f \in \mathcal{F} \) we denote by \( (\pi_f, \lambda_f, \mathcal{H}_f) \) the GNS-construction for \( f \mid A_0 \). Let \( f \in \mathcal{F} \). For any \( a \in \overline{A_0}[\tau] \) we put

\[
< \lambda_f(a), \lambda_f(x) > = f(x^*a), \quad x \in A_0.
\]
Then, since \( f \) is \( \tau \)-continuous, it follows that
\[
|f(x^*a)|^2 = \lim_{\alpha} |f(x^*x_\alpha)|^2 \leq \lim_{\alpha} f(x^*x)f(x_\alpha^*x_\alpha),
\]
for each \( a \in \tilde{A}_0[\tau] \) and \( x \in A_0 \), where \( \{x_\alpha\} \) is a net in \( A_0 \) converging to \( a \) with respect to \( \tau \); it follows that \( \tilde{\lambda}_f(a) \) is well-defined and belongs to the algebraic conjugate dual \( \lambda_f(A_0)^{\dagger} \) of the vector space \( \lambda_f(A_0) \). It is clear that \( \tilde{\lambda}_f \) is a linear map of \( \tilde{A}_0[\tau] \) into the vector space \( \lambda_f(A_0)^{\dagger} \), which is an extension of \( \lambda_f \). Put
\[
D(\pi) := \{(\lambda_f(x))_{f \in F} \in \bigoplus f \in F H_f : x_f \in A_0 \text{ and } \lambda_f(x_f) = 0 \text{ except for finite } f \in F \text{ and for } (\lambda_f(x_f)) \in D(\pi),
\]
\[
< (\tilde{\lambda}_f(a_f)), (\lambda_f(x_f)) > = \sum_{f \in F} < \tilde{\lambda}_f(a_f), \lambda_f(x_f) > = \sum_{f \in F} f(x^*f)a_f, \ a_f \in \tilde{A}_0[\tau].
\]
Then \( (\tilde{\lambda}_f(a_f)) \in D(\pi)^{\dagger} \). Furthermore, for any \( a \in A \), put
\[
\pi(a)(\lambda_f(x_f)) = (\tilde{\lambda}_f(ax_f)), \ (\lambda_f(x_f)) \in D(\pi).
\]
It is easily shown that \( \pi \) is a quasi \( * \)-representation of the quasi \( * \)-algebra \( \tilde{A}_0[\tau] \) over \( A_0 \). Moreover, the \( (\tau - \tau_w) \)-continuity of \( \pi \) follows from
\[
< \pi(a)(\lambda_f(x_f)), \ (\lambda_f(y_f)) > = \sum_{f \in F} f(y^*f)ax_f,
\]
for any \( a \in A \) and \( \lambda_f(x_f) \) and \( \lambda_f(y_f) \) in \( D(\pi) \) and from the \( \tau \)-continuity of \( f \in F \). The faithfulness of \( \pi \) is shown in a similar way as in the proof of Theorem 2.4, (i) \( \Rightarrow \) (v).

(ii) \( \Rightarrow \) (i) Let \( \pi \) be a faithful \( (\tau - \tau_w) \)-continuous quasi \( * \)-representation of \( \tilde{A}_0[\tau] \) and \( a \in \tilde{A}_0[\tau] \cap (-\tilde{A}_0[\tau]) \). Then, there is a net \( \{x_\alpha\} \) in \( (A_0)_{+} \) such that \( x_\alpha \overset{\tau}{\rightarrow} a \). By the \( (\tau - \tau_w) \)-continuity of \( \pi \) we now have
\[
< \pi(a)\xi, \xi > = \lim_{\alpha} \pi(x_\alpha)\xi|\xi > 0 \text{ and similarly } < \pi(-a)\xi, \xi > > 0,
\]
for each \( \xi \in D(\pi) \). Hence, \( < \pi(a)\xi, \xi > = 0 \) for each \( \xi \in D(\pi) \), which implies \( < \pi(a)\xi, \eta > = 0 \) for any \( \xi, \eta \in D(\pi) \), that is \( \pi(a) = 0 \). By the faithfulness of \( \pi \) we have \( a = 0 \). This completes the proof.

It is natural to consider the question: When there exists a faithful \( * \)-representation \( \pi \) of the quasi \( * \)-algebra \( \tilde{A}_0[\tau] \) over \( A_0 \) (into \( L^1(D(\pi), H_\pi) \))? For that, we define the following notion: A subset \( \mathcal{G} \) of \( F \) is said to be separating if \( a \in \tilde{A}_0[\tau] \) with \( f(a) = 0 \), for each \( f \in \mathcal{G} \), implies \( a = 0 \). For example, if \( F \) is separating and \( \mathcal{G} \) is dense in \( F \) with respect to the weak\(^*\)-topology, then \( \mathcal{G} \) is separating.

**Proposition 3.4.** The following statements are equivalent:
(i) There exists a faithful $(\tau - \tau_w)$-continuous $\ast$-representation $\pi$ of the quasi $\ast$-algebra $\mathcal{A}_0[\tau]$ over $\mathcal{A}_0$ (into $L^+(D(\pi), \mathcal{H}_\pi)$).

(ii) $\mathcal{A}_0[\tau]_+ \cap (-\mathcal{A}_0[\tau]_+) = \{0\}$ and $\mathcal{F}_b$ is separating, where

$$\mathcal{F}_b = \{ f \in \mathcal{F} : \forall a \in \mathcal{A}_0[\tau] \exists \gamma_a > 0 \text{ with } |f(a^* x)|^2 \leq \gamma_a f(x^* x), \forall x \in \mathcal{A}_0 \}. $$

Proof. (i) $\Rightarrow$ (ii) By Theorem 3.3 we have $\mathcal{A}_0[\tau]_+ \cap (-\mathcal{A}_0[\tau]_+) = \{0\}$. For each $\xi \in D(\pi)$ we put $f_\xi(a) = (\pi(a)\xi, a \in \mathcal{A}_0[\tau]$. Then it is easily shown that $\{f_\xi : \xi \in D\}$ is contained in $\mathcal{F}_b$ and it is separating by the faithfulness of $\pi$. Hence, $\mathcal{F}_b$ is separating.

(ii) $\Rightarrow$ (i) As shown in the proof of (i) $\Rightarrow$ (ii) in Theorem 3.3, $\mathcal{A}_0[\tau]_+ \cap (-\mathcal{A}_0[\tau]_+) = \{0\}$ for each $f \in \mathcal{F}$ and $a \in \mathcal{A}_0[\tau]$. Take arbitrary $f \in \mathcal{F}_b$ and $a \in \mathcal{A}_0[\tau]$. Then, since

$$|<\tilde{\lambda}_f(a), \lambda_f(x)>|^2 = |f(x^* a)|^2 \leq \gamma_a f(x^* x),$$

for each $x \in \mathcal{A}_0$, it follows from the Riesz theorem that $\tilde{\lambda}_f(a)$ is regarded as an element of $\mathcal{H}_f$. Now we put

$$D(\pi) = \{((\lambda_f(x_f))\}_{x_f \in \mathcal{F}} : x_f \in \mathcal{A}_0 \text{ and } \lambda_f(x_f) = 0 \text{ except for finite } f \in \mathcal{F}_b \}$$

and for $a \in \mathcal{A}_0[\tau]$, $\pi(a)((\lambda_f(ax_f))) = ((\tilde{\lambda}_f(ax_f)))$, $(\lambda_f(x_f)) \in D(\pi)$.

Then, $\pi$ is a $\ast$-representation of $\mathcal{A}_0[\tau]$ into $L^+(D(\pi), \mathcal{H}_\pi)$. Furthermore, it is easily shown that $\pi$ is $(\tau - \tau_w)$-continuous by the $\tau$-continuity of every $f \in \mathcal{F}_b$, and $\pi$ is faithful since $\mathcal{F}_b$ is separating. This completes the proof.

4. Examples

In this section we give some examples, illustrating the results presented in Sections 2 and 3.

Example 4.1. Let $\mathcal{A}[\tau]$ be a pro-$C^*$-algebra, or more generally a $C^*$-like locally convex $\ast$-algebra with a $C^*$-like family $\Gamma = \{p_\lambda\}_{\lambda \in \Lambda}$ of seminorms determining the topology $\tau$. Then, $p_\Gamma \equiv \sup_\lambda p_\lambda$ is a $C^*$-norm on the $C^*$-algebra $\mathcal{A}_0 \equiv D(p_\Gamma) := \{x \in \mathcal{A} : p_\Gamma(x) < \infty\}$ and $\mathcal{A} \equiv \mathcal{A}_0[\tau]$. In this case, $B_\Gamma \equiv \mathcal{U}(p_\Gamma) = \mathcal{U}(p_\Gamma)$. Here we give a concrete example.

Let $\Omega$ be a locally compact space. We consider the following locally convex $\ast$-algebras of functions on $\Omega$ with the usual operations $f + g, \lambda f, fg$ and the complex conjugate as involution:

- $C_0(\Omega)$: the $C^*$-algebra of all continuous functions on $\Omega$ which converge to 0 at the infinite point;
- $C_b(\Omega)$: the $C^*$-algebra of all continuous and bounded functions on $\Omega$;
- $B(\Omega)$: the $C^*$-algebra of all bounded functions on $\Omega$;
$C(\Omega)$: the pro-$C^*$-algebra of all continuous functions on $\Omega$ equipped with the locally uniform topology $\tau_u$ defined by the family $\{\| \cdot \|_K : K$ a compact subset of $\Omega\}$ of $C^*$-seminorms with $\|f\|_K := \sup_{t \in K} |f(t)|$; $F(\Omega)$: the pro-$C^*$-algebra of all functions on $\Omega$ with the simple convergence topology $\tau_s$ defined by the family of $C^*$-seminorms $\{p_t : t \in \Omega\}$ with $p_t(f) := |f(t)|$. Then,

$$
C_0(\Omega) \subset C_b(\Omega) \subset C(\Omega) = \widehat{C_0(\Omega)}[\tau_u] = \widehat{C_b(\Omega)}[\tau_u] \cap \widehat{B(\Omega)}[\tau_s] = \widehat{C_b(\Omega)}[\tau_s] = \mathcal{F}(\Omega).
$$

**Example 4.2.** Let $A[\tau]$ be a $GB^*$-algebra over $B_0$. Then $A[B_0][\| \cdot \|_{B_0}]$ is a $C^*$-algebra and $A[B_0][\tau] = A[\tau]$. In this case, $B_\tau = \mathcal{U}(A[B_0]) = \mathcal{U}(A[B_0])$. The Arens algebra (see [5]) $A = L^\omega[0, 1] := \bigcap_{1 \leq p < \infty} L^p[0, 1]$ is a $GB^*$-algebra with the usual operations $f + g, \lambda f, fg$, usual involution $f^*$ and the topology $\tau_w$ defined by the family $\{\| \cdot \|_p : 1 \leq p < \infty\}$ of the $L^p$-norms; moreover,

$$
A[B_0] = L^\infty[0, 1] \subset L^\omega[0, 1] = \widehat{L^\infty[0, 1]}[\tau_u]; \quad \text{and}
$$

$$
L^\omega[0, 1][\| \cdot \|_p] = L^p[0, 1], \quad 1 \leq p \leq \infty,
$$

where $L^p[0, 1]$ is a Banach quasi $*$-algebra over $L^\infty[0, 1]$.

**Example 4.3.** (1) The $*$-algebra $B(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$ is a locally convex $*$-algebra equipped with the weak topology $\tau_w$. We investigate the structure of $\widehat{B(\mathcal{H})}[\tau_w]$. Let $S(\mathcal{H})$ be the set of all sesquilinear forms on $\mathcal{H} \times \mathcal{H}$. Then $S(\mathcal{H})$ is a complete locally convex space under the weak topology $\tau_w$ defined by the family $\{p_{\xi, \eta}(\cdot) : \xi, \eta \in \mathcal{H}\}$ of sesquilinear forms with $p_{\xi, \eta}(\varphi) = |\varphi(\xi, \eta)|$, $\varphi \in S(\mathcal{H})$. An element $\varphi$ of $S(\mathcal{H})$ is said to be bounded if there exists a constant $\gamma > 0$ such that $|\varphi(\xi, \eta)| \leq \gamma \|\xi\| \|\eta\|$ for each $\xi, \eta \in \mathcal{H}$. Denote by $S_b(\mathcal{H})$ the set of all bounded sesquilinear forms on $\mathcal{H} \times \mathcal{H}$, and put $S(\mathcal{H})_+ \equiv \{\varphi \in S(\mathcal{H}) : \varphi \geq 0 \iff \varphi(\xi, \xi) \geq 0, \forall \xi \in \mathcal{H}\}$ and $S_b(\mathcal{H})_+ \equiv \{\varphi \in S_b(\mathcal{H}) : \varphi \geq 0\}$. It is easily shown that $\varphi \in S_b(\mathcal{H})_+$ iff there exists an element $A$ of $B(\mathcal{H})$ such that $\varphi(\xi, \eta) = \varphi_A(\xi, \eta) := (A\xi)|\eta|$ for any $\xi, \eta \in \mathcal{H}$, and $\varphi \in S_b(\mathcal{H})_+$ iff $A \geq 0$. Hence, $S_b(\mathcal{H})[\tau_w]$ is a locally convex $*$-algebra equipped with the multiplication $\varphi_A\varphi_B := \varphi_{AB}$ and the involution $\varphi_A^* := \varphi_{A^*}$; it is also isomorphic to the locally convex $*$-algebra $B(\mathcal{H})[\tau_w]$ with respect to the map $B(\mathcal{H})[\tau_w] \ni A \mapsto \varphi_A \in S_b(\mathcal{H})[\tau_w]$. So, $B(\mathcal{H})[\tau_w]$ is isomorphic to $\widehat{S_b(\mathcal{H})}[\tau_w] = S(\mathcal{H})$ and it is a locally convex quasi $*$-algebra over $B(\mathcal{H})$ under the multiplications

$$(\varphi \circ \varphi_A)(\xi, \eta) := \varphi(A\xi, \eta), \quad (\varphi_A \circ \varphi)(\xi, \eta) := \varphi(\xi, A^*\eta), \quad \xi, \eta \in \mathcal{H},$$

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for \( A \in \mathcal{B}(\mathcal{H}) \) and \( \varphi \in \hat{S}_b(\mathcal{H})[\tau_w] \). Furthermore, it is easily shown that
\[
\overline{\mathcal{B}(\mathcal{H})}[\tau_w]_+ \cap (-\overline{\mathcal{B}(\mathcal{H})}[\tau_w]_+) = \{0\}.
\]

(2) Let \( D \) be a dense subspace in a Hilbert space \( \mathcal{H} \). We introduce on \( L^1(D,\mathcal{H}) \), the strong\(^*\)-topology \( \tau_D^* \) defined by the family \( \{ p^*_\xi, p^*_x : \xi \in D \} \) of seminorms with \( p^*_\xi(X) := \|X\xi\|, p^*_x(X) := \|X^*\xi\|, X \in L^1(D,\mathcal{H}). \) Then, \( (\mathcal{B}(\mathcal{H}) \uparrow D) \, [\tau_D] = L^1(D,\mathcal{H}), \) but \( (\mathcal{B}(\mathcal{H}) \uparrow D) \, [\tau_{D*}] \) is not a locally convex \(*\)-algebra, and so \( L^1(D,\mathcal{H}) \) is not a locally convex \(*\)-algebra over \( \mathcal{B}(\mathcal{H}) \uparrow D \). We put
\[
B(D) := \{A \uparrow D : A \in \mathcal{B}(\mathcal{H}), AD \subset D \text{ and } A^*D \subset D\}.
\]
Then, \( L^1(D,\mathcal{H}) \) is a quasi \(*\)-algebra over \( B(D) \), but as \( \overline{B(D)}[\tau_{D*}] \subseteq L^1(D,\mathcal{H}) \), in general, \( L^1(D,\mathcal{H})[\tau_{D*}] \) is not necessarily a locally convex quasi \(*\)-algebra over \( B(D) \). Let \( H \) be an unbounded positive self-adjoint operator on \( \mathcal{H} \) with \( H \geq I \), \( H = \int_1^\infty \lambda \, dE_H(\lambda) \) the spectral resolution of \( H \) and \( \mathcal{D}^\infty(H) = \bigcap_{n=1}^\infty \mathcal{D}(H^n) \). Then, for any \( A \in B(\mathcal{H}), E_H(n)AE_H(n) \in \mathcal{B}(\mathcal{D}^\infty(H)), \) for each \( n \in \mathbb{N} \) and for \( n \to \infty \) it converges to \( A \) with respect to \( \tau_{D*}^\infty(H) \); so \( L^1(\mathcal{D}^\infty(H),\mathcal{H})[\tau_{D*}^\infty(H)] \) is a locally convex quasi \(*\)-algebra over \( \mathcal{B}(\mathcal{D}^\infty(H)) \).

**Example 4.4.** Let \( \mathcal{A}_b \) be a unital \( C^* \)-algebra, with norm \( \| \cdot \|_b \) and involution \( \# \). Let \( \mathcal{A}[\| \cdot \|] \) be a right Banach module over the \( C^* \)-algebra \( \mathcal{A}_b \), with isometric involution \( * \) and such that \( \mathcal{A}_b \subset \mathcal{A} \). Set \( \mathcal{A}_* := (\mathcal{A}_b)^* \). We say that \( \{\mathcal{A}, *, \mathcal{A}_b, b\} \) is a \( CQ^* \)-algebra if

(i) \( \mathcal{A}_* \) is dense in \( \mathcal{A} \) with respect to its norm \( \| \cdot \| \);
(ii) \( \mathcal{A}_0 \equiv \mathcal{A}_b \cap \mathcal{A}_* \) is dense in \( \mathcal{A}_b \) with respect to its norm \( \| \cdot \|_b \);
(iii) \( (xy)^* = y^*x^* \), \( \forall x, y \in \mathcal{A}_0 \);
(iv) \( \|x\|_b = \sup_{a \in \mathcal{A}_b, \|a\| \leq 1} \|ax\|, \ x \in \mathcal{A}_b \).

Since \( * \) is isometric, it is easy to see that the space \( \mathcal{A}_* \) itself is a \( C^* \)-algebra with respect to the involution \( x^\# \equiv (x^*)^* \) and the norm \( \|x\|_b \equiv \|x^\#\|_b \). A \( CQ^* \)-algebra is called proper if \( \mathcal{A}_b = \mathcal{A}_0 \). For \( CQ^* \)-algebras we refer to [9, 10].

Let \( \{\mathcal{A}, *, \mathcal{A}_b, b\} \) be a proper \( CQ^* \)-algebra. Then we have
\[
\|xy\| \leq \|x\|_b \|y\|_b, \quad \|xy\| \leq \|y\| \|x\|_b, \quad \|x^\#\| = \|x\| \text{ and } (xy)^* = y^*x^*,
\]
for any \( x, y \in \mathcal{A}_b \), and so \( \mathcal{A}[\| \cdot \|] \) is a locally convex \(*\)-algebra with the involution \( * \). Furthermore, since \( \mathcal{A} = \overline{\mathcal{A}_b[\| \cdot \|]} \), it follows that \( \mathcal{A}[\| \cdot \|] \) is a locally convex quasi \(*\)-algebra over \( \mathcal{A}_b \). Consider the set \( S_\varphi(A)_+ \) of all sesquilinear forms \( \varphi \) on \( \mathcal{A} \times \mathcal{A} \) such that

(i) \( \varphi(a, a) \geq 0, \ \forall a \in \mathcal{A} \);
(ii) \( \varphi(ax, y) = \varphi(x, a^*y), \ \forall a \in \mathcal{A}, \forall x, y \in \mathcal{A}_b \).
(i) \(|\varphi(a,b)| \leq \|a\|\|b\|\), \quad \forall a, b \in \mathcal{A}.

Then, \((\mathcal{A}, \ast, \mathcal{A}_0, \mathbf{b})\) is called *-semisimple if \(a \in \mathcal{A}\) and \(\varphi(a,a) = 0\), for every \(\varphi \in S_\mathcal{A}(\mathcal{A})_{++}\), implies \(a = 0\). Suppose \((\mathcal{A}, \ast, \mathcal{A}_0, \mathbf{b})\) is a *-semisimple proper CQ*-algebra. Then \(\mathcal{A}_0 \cap (-\mathcal{A}_0) = \{0\}\). Indeed, for any \(\varphi \in S_\mathcal{A}(\mathcal{A})_{++}\) we define a strongly positive linear functional on the quasi *-algebra \(\mathcal{A}\) over \(\mathcal{A}_0\) by \(f_\varphi(a) = \varphi(a, 1), a \in \mathcal{A}\). Take an arbitrary \(h \in \mathcal{A}_0 \cap (-\mathcal{A}_0)\). Then,

\[
f_\varphi(h) = \lim_{n \to \infty} f_\varphi(x_n) \geq 0,
\]

where \(\{x_n\} \subset (\mathcal{A}_0)_{++}\) converges to \(h\) with respect to \(\|\cdot\|\). Thus, \(f_\varphi(h) = 0\), for each \(\varphi \in S_\mathcal{A}(\mathcal{A})_{++}\). We want to prove that \(\varphi(h, h) = 0\) for each \(\varphi \in S_\mathcal{A}(\mathcal{A})_{++}\). Let \(x \in \mathcal{A}_0\) with \(\|x\| \leq 1\). Then, we may define an element \(\varphi_x\) of \(S_\mathcal{A}(\mathcal{A})_{++}\) by \(\varphi_x(a, b) = \varphi(ax, bx)\) with \(a, b \in \mathcal{A}\). Hence, \(\varphi(hx, x) = 0\) for each \(x \in \mathcal{A}_0\), which implies that \(\varphi(hx, y) = 0\) for all \(x, y \in \mathcal{A}_0\). Thus,

\[
\varphi(h, h) = \lim_{n \to \infty} \varphi(h, x_n) = 0, \quad \forall \varphi \in S_\mathcal{A}(\mathcal{A})_{++}, \text{ therefore } h = 0,
\]

from the *-semisimplicity of \((\mathcal{A}, \ast, \mathcal{A}_0, \mathbf{b})\).

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