CQ*-Algebras: Structure Properties

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1. Introduction

Quasi *-algebras were introduced by Lassner [1, 2] with the purpose of providing a reasonable mathematical environment where properly dealing with the thermodynamical limit of certain quantum statistical problems which did not fit into the set-up of the algebraic formulation of quantum theories developed by Haag and Kastler [3].

In [4] we begun a systematic analysis of a special class of quasi *-algebras, the so called CQ*-algebras, taking particular care for those mathematical aspects which are more relevant for applications.

A CQ*-algebra is, roughly speaking, a complete normed quasi *-algebra \( A \) containing two dense C*-algebras \( R A \) and \( L A \) (each one with respect to its own norm and its own involution) mapped one into the other by the involution of \( A \). Typical examples of this structure are provided by the family of bounded operators in a scale of Hilbert spaces, in the non-commutative case [4], and by \( L^p \)-spaces on (locally) compact Hausdorff measure spaces [5], in the commutative case.

A CQ*-algebra can also be viewed as a partial *-algebra [6]-[10] whose lattice of multipliers consists only of four elements \( \{ A, RA, LA, A_0 \} \), where \( A_0 = RA \cap LA \). The first point of interest for us is to investigate the possibility of refining the lattice of multipliers. This is done by introducing two different notions of multiplication. The strong multiplication is defined invoking the familiar notion of closable linear map whereas the weak one is obtained via an appropriately defined family of sesquilinear forms. In both cases they mimick the notion of strong and weak multiplication discussed in [6], [7] and [9] for closable operators. All this is done in view of extending to CQ*-algebras the well-known functional calculus for C*-algebras: the first thing we need for this purpose is in fact to have at hand the largest possible set of invertible elements.

As a matter of fact, both the strong and the weak multiplication are well-behaved in the case the CQ*-algebra under consideration is *-semisimple. We will introduce this concept by generalizing one of the possible equivalent characterizations of the *-semisimplicity for C*-algebras, the one in terms of the Gel’fand seminorms.
*-Semisimple CQ*-algebras share with C*-algebras a lot of topological properties, described by several different norms which is possible to introduce on them. These norms coincide for C*-algebras and we show that this fact is indeed characteristic of C*-algebras.

2. Preliminaries and examples

Throughout the paper we will extensively use the notion of partial *-algebra [6, 9].

A partial *-algebra is a vector space $A$ with involution $A \to A^*$ [i.e. $(A + \lambda B)^* = A^* + \overline{\lambda}B^*$; $A = A^{**}$] and a subset $\Gamma \subset A \times A$ such that

(i) $(A, B) \in \Gamma$ implies $(B^*, A^*) \in \Gamma$ ; (ii) $(A, B)$ and $(A, C) \in \Gamma$ imply $(A, B + \lambda C) \in \Gamma$ ; and (iii) if $(A, B) \in \Gamma$, then there exists an element $AB \in A$ and for this multiplication the distributive property holds in the following sense: if $(A, B) \in \Gamma$ and $(A, C) \in \Gamma$ then

$$AB + AC = A(B + C)$$

Furthermore $(AB)^* = B^*A^*$.  

The product is not required to be associative.

The partial *-algebra $A$ is said to have a unit if there exists an element $\mathbb{I}$ (necessarily unique) such that $\mathbb{I}^* = \mathbb{I}$, $(\mathbb{I}, A) \in \Gamma$, $\mathbb{I}A = A\mathbb{I} = A$, $\forall A \in A$.

If $(A, B) \in \Gamma$ then we say that $A$ is a left multiplier of $B$ [and write $A \in L(B)$] or $B$ is a right multiplier of $A$ [B $\in R(A)$]. For $S \subset A$ we put $LS = \bigcap_{A \in S} L(A)$; the set $RS$ is defined in analogous way. The set $MS = LS \cap RS$ is called the set of universal multipliers of $S$.

Following Lassner [1],[2], we call quasi *-algebras a special family of partial *-algebras: namely, those for which the set $MA$ of universal multipliers is a *- algebra. We give the complete definition for reader's convenience.

Let $A$ be a linear space and $A_r$ a *-algebra contained in $A$. We say that $A$ is a quasi *-algebra with distinguished *-algebra $A_r$ (or, simply, over $A_r$) if (i) the right and left multiplications of an element of $A$ and an element of $A_r$ are always defined and linear; and (ii) an
involution $^*$ (which extends the involution of $\mathcal{A}$) is defined in $\mathcal{A}$ with the property $(AB)^* = B^*A^*$ whenever the multiplication is defined.

A quasi $^*$-algebra $(\mathcal{A}, \mathcal{A}^*)$ is said to have a unit $I$ if there exists an element $I \in \mathcal{A}$ such that $A I = I A = A, \forall A \in \mathcal{A}$.

A quasi $^*$-algebra $(\mathcal{A}, \mathcal{A}^*)$ is said to be a topological quasi $^*$-algebra if in $\mathcal{A}$ is defined a locally convex topology $\xi$ such that (a) the involution is continuous and the multiplications are separately continuous; and (b) $\mathcal{A}$ is dense in $\mathcal{A}[\xi]$.

Following [11], if $(\mathcal{A}[\xi], \mathcal{A})$ is a topological quasi $^*$-algebra, by $\xi_0$ we will denote the weakest locally convex topology on $\mathcal{A}$, such that for every bounded set $\mathcal{M} \subset \mathcal{A}[\xi]$ the family of maps $B \to AB, B \to BA; A \in \mathcal{M}$ from $\mathcal{A}[\xi]$ into $\mathcal{A}[\xi]$ is equicontinuous. In this case $\mathcal{A}[\xi]$ is a locally convex $^*$-algebra. The topology $\xi_0$ will be called the reduced topology of $\xi$.

In [4] we considered a special class of quasi $^*$-algebras, called CQ$^*$-algebras, which arise as completions of C$^*$-algebras. Let us begin with a purely algebraic definition.

**Definition 2.1.** A rigged quasi $^*$-algebra $\mathcal{A}$ is a partial $^*$-algebra for which there exist two vector subspaces $\mathcal{A}^\flat$ and $\mathcal{A}^\sharp$ such that

(i) $(\mathcal{A}^\flat)^* = \mathcal{A}_2$

(ii) $\Gamma = \{(A, B) \in \mathcal{A} \times \mathcal{A} : A \in \mathcal{A}_2 \text{ or } B \in \mathcal{A}_2\}$

(iii) both $\mathcal{A}^\flat$ and $\mathcal{A}_2$ are algebras with respect to the partial multiplication $(A, B) \in \Gamma \to AB \in \mathcal{A}$ defined in $\mathcal{A}$

The multiplication $(A, B) \in \Gamma \to AB \in \mathcal{A}$ is supposed to be (weakly) semi-associative; i.e. $(AB)C = A(BC) \forall A \in \mathcal{A}$ and $\forall B, C \in \mathcal{A}$.

**Definition 2.2.** A rigged quasi $^*$-algebra $\{\mathcal{A}, *, \mathcal{A}_\flat, \mathcal{A}_2\}$ is called a CQ$^*$-algebra if

(i) $\mathcal{A}$ is a Banach space under the norm and $\| A^* \| = \| A \| \forall A \in \mathcal{A}$

(ii) $\mathcal{A}_\flat$ is a C$^*$-algebra with respect to the norm $\| \|_\flat$ and to the involution $^\flat$
(iii) $\mathcal{A}_\sharp$ carries the norm $\|\|$, defined by $\|A\|_{\sharp} \equiv \|A^*\|_{\flat}$ (thus the involution $^*$ is an isometric anti-isomorphisms of $\mathcal{A}_\flat$ onto $\mathcal{A}_\sharp$) and $A^*_{\sharp} = A_{\flat}^* \ \forall A \in \mathcal{A}_\sharp$

(iv) $\|B\|_{\flat} = \sup_{\|A\| \leq 1} \|AB\|

(v) $\mathcal{A}_0 = \mathcal{A}_\flat \cap \mathcal{A}_\sharp$ is $\|\|_{\flat}$-dense in $\mathcal{A}_\flat$ and $\mathcal{A}_\flat$ is $\|\|_{\flat}$-dense in $\mathcal{A}$.

Throughout the paper we will always assume that $\mathcal{A}$ has a unit $I \in \mathcal{A}$. Moreover, by (ii) of Definition 2.1, $\mathcal{A}_\flat$ (resp., $\mathcal{A}_\sharp$) coincides with the set $RA$ (resp., $LA$) of the right (resp., left) multipliers of $\mathcal{A}$. For this reason, we will often write $RA$ instead of $\mathcal{A}_\flat$, etc.

**Example 2.3.** Operators on scales of Hilbert spaces. — Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle \ldots \rangle$ and $\lambda(\ldots)$ a positive sesquilinear closed form defined on a dense domain $\mathcal{D}_\lambda \subset \mathcal{H}$. Then $\mathcal{D}_\lambda$ becomes a Hilbert space, that we denote by $\mathcal{H}_\lambda$, with respect to the scalar product

$$\langle f, g \rangle_\lambda = \langle f, g \rangle + \lambda(f, g) \quad (1)$$

Let $\mathcal{H}_\overline{\lambda}$ be the Hilbert space of conjugate linear forms on $\mathcal{H}_\lambda$.

This is the canonical way to get a scale of Hilbert spaces ([14], VIII.6)

$$\mathcal{H}_\lambda \overset{i}{\rightarrow} \mathcal{H} \overset{j}{\rightarrow} \mathcal{H}_\overline{\lambda} \quad (2)$$

where $i$ and $j$ are continuous embeddings with dense range. In fact, the identity map $i$ embeds $\mathcal{H}_\lambda$ in $\mathcal{H}$ and the map $j: \psi \in \mathcal{H} \rightarrow \psi(\phi) \in \mathcal{H}_\overline{\lambda}$, where $j(\psi)(\phi) = \langle \phi, \psi \rangle$, $\forall \phi \in \mathcal{H}_\lambda$ is a linear imbedding of $\mathcal{H}$ into $\mathcal{H}_\overline{\lambda}$. Identifying $\mathcal{H}_\lambda$ and $\mathcal{H}$ with their respective images in $\mathcal{H}_\overline{\lambda}$ we can read (2) as a chain of topological inclusions

$$\mathcal{H}_\lambda \subset \mathcal{H} \subset \mathcal{H}_\overline{\lambda}$$

The representation theorem for sesquilinear forms implies ([15], Ch.VI, Sect.2) the existence of a selfadjoint positive operator $H$ such that $D((1 + H)^{1/2}) = \mathcal{D}_\lambda = \mathcal{H}_\lambda \subseteq \mathcal{H}$ and

$$\langle f, g \rangle_\lambda = \langle (1 + H)^{1/2}f, (1 + H)^{1/2}g \rangle \quad \forall f, g \in \mathcal{D}_\lambda \quad (3)$$

The operator $R = (1 + H)^{1/2}$ has a bounded inverse $R^{-1}$ which maps $\mathcal{H}$ into $\mathcal{H}_\lambda$. As a result, we can write:

$$\langle f, g \rangle_\lambda = \langle Rf, Rg \rangle = \langle Uf, Ug \rangle_\overline{\lambda} \quad \forall f, g \in \mathcal{H}_\lambda$$
Here $U$ is the operator from $H_\lambda$ onto $H_\overline{\lambda}$ whose existence is ensured by the Riesz lemma.

Let $B(H_\lambda, H_\overline{\lambda})$ be the Banach space of bounded operators from $H_\lambda$ into $H_\overline{\lambda}$ and let us denote with $\| A \|_{\lambda \overline{\lambda}}$ the natural norm of $A \in B(H_\lambda, H_\overline{\lambda})$.

In $B(H_\lambda, H_\overline{\lambda})$ we can introduce an involution in the following way: to each element $A \in B(H_\lambda, H_\overline{\lambda})$ we associate the linear map $A^*$ from $H_\lambda$ into $H_\overline{\lambda}$ defined by the equation

$$< A^* f, g > = < A g, f > \quad \forall f, g \in H_\lambda$$

As can be easily proved $A^* \in B(H_\lambda, H_\overline{\lambda})$ and $\| A^* \|_{\lambda \overline{\lambda}} = \| A \|_{\lambda \overline{\lambda}} \forall A \in B(H_\lambda, H_\overline{\lambda})$.

Let $B(H_\lambda)$ denotes the $C^*$-algebra of bounded operators on $H_\lambda$ (the usual involution of $B(H_\lambda)$ will be denoted here as $\flat$) and $B(H_\overline{\lambda})$ the $C^*$-algebra of bounded operators on $H_\overline{\lambda}$ (the natural involution of $B(H_\overline{\lambda})$ is denoted as $\sharp$). Then, both $B(H_\lambda)$ and $B(H_\overline{\lambda})$ are contained in $B(H_\lambda, H_\overline{\lambda})$ and $A \in B(H_\lambda)$ if, and only if, $A^* \in B(H_\overline{\lambda})$. Moreover $B^{\sharp*} = B^{\flat*} \forall B \in B(H_\lambda)$.

Defining the algebraic operations in the natural way, it is quite easy to show that $(B(H_\lambda, H_\overline{\lambda}), *, B(H_\lambda), \flat)$ is a rigged quasi $*$-algebra. The distinguished $*$-algebra of $B(H_\lambda, H_\overline{\lambda})$ is

$$B^+(H_\lambda) = \{ A \in B(H_\lambda, H_\overline{\lambda}) : A, A^* \in B(H_\lambda) \}$$

Actually, $(B(H_\lambda, H_\overline{\lambda}), *, B(H_\lambda), \flat)$ is a CQ$*$- algebra if $B(H_\lambda, H_\overline{\lambda})$ and $B(H_\lambda)$ carry their natural norms. In fact, $B^+(H_\lambda)$ is dense in $B(H_\lambda, H_\overline{\lambda})$ and the other requirements of Definition 2.2 are also fulfilled. For the details see [4, Example 3.3]

**Example 2.4.** Hilbert algebras — Let $A_0$ be an achieved left Hilbert algebra with identity $e$ and involution $\sharp$ and let $H$ the Hilbert space obtained by completing $A_0$ with respect to its own scalar product. Then, as is known [16, Ch.10], the commutant $A_0'$ of $A_0$ is an achieved right Hilbert algebra in $H$ with (the same) identity and involution $\flat$. The involution in $H$ is defined by the modular conjugation operator $J$. 
For shortness we put $\mathcal{H}_b = \mathcal{A}_0'$ and $\mathcal{H}_\sharp = \mathcal{A}_0$. It is easy to check that $(\mathcal{H}, J, \mathcal{H}_b, b)$ is a rigged quasi $*$-algebra in the sense of Definition 2.1.

As for the norms, one defines for $\eta \in \mathcal{H}_b$,\[
\| \eta \|_b = \| \pi_0'(\eta) \| = \sup_{\| \xi \| \leq 1} \| \xi \eta \|
\]
where $\pi_0'(\eta)$ denotes the regular $*$-representation of $\mathcal{A}_0'$ in $\mathcal{B}(\mathcal{H})$. We also define $\| \xi \|_\sharp = \| J\xi \|_b$, $\forall \xi \in \mathcal{H}_\sharp$.

Is $(\mathcal{H}, J, \mathcal{H}_b, b)$ a CQ*-algebra? First of all, we observe that conditions (i) and (iv) of Definition 2.2 are obviously fulfilled, whereas condition (iii) follows from the known equality $(J\xi)^\flat = J\xi \sharp$, $\forall \xi \in \mathcal{H}_\sharp$. As for (ii), the C*-property for the norm $\| \|_b$ is easily obtained from the fact that $\pi_0'$ is a $*$-representation of $\mathcal{H}_b$ into $\mathcal{B}(\mathcal{H})$.

To show the completeness of $\mathcal{H}_b = \mathcal{A}_0'$ one has to take into account the equality:

$\mathcal{A}_0' = \{ \eta \in D(S^*) : \pi_0'(\eta) \text{ is bounded} \}$

where $S$ is, as usual, the closure of the operator $S_0$ defined on the dense domain $\mathcal{A}_0^2$ by $\eta \in \mathcal{A}_0^2 \mapsto \eta^2 \in \mathcal{H}$.

Now, if $\{ \eta_k \}$ is a $\| \|_b$-Cauchy sequence in $\mathcal{H}_b$, since $e \in \mathcal{A}_0'$, one can find an element $\eta \in \mathcal{H}$ such that $\eta_k$ converges to $\eta$ with respect to the Hilbert norm; moreover since, as is known, for each $\eta \in \mathcal{A}_0'$, $\eta^\flat = S^*\eta$, the sequence $\{ S^*\eta_k \}$ is also convergent. Therefore $\eta \in D(S^*)$. The fact that $\pi_0'(\eta)$ is bounded follows easily from the norm completeness of $\mathcal{B}(\mathcal{H})$.

To conclude that $(\mathcal{H}, J, \mathcal{H}_b, b)$ is a CQ*-algebra, we should prove the density of $\mathcal{H}_b \cap \mathcal{H}_\sharp$ in $\mathcal{H}_b$ with respect to $\| \|_b$. We do not have a definite result in this direction; however in [16, Sect. 10.19] it is shown that the set

$\{ f_r(\Delta)f_s(\Delta^{-1})\eta : \eta \in \mathcal{H}_b, r, s > 0 \}$,

where $f_m(x) = \exp (-mx)$ and $\Delta$ is the modular operator, is contained in $\mathcal{H}_b \cap \mathcal{H}_\sharp$. This set is, in a sense, quite rich; indeed, a simple application of the spectral theorem for the operator $\Delta$ and of the Lebesgue dominated convergence theorem shows that $f_r(\Delta)f_s(\Delta^{-1})\eta$ converges to $\eta$ with respect to the Hilbert norm, for each $\eta \in \mathcal{H}_b$. We leave a deeper analysis of these points to a further paper.
Extensions of the notion of (left) Hilbert algebra in the framework of partial *-algebras have been studied by Inoue in [17].

The general structure of CQ*-algebras is simplified a lot for the so-called proper CQ*-algebras.

**Definition 2.5.** A CQ*-algebra \( \{A, *, RA, ♭\} \) is called proper if \( RA = LA \) and if \( A♭ = A♯, \forall A \in RA \).

In [4] it is proved that from the above definition it follows that

(i) \( ||A||_♯ = ||A||_♭, \forall A \in RA \);

(ii) all the abelian CQ*-algebras (i.e. \( RA = LA \) and \( AB = BA \forall A \in \mathcal{A}, B \in RA \)) are proper.

In [4] we have also proved the following constructive Proposition:

**Proposition 2.6.** Let \( \mathcal{C} \) be a C*-algebra with norm \( ||| \) and involution *.

Let \( || \) be another norm on \( \mathcal{C} \), weaker than \( ||| \) and such that

(i) \( ||A|| = || A^*||, \forall A \in \mathcal{C} \);

(ii) \( ||AB|| ≤ ||A||_1 ||B||_1, \forall A, B \in \mathcal{C} \).

Then the completion \( \hat{\mathcal{C}} \) of \( \mathcal{C} \), with its natural norm, is a proper CQ*-algebra over \( \mathcal{C} \), with \( * = ♭ \).

We will now give some examples of proper CQ*-algebras.

**Example 2.7.** \( L_ρ \)-spaces. —

Let \( \mu \) be a measure in a non-empty point set \( X \). Let \( M^+ \) be the collection of all the \( \mu \)-measurable functions on \( X \). We assume that to each \( f \in M^+ \) it corresponds a number \( ρ(f) \in [0, \infty] \) such that:

i) \( ρ(f) = 0 \) iff \( f = 0 \) a.e. in \( X \);

ii) \( ρ(f_1 + f_2) ≤ ρ(f_1) + ρ(f_2) \);

iii) \( ρ(af) = aρ(f), \forall a \in \mathbb{R}_+ \);

iv) let \( f_n \in M^+ \) and \( f_n \uparrow f \) a.e. in \( X \). Then \( ρ(f_n) \uparrow ρ(f) \).

Following [13] we call \( ρ \) a function norm. Let us define

\[ L_ρ \equiv \{ f \in M^+ : ρ(f) < \infty \} \] .

With this definition it has been proved in [13] that the space \( L_ρ \) is a Banach space, that is it is complete, with respect to the norm \( ||f||_ρ = ρ(|f|) \).
Some $L_\rho$ spaces generate examples of abelian CQ*-algebras.

(A) Let $(X, \mu)$ be a measure space with $\mu$ a regular Borel measure on the compact Hausdorff space $X$. As usual, we denote by $L^p(X, d\mu)$ the Banach space of all (equivalence classes of) measurable functions $f : X \to \mathbb{C}$ such that

$$\| f \|_p \equiv \left( \int_X |f|^p \, d\mu \right)^{1/p} < \infty.$$ 

On $L^p(X)$ we consider the natural involution $f \in L^p(X) \mapsto f^* \in L^p(X)$ with $f^*(x) = \overline{f(x)}$. Clearly $L^p$ is an $L_\rho$ space (with $\| \|_p \equiv \| \|$).

We denote with $C(X)$ the C*-algebra of continuous functions defined on $X$.

The pair $(L^p(X, \mu), C(X))$ provides the basic commutative example of topological quasi *-algebra.

It turns out also that $(L^p(X, \mu), C(X))$ is a proper abelian CQ*-algebra, for any $p \geq 1$, since the p-norm satisfies all the conditions of Proposition 2.6. These spaces have been analyzed with a certain care in [5].

(B) Let $X$ be a compact Hausdorff space and $M = \{\mu_\alpha, \alpha \in I\}$ a family of Borel measures on $X$, for which there exists a constant $C > 0$ such that $\mu_\alpha(X) \leq C \, \forall \alpha \in I$. Let $\| \|_{p,\alpha}$ be the norm on $L^p(X, \mu_\alpha)$. Of course each norm is related to a particular function norm $\rho_{p,\alpha}(f)$. Let us define, for $\phi \in C(X)$

$$\| \phi \|_{p,I} \equiv \sup_{\alpha \in I} \| \phi \|_{p,\alpha}.$$ 

In [13] it is shown that the map $\rho_{p,I}$ related to this norm still satisfies all the requirements of a function norm so that the completion of $C(X)$ with respect to $\| \|_{p,I}$, $L^p_I(X, M)$, is a Banach space. Furthermore $L^p_I(X, M)$ is contained in the intersection of all the $L^p(X, \mu_\alpha)$ spaces.

Moreover, in the hypothesis above, it is easy to prove that $\| \|_{p,I}$ also satisfies the conditions of the Proposition 2.6. Therefore $(L^p_I(X, M), C(X))$ is an abelian proper CQ*-algebra.
(C) Let $X, M$ and $\rho_{p,\alpha}$ be as above. For a sequence $\{a_n\}$ of positive constants, we define

$$\rho_p(f) \equiv \sum_{n \in I} a_n \rho(f)_{p,n}.$$ 

Then the space $L_p(X, M)$ (the completion of $C(X)$ with respect to the norm $\|\|_p$ generated by $\rho_p$) is a Banach space which, if the sequence $\{a_n\}$ is summable, contains the space $L^p_I(X, M)$ of the previous example. Again, $(L_p(X, M), C(X))$ is an abelian proper CQ*-algebra.

Example 2.8. Non-commutative $L^p$-spaces. — Let $A_0$ be a Hilbert algebra with unit $e$, $\pi_0$ the left regular representation of $A_0$ in its norm-completion $H$ and $\mathfrak{U}(A_0)$ the left von Neumann algebra of $A_0$. Let us denote by $\tau_0$ the natural trace on $\mathfrak{U}(A_0)^+$ (the positive cone of $\mathfrak{U}(A_0)$).

If $T$ is a measurable operator in Segal’s sense [18] and $T \geq 0$ one defines (we refer in the following to [19, Sect. 3] for definitions and theorems)

$$\mu(T) = \sup\{\tau_0(\pi_0(\xi)); 0 \leq \pi_0(\xi) \leq T, \xi \in (A_0)_b^2\}$$

where $(A_0)_b^2 = \{x \in H : \pi_0(x) \in B(H)\}$. Let $L^p(\tau_0), 1 \leq p < \infty$ be the space of all measurable operators $T$ such that $\mu(|T|^p) < \infty$. Then, $L^p(\tau_0)$ is a Banach space with respect to the norm $\|T\|_p = \mu(|T|^p)^{1/p}$. $L^\infty(\tau_0)$ is identified with $\mathfrak{U}(A_0)$ with its own norm. Since,

$$\|T T^*\|_p = \mu(T), \quad \forall T \in L^p(\tau_0)$$

and

$$\|T S\|_p \leq \|T\|_p \|S\|_\infty, \quad \forall T \in L^p(\tau_0), S \in \mathfrak{U}(A_0)$$

and $\mathfrak{U}(A_0)$ is dense in $L^p(\tau_0)$, applying Proposition 2.6, we get that $L^p(\tau_0)$ is a (non-abelian) proper CQ*-algebra over $\mathfrak{U}(A_0)$.

Example 2.9. Let $A_0$ be a C*-algebra (with unit $\mathbb{I}$) with respect to the norm $\|\|_0$ and the involution $^*$. Let $\Phi$ be a linear map of $A_0$ into itself with $\Phi(A^*) = \Phi(A)^*$, $\forall A \in A_0$. Suppose that the following inequality is fulfilled, for all $A, B \in A_0$

$$\|\Phi(AB)\|_0 \leq \|\Phi(A)\|_0 \|B\|_0.$$

(4)
Let us assume that $\| \Phi(I) \| = \not\in$ and define a new norm on $\mathcal{A}_0$ by

$$\| A \| = \| \Phi(A) \|_0 .$$

It is easy to verify that this norm satisfies the condition of Proposition 2.6. Therefore, the $\| \|\,$-completion $\mathcal{A}$ of $\mathcal{A}_0$ is a proper CQ*-algebra over $\mathcal{A}_0$.

Of course, the inequality (4) automatically holds if $\Phi$ is a *-homomorphism, [12]. However in this case the two norms coincide, as always when $\| \|\,$ is a Banach algebra norm on $\mathcal{A}_0$.

3. The Weak- and Strong-Multiplication

In this Section we will focus our attention on the problem of refining the multiplication in a CQ*-algebra (in the sense of obtaining a richer lattice of multipliers). This is already a significant question in some very simple situations. It is clear, for instance, that in $(L^p(X), C(X))$ the multiplication is defined not only between elements of $\mathcal{A} = L^p(X)$ and elements belonging to $\mathcal{A}_0 = C(X)$: indeed, any essentially bounded discontinuous functions with support in $X$ can be multiplied with any function of $L^p(X)$ and the result is again in $L^p(X)$. We will show here that it is possible to introduce in a CQ*-algebra two different multiplications, both extending the usual one, and we discuss some of their properties. The first one, called the strong multiplication and indicated with $\bullet$, is obtained via a closure procedure. The second one, called weak multiplication, $\circ$, is defined via a suitable family of sesquilinear forms.

Let $(\mathcal{A}, *, RA, \|\|)$ be an arbitrary CQ*-algebra. Given $A \in \mathcal{A}$ we consider the linear map $L_A : B \in RA \mapsto AB \in \mathcal{A}$. Since $\|AB\| \leq \|A\| \|B\|$, $L_A$ is continuous from $RA(\|\|_b)$ into $\mathcal{A}(\|\|)$, while, in general, it is not continuous from $RA(\|\|)$ into $\mathcal{A}(\|\|)$.

**Definition 3.1.** We say that $A \in \mathcal{A}$ is closable to the right if $L_A$ is closable as a map from $RA$ into $\mathcal{A}$.

The closability of $L_A$ means that $\forall \{B_n\} \subset RA$ such that $B_n \rightharpoonup 0$ in $\mathcal{A}$ and $AB_n \rightharpoonup Y \in \mathcal{A}$ then $Y = 0$. 
If $A \in \mathcal{A}$ is closable to the right we define the domain of its closure

$$D(\overline{L_A}) \equiv \{B \in \mathcal{A} : \exists \{B_n\} \subset R\mathcal{A} : B_n \rightharpoonup B$$

and such that $AB_n$ is $\|\|$-converging\}

and, for $B \in D(\overline{L_A})$

$$\overline{L_A}(B) \equiv \| - \lim_{n \to \infty} AB_n$$ \tag{5}

Since $D(\overline{L_A}) \supseteq \mathcal{A}_0$ then this set is dense in $\mathcal{A}$.

Of course, in the same way, $\forall A \in \mathcal{A}$ one can consider a right multiplication map $R_A$ defined by $R_A : B \in L\mathcal{A} \to BA \in \mathcal{A}$.

The domain of the closure of $R_A$ is now the set

$$D(\overline{R_A}) \equiv \{B \in \mathcal{A} : \exists \{B_n\} \subset L\mathcal{A} : B_n \rightharpoonup B$$

and such that $B_nA$ is $\|\|$-converging\}

and, for $B \in D(\overline{R_A})$

$$\overline{R_A}(B) \equiv \| - \lim_{n \to \infty} B_nA$$ \tag{6}

The right and left multiplications are linked with each other by the following

**Lemma 3.2.** Given $A \in \mathcal{A}$, $R_A$ is closable if, and only if, $L_A^*$ is closable. Moreover $D(\overline{R_A})^* = D(\overline{L_A})$

It is useful to remark that any element $A$ in $R\mathcal{A}$ ($L\mathcal{A}$) is closable to the right (left) and that $D(\overline{L_A}) = \mathcal{A}$ ($D(\overline{R_A}) = \mathcal{A}$).

If $L_A$ is closable and $B \in D(\overline{L_A})$ and if, at the same time, $R_B$ is also closable and $A \in D(\overline{R_B})$, then one would expect that the equality $\overline{L_A}(B) = \overline{R_B}(A)$ holds. This is, indeed, true and will be proved in Proposition 3.18, making use of a weaker notion of multiplication .

**Definition 3.3.** A CQ*-algebra ($\mathcal{A}, \ast, R\mathcal{A}, b$) is said to be **fully-closable** if $L_A$ is closable $\forall A \in \mathcal{A}$.

Due to 3.2 a CQ*-algebra ($\mathcal{A}, \ast, R\mathcal{A}, b$) is fully-closable if, and only if, $R_A$ is closable $\forall A \in \mathcal{A}$.

The fully-closability of a CQ*-algebra seems a very strong requirement. We are going to discuss some equivalent condition and also to discuss an example.
Let $\mathcal{A}'$ denote the dual Banach space of $\mathcal{A}$. For $\eta \in \mathcal{A}'$ and $A \in \mathcal{A}$ we put $\eta_A^L(B) \equiv \eta(AB)$, $\forall B \in RA$. We observe that $\eta_A^L$ is continuous on $RA(\|\|)$.

**Proposition 3.4.** Let $A \in \mathcal{A}$. The map $L_A$ is closable if, and only if, the set

$$F_A \equiv \{\eta \in \mathcal{A}' : \eta_A^L \text{ is continuous on } RA(\|\|)\}$$

is $\sigma(\mathcal{A}', \mathcal{A})$-dense in $\mathcal{A}'$.

**Proof.** This is nothing but an application of a well known theorem on the existence of closed extensions of linear maps [20, Ch.7, sect.36.3] $\square$

**Example 3.5.** We now use the above proposition to prove that the CQ*-algebra $(L^p(X, d\mu), C(X))$, with $2 \leq p < \infty$, discussed in Example 2.7, is fully-closable.

We call $p'$ the index conjugate to $p$, $(p^{-1} + p'^{-1} = 1)$. The dual of the space $L^p(X, d\mu)$ is therefore $L^{p'}(X, d\mu)$. For $f \in L^p(X, d\mu)$ and $g \in L^{p'}(X, d\mu)$ we put $g_f(\phi) \equiv \int_X f\phi g \, d\mu$, $\phi \in C(X)$.

The functional $g_f$ is continuous on $C(X)$ with respect to the norm $\|\|$ if, and only if, $fg \in L^{p'}(X, d\mu)$. Therefore the set $F_f$ of Proposition 3.4 is

$$F_f \equiv \{g \in L^{p'}(X, d\mu) : fg \in L^{p'}(X, d\mu)\}$$

If $p \geq 2$, this set is $\|\|$-dense in $L^{p'}(X, d\mu)$ since it contains $C(X)$. A fortiori $F_f$ is $\sigma(L^{p'}, L^p)$-dense in $L^{p'}$.

We conclude that the CQ*-algebras $(L^p(X, d\mu), C(X))$ for any $p \geq 2$ are fully-closable.

The same conclusion can be obtained also from the very first definition of fully-closability. As a matter of fact, we showed in [5] that the same statement holds for $p > 1$ without limitation on $X$ and, if $\mu(X) < \infty$, also for $p = 1$.

We introduce now right approximate identities of a CQ*-algebra. This notion extends the concept of approximate identities of a C*-algebra and will imply our CQ*-algebra to be fully-closable.
Definition 3.6. Let \((\mathcal{A}, *, RA, b)\) be a CQ*-algebra. A right approximate identity is a net \(\{E_\alpha\}\) of elements of \(RA\) such that

i) \(E_\alpha\) is a bilateral approximate identity of \(RA\);

ii) \(\lim_\alpha \|AE_\alpha - A\| = 0 \text{ \forall } A \in \mathcal{A}\).

We further say that the right approximate identity \(\{E_\alpha\}\) is regularizing if \(AE_\alpha \in RA \text{ \forall } A \in \mathcal{A}\).

The definition of left approximate identity is an obvious modification of the previous one.

Remark 3.7. (1)– If \(\{E_\alpha\}\) is a right approximate identity, then \(\{E_\alpha^*\}\) is a left approximate identity.

(2)– If \(\{E_\alpha\}\) is a regularizing left approximate identity, then it is easy to show by a simple limit argument that, for each \(\alpha\), \(E_\alpha(AB) = (E_\alpha A)B \text{ \forall } A \in \mathcal{A}, B \in RA\).

Proposition 3.8. Any CQ*-algebra has a right approximate identity (and then also a left approximate identity).

Proof. \(RA\) is a C*-algebra, then it has an increasing approximate identity bounded by 1 (i.e., \(\|E_\alpha\|_b \leq 1\)). Let \(A \in \mathcal{A}\) and \(\{A_n\} \subset RA\) be a \(\|\|\|-\text{converging to } A\); then we get

\[
\|A - AE_\alpha\| \leq \|A - A_n\| + \|A_n - A_nE_\alpha\| + \|A_nE_\alpha - AE_\alpha\| \\
\quad \leq \|A - A_n\| + \|A_n\|\|I - E_\alpha\|_b + \|A - A_n\|\|E_\alpha\|_b \rightarrow 0'
\]

since \(\|I - E_\alpha\|_b \rightarrow 0\) and \(\|E_\alpha\|_b \leq 1\).

By means of regularizing approximate identities one can give a sufficient condition for a CQ*-algebra to be fully closable.

Proposition 3.9. If a CQ*-algebra \((\mathcal{A}, *, RA, b)\) has a left regularizing approximate identity, then \(\mathcal{A}\) is fully closable.

Proof. Let \(\{E_\alpha\}\) be a regularizing left approximate identity of \(\mathcal{A}\) and let us define \(\eta_\alpha(X) \equiv \eta(E_\alpha X)\), for \(\eta \in \mathcal{A}'\) and \(\forall X \in \mathcal{A}\). Due to the continuity of \(\eta\), we have \(\eta_\alpha(X) \rightarrow \eta(X)\), \(\forall X \in \mathcal{A}\). Now, if \(A \in \mathcal{A}\) and \(B \in RA\), we define \(\eta^L_{\alpha,A}(B) \equiv \eta_\alpha(AB) = \eta(E_\alpha(AB)) = \eta((E_\alpha A)B)\). Then,

\[
|\eta^L_{\alpha,A}(B)| = |\eta_\alpha(AB)| \leq \|E_\alpha A\|_b \|B\|.
\]
This implies that $\eta_a \in F_A, \forall A \in \mathcal{A}$. Therefore $F_A$ is $\sigma(\mathcal{A}', \mathcal{A})$-dense in $\mathcal{A}', \forall A \in \mathcal{A}$.

We now introduce a different multiplication, which we call weak.

**Definition 3.10.** Let $(\mathcal{A}, *, R\mathcal{A}, b)$ be a $CQ^*$-algebra. We denote as $\mathcal{S}(\mathcal{A})$ the set of sesquilinear forms $\Omega$ on $\mathcal{A} \times \mathcal{A}$ with the following properties:

(i) $\Omega(A, A) \geq 0 \forall A \in \mathcal{A}$;
(ii) $\Omega(AB, C) = \Omega(B, A^*C) \forall A \in \mathcal{A}, \forall B, C \in R\mathcal{A}$;
(iii) $|\Omega(A, B)| \leq \|A\|\|B\| \forall A, B \in \mathcal{A}$.

Given $\Omega \in \mathcal{S}(\mathcal{A})$ we define the following positive sesquilinear form on $\mathcal{A} \times \mathcal{A}$:

$$\Omega^*(X, Y) \equiv \Omega(Y^*, X^*) \quad \forall X, Y \in \mathcal{A}. \quad (7)$$

Then $\Omega^*$ satisfies conditions (i) and (iii) of Def. 3.10, while condition (ii) should be substituted with the following one:

(ii') $\Omega^*(BA, C) = \Omega^*(B, CA^*) \forall A \in \mathcal{A}, \forall B, C \in L\mathcal{A}$.

If we call $\mathcal{S}(\mathcal{A})^*$ the set of sesquilinear forms on $\mathcal{A} \times \mathcal{A}$ satisfying (i), (ii') and (iii), it is easy to prove that $\Omega$ belongs to $\mathcal{S}(\mathcal{A})$ if, and only if, $f \Omega^*$ belongs to $\mathcal{S}(\mathcal{A})^*$.

Moreover, if $\Omega \in \mathcal{S}(\mathcal{A})$ and $B \in R\mathcal{A}$, with $\|B\|_b \leq 1$, we set

$$\Omega_B(X, Y) \equiv \Omega(XB, YB) \quad \forall X, Y \in \mathcal{A}. \quad (8)$$

It is easy to prove that $\Omega_B$ still belongs to $\mathcal{S}(\mathcal{A})$.

**Remark 3.11.** – It is well known that to any bounded sesquilinear form $\Omega$ on $\mathcal{A} \times \mathcal{A}$ it corresponds a continuous linear map $T_\Omega \in \mathcal{B}(\mathcal{A}, \mathcal{A}')$, defined by the formula

$$< A, T_\Omega(B) > \equiv \Omega(A, B) \quad \forall A, B \in \mathcal{A},$$

where $\mathcal{A}'$ is the conjugate dual of $\mathcal{A}$ with respect to the form $< \cdot, \cdot >$.

In particular, if $\Omega$ belongs to $\mathcal{S}(\mathcal{A})$, the corresponding $T_\Omega$ satisfies the following properties:
(i) \(< A, T_\Omega(A) \geq 0, \forall A \in \mathcal{A}\);
(ii) \(< AB, T_\Omega(C) \geq < B, T_\Omega(A^*C) >, \forall A \in \mathcal{A}, \forall B, C \in RA; \)
(iii) \(\| T_\Omega \| \leq 1\) as an operator from \(\mathcal{A}\) into \(\mathcal{A}'\).

The next Proposition shows that normalized elements of \(S(\mathcal{A})\) give rise to states, in the usual sense, on \(RA\).

**Proposition 3.12.** Let \(\Omega \in S(\mathcal{A})\) with \(\Omega(\mathbb{I}, \mathbb{I}) = k\), then the linear functional \(\omega_\Omega\) on \(RA\) defined by

\[
\omega_\Omega(X) = \Omega(X, \mathbb{I}), \quad X \in RA
\]

is positive in \(RA\); i.e. \(\omega_\Omega(X^*X) = \Omega(X, X^{\flat*}) \geq 0, \forall X \in RA\).

**Proof.** By the Schwarz inequality we get

\[
|\omega_\Omega(X)| = |\Omega(X, \mathbb{I})| \leq \Omega(X, X)^{\flat/\flat^*}\Omega(\mathbb{I}, \mathbb{I})^{\flat/\flat^*} \leq \|X\|_b.
\]

Therefore \(\| \omega_\Omega \| \leq 1\); on the other hand,

\[
\| \omega_\Omega \| \geq \omega_\Omega(\mathbb{I}) = \Omega(\mathbb{I}, \mathbb{I}) = \|k\|.
\]

Thus \(\| \omega_\Omega \| = \omega_\Omega(\mathbb{I})\). Hence, \(\omega_\Omega\) is positive. \(\square\)

**Remark 3.13.** Since \(\omega_\Omega\) is positive, one has

\[
\omega_\Omega(X^*) = \overline{\omega_\Omega(X)} = \Omega(\mathbb{I}, X)
\]

and therefore \(\Omega(X^*, \mathbb{I}) = \Omega(X^*, \mathbb{I}), \forall X \in RA\). From this it follows easily that

\[
\Omega(X^{\flat*} B^{\flat*}, C) = \Omega(X^{\flat*} B, C^{\flat*}), \forall X, B, C \in RA.
\]

From now on we will study certain 'weak' properties related to a convenient family of sesquilinear forms on \(\mathcal{A} \times \mathcal{A}\). Of course both \(S(\mathcal{A})\) and \(S(\mathcal{A})^*\) are good candidates, as well as their intersection. However, since we have in mind essentially vector states, which may satisfy (ii) but not (ii'), we will consider only the family \(S(\mathcal{A})\). There is no other reason for this 'symmetry breaking'.

Let \((\mathcal{A}, *, RA, b)\) be an arbitrary CQ*-algebra and let \(X, Y \in \mathcal{A}\).
Definition 3.14. We say that \( X \) (\( Y \)) is a weak left (right) multiplier of \( Y \) (\( X \)), if there exists a unique element \( Z \in \mathcal{A} \) such that
\[
\Omega(YB, X^*C) = \Omega(ZB, C) \quad \forall \Omega \in \mathcal{S}(\mathcal{A}), \forall B, C \in RA. \tag{9}
\]
In this case we write \( Z = X \circ Y \) and \( X \in L_w(Y) \) or \( Y \in R_w(X) \).

It is straightforward to prove that, if the usual product is defined, then this coincides with the weak one. More explicitly, if \( X \in \mathcal{A}, C \in RA \), then \( X \in L_w(C) \) and \( X \circ C = XC \).

At this point, let us now define the following subset \( \Gamma_w \subset \mathcal{A} \times \mathcal{A} \)
\[
\Gamma_w \equiv \{(X, Y) \in \mathcal{A} \times \mathcal{A} : \exists ! Z \in \mathcal{A} : \Omega(YB_1, X^*B_2) = \Omega(ZB_1, B_2) \quad \forall \Omega \in \mathcal{S}(\mathcal{A}), \forall B_1, B_2 \in RA \} \tag{10}
\]
As usual we put, for \( X \in \mathcal{A}, R_w(X) = \{Y \in \mathcal{A} : (X, Y) \in \Gamma_w \} \) and we define in similar way \( L_w(X) \).

Remark 3.15. It is clear that if \( \mathcal{A} \) satisfies the following condition:
\[
\Omega(XB, C) = 0 \quad \forall \Omega \in \mathcal{S}(\mathcal{A}); \forall B, C \in RA \implies X = \mathbb{I}
\]
or the equivalent one
\[
\Omega(X, X) = 0 \quad \forall \Omega \in \mathcal{S}(\mathcal{A}) \implies X = \mathbb{I}
\]
then the element \( Z \) in Definition 3.14, if it exists, is necessarily unique.

We will come back to these two conditions in the next Section.

Proposition 3.16. Let \( (\mathcal{A}, *, RA, \delta) \) be a \( CQ^* \)-algebra. Then \( (\mathcal{A}, \Gamma_w, \circ) \) is a partial *-algebra and \( RA \subset R_w(A) \).

This result easily follows from the definition of partial *-algebra, see [9]. This partial *-algebra is, in general, non-associative.

Proposition 3.17. If \( X \) is closable to the right and if \( Y \in D(\overline{L_X}) \) then \( X \in L_w(Y) \) and \( X \circ Y = \overline{L_X}(Y) \).

Proof. Indeed, if \( Y \in D(\overline{L_X}) \), then there exists a sequence \( \{Y_n\} \subset RA \) such that \( Y_n \to Y \) and \( XY_n \to Z \) in the norm of \( \mathcal{A} \). Then, for \( \Omega \in \mathcal{S}(\mathcal{A}) \) and \( B_1, B_2 \in RA \) we get
\[
|\Omega((XY_n - Z)B_1, B_2)| \leq ||XY_n - Z|| \|B_1\| \|B_2\|b.
\]
It follows that $\Omega((XY_n)B_1, B_2) \to \Omega(ZB_1, B_2)$ $\forall \Omega \in \mathcal{S}(\mathcal{A})$, $\forall B_1, B_2 \in RA$. Therefore $\Omega(YB_1, X^*B_2) = \lim_{n \to \infty} \Omega(Y_nB_1, X^*B_2) = \lim_{n \to \infty} \Omega(XY_nB_1, B_2) = \Omega(ZB_1, B_2)$ \hfill \Box

Making use of the above Proposition, and of the analogous statement for the left-closability, we can deduce the following statement, which easily follows from the uniqueness of the product $\circ$.

**Proposition 3.18.** Let $X$ be closable to the right and $Y \in D(L_X)$. Let furthermore $Y$ be closable to the left and $X \in D(R_Y)$. Then $\overline{R_Y}(X) = \overline{L_X}(Y) = X \circ Y$.

At this point we can define the strong product in the following way

**Definition 3.19.** In the hypotheses of Proposition 3.18 we define $X \bullet Y \equiv \overline{R_Y}(X) = \overline{L_X}(Y)$.

An obvious consequence is that the weak and the strong products coincide whenever they are both defined.

**Remark 3.20.** $(L^p(X, \mu), C_0(X))$ is a very simple instance where the strong and weak multiplication coincide, [5].

It is a natural question to ask whether $\mathcal{A}$ is a partial $*$-algebra with respect to the strong multiplication too. More precisely, if $\mathcal{A}$ is fully closable, then we can define

$$\Gamma_s \equiv \{(X, Y) \in \mathcal{A} \times \mathcal{A} : Y \in D(L_X) \text{ and } X \in D(R_Y)\} \quad (11)$$

(In analogy with the case of the weak-multiplication, we define , for $X \in \mathcal{A}$, $R_s(X) = \{Y \in \mathcal{A} : (X, Y) \in \Gamma_s\}$ ,etc.)

Is then $(\mathcal{A}, \Gamma_s, \bullet)$ a partial $*$-algebra? The answer is, in general, negative because of the possible lack of the distributivity. This unpleasant feature depends on the following fact: if $(A, B) \in \Gamma_s$ and $(A, C) \in \Gamma_s$ then certainly $B + C \in D(L_A)$; but on the other side, we only get $A \in D(R_B) \cap D(R_C)$ and this is, in general, different from $D(R_{B+C})$ (this is the same pathology discussed in [6, Add./Err.] and has the same topological motivations). The conditions $(X, Y) \in \Gamma_s \Leftrightarrow (Y^*, X^*) \in \Gamma_s$ and $(X \bullet Y)^* = Y^* \bullet X^*$ are, on the contrary, always fulfilled.

The definition of strong multiplication allows also to give the following weaker form of the associative law in $(\mathcal{A}, \Gamma_w, \circ)$:
Proposition 3.21. If $A \in D(R_B)$, $(A \cdot B, C) \in \Gamma_w$ and $(B, C) \in \Gamma_w$ then $(A, B \circ C) \in \Gamma_w$ and $(A \cdot B) \circ C = A \circ (B \circ C)$.

Proof. First we recall that $A \in D(R_B)$ if, and only if, $A^* \in D(L_B^*)$. Therefore there exists a sequence $\{R_n\} \subset L_A : R_n \to A^*$ and $B^*R_n$ is convergent in $A$.

If $\Omega \in S(A), S_1, S_2 \in RA$ we get

$$\Omega((A \cdot B) \circ C)S_1, S_2) = \Omega((A \circ B) \circ C)S_1, S_2) = \Omega(CS_1, (A \circ B)^*S_2) = \Omega(CS_1, B^* \circ A^*S_2) = \lim_{n \to \infty} \Omega(CS_1, B^*R_nS_2) = \lim_{n \to \infty} \Omega((B \circ C)S_1, R_nS_2) = \Omega((B \circ C)S_1, A^*S_2) = \Omega(A \circ (B \circ C)S_1, S_2)$$

4. *-Semisimple CQ*-algebras

Lemma 4.1. Let $(A, *, RA, b)$ be a CQ*-algebra. Let us consider the following three sets

$$\mathcal{R}_1 = \{X \in A : \Omega(X, X) = 0 \ \forall \Omega \in S(A)\}$$
$$\mathcal{R}_2 = \{X \in A : \Omega(XB, C) = 0 \ \forall \Omega \in S(A), \forall B, C \in RA\}$$
$$\mathcal{R}_3 = \{X \in A : \Omega(XB, XC) = 0 \ \forall \Omega \in S(A), \forall B, C \in RA\}$$

Then $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}_3 \equiv \mathcal{R}^{(*)}$.

The set $\mathcal{R}^{(*)}$ is called the *-radical of $A$.

Proof. The inclusion $\mathcal{R}_3 \subseteq \mathcal{R}_2$ follows immediately from the Schwarz inequality.

Next we will show that $\mathcal{R}_2 \subseteq \mathcal{R}_1$. Let $X \in A$, and $\{X_n\}$ be a sequence of elements in $RA |||−$ converging to $X$. If $\Omega \in S(A)$, then $\Omega(X, X_n) = 0 \ \forall n$, and the $|||−$ continuity of the sesquilinear form $\Omega$, implies that $\Omega(X, X) = 0$.

Finally, the inclusion $\mathcal{R}_1 \subseteq \mathcal{R}_3$ follows easily from the Schwarz inequality making use of the form $\Omega_B$ defined in (8).

Lemma 4.2. The *-radical $\mathcal{R}^{(*)}$ of a CQ*-algebra $(A, *, RA, b)$ has the following properties:
(i) $\mathfrak{R}(\ast)$ is a linear subspace of $\mathcal{A}$
(ii) If $X \in \mathfrak{R}(\ast)$ then $X^\ast \in \mathfrak{R}(\ast)$
(iii) $\mathfrak{R}(\ast) \cap \mathcal{R} \mathcal{A}$ is a right ideal of $\mathcal{R} \mathcal{A}$
(iv) If $X \in L \mathcal{A}$, $Y \in \mathfrak{R}(\ast)$ and $Z \in \mathcal{R} \mathcal{A}$ then $XY \in \mathfrak{R}(\ast)$ and $YZ \in \mathfrak{R}(\ast)$

The proof is straightforward.

Remark 4.3. The property (iii) of Lemma 4.2 has a left counterpart: $\mathfrak{R}(\ast) \cap L \mathcal{A}$ is a left ideal of $L \mathcal{A}$ and, analogously, $\mathfrak{R}(\ast) \cap \mathcal{A}_0$ is a $\ast$-ideal of $\mathcal{A}_0$

It is rather natural to call quasi $\ast$-ideal a subset of $\mathcal{A}$ which has the properties (i)-(iv) of Lemma 4.2.

Definition 4.4. We call $\ast$-semisimple any CQ*-algebra $(\mathcal{A}, \ast, \mathcal{R} \mathcal{A}, \mathfrak{b})$ such that $\mathfrak{R}(\ast) = \{0\}$.

For a $\ast$-semisimple CQ*-algebra $(\mathcal{A}, \ast, \mathcal{R} \mathcal{A}, \mathfrak{b})$ the set
\[
\mathfrak{R}_4 = \{X \in \mathcal{A} : X^\ast \circ X \text{ is well-defined and } X^\ast \circ X = 0\}
\]
coincides with the $\ast$-radical and, therefore, reduces to $\{0\}$. We will be mainly concerned with such CQ*-algebras. The reason is that the $\ast$-semisimplicity turns out to be a structure property which simplifies the (heavy) general framework developed in [4]. Moreover many interesting examples of CQ*-algebras are indeed $\ast$-semisimple as we shall see later.

We recall that the notion of $\ast$-semisimplicity in the ordinary Banach $\ast$-algebras theory can be formulated in terms similar to those used here. Our set-up is indeed an extension of the Gel’fand description of the $\ast$-semisimplicity. Actually, if $\mathcal{A}_0$ is a Banach $\ast$-algebra then $\mathcal{A}_0$ is $\ast$-semisimple if, when $A \in \mathcal{A}_0$ is such that $\omega(A^\ast A) = 0$ for all $\omega$ in the set $\mathcal{P}(\mathcal{A}_0)$ of all positive functionals with norm less or equal to 1, $A = 0$ results.

If $\mathcal{A}_0$ is $\ast$-semisimple then the Gelfand seminorm
\[
\|A\|_2^\ast = \sup_{\omega \in \mathcal{P}(\mathcal{A}_0)} \omega(A^\ast A)
\]
is actually a norm which satisfies the C*-property.
Given a CQ*-algebra \((A, *, RA, b)\) and a sesquilinear form \(\Omega \in \mathcal{S}(A)\), we define the positive linear functional \(\omega_\Omega(A) \equiv \Omega(A, I)\) where \(A\) is taken in \(A_0\), [4]. It is easy to see that any such \(\omega_\Omega\) belongs to \(P(A_0)\).

**Proposition 4.5.** Let \((A, *, RA, b)\) be a CQ*-algebra. If \(A\) is *-semisimple then \(A_0\) is *-semisimple.

**Proof.** Let \(\omega(A^*A) = 0 \forall \omega \in \mathcal{P}(A_0)\). Then, in particular, we will have \(\omega_\Omega(A^*A) = 0 \forall \Omega \in \mathcal{S}(A)\). This implies that, for all such \(\Omega\), \(\Omega(A, A) = 0\), that is \(A = 0\). \(\square\)

**Definition 4.6.** Given a *-semisimple CQ*-algebra \((A, *, RA, b)\) we define
\[
\|X\|_\alpha^2 \equiv \sup_{\Omega \in \mathcal{S}(A)} \Omega(X, X).
\]
and
\[
\|X\|_\alpha = \max \{\|X\|, \|X^*\|\}.
\]
We say that the CQ*-algebra \((A, *, RA, b)\) is regular if \(\|X\| = \|X\|_\alpha \forall X \in \mathcal{A}\).

The Gelfand seminorm (12) can be compared with the other norms, \(\|\|\) and \(\|\|_0\) which enter in our structure. Actually, it is easy to prove the following

**Corollary 4.7.** Let \((A, *, RA, b)\) be a regular CQ*-algebra. Then \(\|A\| \leq \|A\|_* \leq \|A\|_0, \forall A \in A_0\).

Due to the definition of \(\mathcal{S}(A)\), the inequality \(\|X\|_\alpha \leq \|X\|, \forall X \in \mathcal{A}\) holds. The regular CQ*-algebras also satisfy the converse inequality. Moreover, it is easily seen that if \(\mathcal{A}\) is regular then we also have \(\|A\| = \|A\|_\alpha, \forall A \in \mathcal{A}\).

Let us now give some examples of *-semisimple CQ*-algebras.

**Example 4.8.** We start considering an abelian example, that is \((L^p(X, \mu), C(X))\), where \((X, \mu)\) is a measure space with \(X\) a compact Hausdorff space and \(p \geq 2\). We know from [4, 5] that \((L^p(X, \mu), C(X))\) is an abelian proper CQ*-algebra with \(b = \sharp = *\). We will first show that for
all \( f \in L^p(X, d\mu) \) there exists a sesquilinear form \( \Omega_f \in \mathcal{S}(L^p(X)) \) such that \( \Omega_f(f, f) = \|f\|_p^2 \). In the following we fix for simplicity \( X = [0, 1] \).

Given \( f \in L^p(X, d\mu) \), we define for all \( \psi, \phi \in L^p(X, d\mu) \) a sesquilinear form \( \Omega_f \) as

\[
\Omega_f(\psi, \phi) \equiv \|f\|_p^{2-p} \int_0^1 \psi(x) \overline{\phi(x)} |f(x)|^{p-2} \, dx
\]

It is easy to verify the first two conditions of Definition 3.10. The last condition requires twice the use of the Holder inequality

\[
|\Omega_f(\psi, \phi)| \leq \|f\|_p^{2-p} \|\psi\|_p \|\phi\|_p \|f\|_p \leq \|\psi\|_p \|\phi\|_p.
\]

Therefore \( \Omega_f \in \mathcal{S}(\mathcal{A}) \). Moreover, from the definition itself, \( \Omega_f(f, f) = \|f\|_p^2 \).

This fact immediately implies the *-semisemplicity of \( (L^p(X, d\mu), C(X, d\mu)) \).

As for the regularity, we already know that \( \|f\|_\alpha \leq \|f\|_p \forall f \in L^p(X, d\mu) \). To prove the regularity of the CQ*-algebra we have to prove the converse inequality. This easily follows from the above property; in fact we have \( \|f\|_\alpha = \sup_{\Omega \in \mathcal{S}(\mathcal{A})} \Omega(f, f) \geq \Omega_f(f, f) = \|f\|_p \).

For \( p < 2 \), the *-semisemplicity fails. More details can be found in [5].

**Example 4.9.** The second example is again proper but not abelian.

Let \( (\mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_\bar{\lambda}), *, \mathcal{B}(\mathcal{H}_\lambda), \overline{\circ}) \) be the CQ*-algebra considered in [4, Example 3.3]. Let \( \mathcal{B}^+(\mathcal{H}_\lambda) = \mathcal{B}(\mathcal{H}_\lambda) \cap \mathcal{B}(\mathcal{H}_\bar{\lambda}) \) and set

\[
\mathcal{B}^+_\gamma(\mathcal{H}_\lambda) = \{ X \in \mathcal{B}^+(\mathcal{H}_\lambda) : \lambda(X\{,\}) = \lambda(\{,X^*\}) \ \forall \{,\} \in \mathcal{H}_\lambda \}
\]

i.e., the set of elements of \( \mathcal{B}^+(\mathcal{H}_\lambda) \) commuting with \( \lambda \). It is readily checked that

\[
\mathcal{B}^+_\gamma(\mathcal{H}_\lambda) = \{ X \in \mathcal{B}^+(\mathcal{H}_\lambda) : X^* = X^\overline{\circ} \}
\]

It turns out that \( \mathcal{B}^+_\gamma(\mathcal{H}_\lambda) \) is a C*-algebra (with respect to the norm of \( \mathcal{B}(\mathcal{H}_\lambda) \)) and then the \( \|\|_{\lambda,\overline{\lambda}} \)-closure \( \mathcal{B}_\gamma(\mathcal{H}_\lambda, \mathcal{H}_\overline{\lambda}) \) of \( \mathcal{B}^+_\gamma(\mathcal{H}_\lambda) \) in \( \mathcal{B}(\mathcal{H}_\lambda, \mathcal{H}_\overline{\lambda}) \) is a proper CQ*-algebra with \( \overline{\circ} = * \) (Proposition 2.6).

Let us now prove that \( \mathcal{B}_\gamma(\mathcal{H}_\lambda, \mathcal{H}_\overline{\lambda}) \) is *-semisimple.

Let \( f \in \mathcal{H}_\lambda \) with \( \|f\|_\lambda = 1 \). For \( X, Y \in \mathcal{B}_\gamma(\mathcal{H}_\lambda, \mathcal{H}_\overline{\lambda}) \) set

\[
\Omega_f(X, Y) = \langle Xf, Yf \rangle_{\overline{\lambda}}.
\]
Then it is easy to check that \( \Omega_f \in \mathcal{S}(\mathcal{B}_\gamma(\mathcal{H}_\lambda, \mathcal{H}_\lambda)) \) (condition (ii) of Definition 3.10 follows from a simple limit argument). It is clear that if \( \Omega_f(X, X) = \|Xf\|_\lambda^2 = 0 \ \forall f \in \mathcal{H}_\lambda \), then \( X = 0 \). This proves our claim.

The next proposition shows that the *-semisimplicity has relevant consequences also for the algebraic structure of \( \mathcal{A} \).

**Proposition 4.10.** If a CQ*-algebra \( \mathcal{A}, Basis, RA, \mathord{,} \) is *-semisimple, then \( \mathcal{A} \) is fully-closable.

**Proof.** Let \( A \in \mathcal{A} \) and \( \{C_n\} \subset RA \) a sequence \( \|\| - \)-converging to zero and such that \( \|\| - \lim_{n \to \infty} AC_n = Y \). Then, if \( \Omega \in \mathcal{S}(\mathcal{A}) \) and \( B_1, B_2 \in RA \), we get

\[
|\Omega(YB_1, B_2)| \leq |\Omega((Y - AC_n)B_1, B_2)| + |\Omega(C_nB_1, A^*B_2)|
\]

\[
\leq \|Y - AC_n\|\|B_1\|\|B_2\| + \|C_n\|\|B_1\|\|A^*B_2\| \to 0
\]

Therefore \( \Omega(YB_1, B_2) = 0 \), \( \forall \Omega \in \mathcal{S}(\mathcal{A}), \forall B_1, B_2 \in RA \).

The *-semisimplicity of \( \mathcal{A} \) and Lemma 4.1, imply \( Y = 0 \). This proves the statement. \( \square \)

It is worth mentioning, at this point, that for *-semisimple abelian CQ*-algebras a generalization of the well-known Gel’fand theorem on the representation of an abelian C*-algebra as a C*-algebra of functions can be proved, supporting the idea that the notion of *-semisimplicity is the right one in order to get significant structure properties [5].

It is sometimes convenient to consider also a stronger notion (equivalent to *-semisimplicity for proper CQ*-algebras). Let \( \mathcal{S}_0(\mathcal{A}) \) denote the subset of \( \mathcal{S}(\mathcal{A}) \) consisting of those elements \( \Omega \) satisfying also the following additional condition:

\[
\Omega(X^*B, C) = \Omega(X^0B, C), \quad \forall X, B, C \in RA.
\]  

(15)

We define **strongly *-semisimple** a CQ*-algebra \( \mathcal{A}, Basis, RA, \mathord{,} \) such that the following condition holds:

If \( \Omega(X, X) = 0, \ \forall \Omega \in \mathcal{S}_0(\mathcal{A}) \) then \( X = 0 \).  

(16)

It turns out that any strongly *-semisimple CQ*-algebra \( \mathcal{A}, Basis, RA, \mathord{,} \) is proper and * = \( \mathord{,} \) (this follows immediately from the fact that the
strong *-radical can be characterized in analogy to \( \mathfrak{R}_2 \) in Lemma 4.1) and conversely.

Further, we mention the fact that sesquilinear forms of \( S(\mathcal{A}) \) satisfying (15) drastically simplify the GNS-construction discussed in [4]. Indeed, if \( (\mathcal{A}, *, RA, \check{\cdot}) \) is a CQ*-algebra and \( \Omega \in S_0(\mathcal{A}) \), then one also has:

\[
\Omega(B, B) = \Omega(B, B^{**}), \quad \forall B \in RA
\]

and this equality makes easier to check the conditions given in [4] for the general case. Let us now sketch the construction. Let \( \mathcal{K} = \{ A \in \mathcal{A} : \Omega(A, A) = 0 \} \). Let us consider the linear space \( \mathcal{A}/\mathcal{K} \); an element of this set will be denoted as \( \lambda_\Omega(A), A \in \mathcal{A} \). Clearly, \( \mathcal{A}/\mathcal{K} = \lambda_\Omega(\mathcal{A}) \) is a pre-Hilbert space with respect to the scalar product \( \langle \lambda_\Omega(A), \lambda_\Omega(B) \rangle = \Omega(A, B), A, B \in \mathcal{A} \). We denote by \( \mathcal{H}_\Omega \) the Hilbert space obtained by the completion of \( \lambda_\Omega(\mathcal{A}) \). Then \( \Omega \) is invariant in the sense of [10]. This means, in this case, that \( \Omega \) satisfies condition (ii) of Definition 3.10 and that \( \lambda_\Omega(RA) \) is dense in \( \mathcal{H}_\Omega \). Indeed, let \( \lambda_\Omega(A) \in \lambda_\Omega(\mathcal{A}) \) and let \( \{ A_n \} \) be a sequence in \( RA \) converging to \( A \) in the norm of \( \mathcal{A} \). Then from the inequality

\[
\Omega(A - A_n, A - A_n) \leq \| A - A_n \|^2
\]

it follows that \( \lambda_\Omega(A_n) \to \lambda_\Omega(A) \) in \( \mathcal{H}_\Omega \).

If we put

\[
\pi_\Omega(A) = \lambda_\Omega(A)B = \lambda_\Omega(AB) \quad B \in RA,
\]

then \( \pi_\Omega(A) \) is a well-defined closable operator with domain \( \lambda_\Omega(RA) \) in \( \mathcal{H}_\Omega \). More precisely it is an element of the partial O*-algebra \( \mathcal{L}^+(\lambda_\Omega(RA), \mathcal{H}_\Omega) \) [9, 10]. The map \( A \mapsto \pi_\Omega(A) \) is a *-representation of partial *-algebras in the sense of [10]. We define now the following set:

\[
\mathcal{D}_\Omega = \left\{ A \in \mathcal{A} : \sup_{B \in RA, \Omega(B,B) \neq 0} \frac{\Omega(AB, AB)}{\Omega(B, B)} < \infty \right\}
\]

then

(i) \( \mathcal{D}_\Omega \) is a linear space;

(ii) \( \mathcal{D}_\Omega \supset RA \);

(iii) if \( A \in \mathcal{D}_\Omega \) and \( B \in RA \), then \( AB \in \mathcal{D}_\Omega \)
If \( D_\Omega = A \) then \( \Omega \) is *admissible* in the sense of [4].

From the definition itself, it follows easily that \( \pi_\Omega(D_\Omega) \subseteq B(\mathcal{H}_\Omega) \), i.e. each element of \( D_\Omega \) is represented by a bounded operator in Hilbert space.

5. Norms on a *-semisimple CQ*-algebra

As shown in [4], the topological structure of a CQ*-algebra \((A, *, RA, \flat)\) is described in terms of four, generally different, norms: \( \|\|_0, \|\|_\flat, \|\|_\sharp \).

In this Section we introduce more inequivalent norms which are very useful to investigate some structure properties of the *-semisimple CQ*-algebras. We already defined the norm \( \|\|_\alpha \) in (13). We now define

\[
\|X\|_\beta \equiv \sup \{ |\Omega(XB, B)|; \Omega \in \mathcal{S}(A), B \in RA, \|B\|_\flat \leq 1 \}
\]

(17)

Lemma 4.1 and the fact that \( A \) is *-semisimple ensure that \( \|\|_\beta \) is really a norm. Moreover, making use of the polarization identity, it is possible to prove that it generates the same topology as the one defined by the following norm

\[
\|X\|_{\beta'} \equiv \sup \{ |\Omega(XB_1, B_2)|; \Omega \in \mathcal{S}(A), \|B_1\|_\flat, \|B_2\|_\flat \leq 1 \}.
\]

Here \( B_1 \) and \( B_2 \) are both taken in \( RA \).

Let us remind that if \( \Omega \) belongs to \( \mathcal{S}(A) \) and \( B \in RA \), with \( \|B\|_\flat \leq 1 \), then the form \( \Omega_B \) still belongs to \( \mathcal{S}(A) \). The following equivalent definition can therefore be given:

\[
\|X\|_\alpha^2 = \sup_{\Omega \in \mathcal{S}(A), \|B\|_\flat \leq 1} \Omega(XB, XB) \tag{18}
\]

and

\[
\|X\|_\beta \equiv \sup_{\Omega \in \mathcal{S}(A)} |\Omega(X, I)| \tag{19}
\]

Furthermore, it is easy to prove that \( \|X\|_\beta = \|X^*\|_\beta \) and \( \|X\|_{\alpha} = \|X^*\|_{(\alpha)} \) \( \forall X \in A \). Moreover we can prove that

\[
\|X\|_\beta \leq \|X\|_{\alpha} \forall X \in A.
\]
A short remark is in order: the above inequality holds if $\|I\| = 1$. This does not hold in general, as it is easy to see looking at the abelian CQ*-algebra $(L^p(X, d\mu), C(X))$, with $1 < \mu(X) < \infty$. However, in this simple case, we can consider the equivalent (in the sense of the norm) CQ*-algebra $(L^p(X, d\nu), C(X))$, where $\nu(E) = \frac{\mu(E)}{\mu(X)}$, $\forall E \subset X$. It is easy to see that in this space now $\|I\| = 1$. We will always assume that $\|I\| = 1$, even for non abelian CQ*-algebras.

With this in mind, it is easy to prove the following inequality:

$$\|X\|_\beta \leq \|X\|_{(\alpha)} \leq \|X\|, \forall X \in A$$

where the last inequality follows from the definition of the $\| \|_{(\alpha)}$ and from the property (iii) of the set $S(A)$.

Two interesting properties of the norm (17) are given by the following

**Proposition 5.1.** Let $(A, *, RA, b)$ be a $*$-semisimple CQ*-algebra. If $X \circ Y$ is well-defined for a certain pair $X, Y \in A$, then

$$\|X \circ Y\|_\beta \leq \|X^*\|_{(\alpha)} \|Y\|_{(\alpha)} \leq \|X\|_{(\alpha)} \|Y\|_{(\alpha)} \leq \|X\| \|Y\|$$

Moreover, if $X^* \circ X$ is defined for a certain $X \in A$ then

$$\|X\|_{(\alpha)}^2 = \|X^* \circ X\|_\beta \leq \|X^* \circ X\|_{(\alpha)}$$

The inequality (21) easily follows from the definition of the weak multiplication, the Schwarz inequality for the positive sesquilinear forms and from the definition of the $\alpha$-norm. The second statement is again a consequence of the above ingredients and of (21).

The next proposition shows that if the extreme norms coincide then the structure is, say, trivialized.

**Proposition 5.2.** Given a $*$-semisimple CQ*-algebra $(A, *, RA, b)$. If $\|X\|_\beta = \|X\|$, $\forall X \in A$, then $A$ is a C*-algebra with respect to the multiplication $\circ$.

**Proof.** We start proving that in the above hypothesis the set $R_w(X)$ is $\| \|_ -$closed $\forall X \in A$. Let $\{Y_n\} \subseteq R_w(X)$, with $Y_n$ converging in $\| \|$ to a certain $Y$. To prove that $Y \in R_w(X)$ we start observing that, if
\(\Omega \in \mathcal{S}(A), \ B_1, B_2 \in RA\) then \(\Omega(YB_1, X^*B_2) = \| - \lim_{n \to \infty} \Omega((X \circ Y_n)B_1, B_2)\)

Moreover, if \(\|X\|_\beta = \|X\|\), using Proposition 5.1 we deduce that

\[
\|X \circ (Y_n - Y_m)\| = \|X \circ (Y_n - Y_m)\|_\beta \leq \|X^*\|_\alpha \|(Y_n - Y_m)\|_\alpha = \|X\|\|(Y_n - Y_m)\| \to 0.
\]

This implies that \(\{X \circ Y_n\}\) is a \(\|\|\)-Cauchy sequence in \(A\). Therefore there exists a \(Z \in A\) which is the \(\|\|\) limit of this sequence.

From the previous step we conclude that

\[
\Omega(YB_1, X^*B_2) = \Omega(ZB_1, B_2) \quad \forall \Omega \in \mathcal{S}(A), \ \forall B_1, B_2 \in RA.
\]

This implies that \(Y \in R_w(X)\), so that the set \(R_w(X)\) is closed.

We now notice that \(R_w(A) \equiv \cap_{X \in A} R_w(X)\) is \(\|\|\)-closed in \(A\); but it is also dense since \(RA \subset R_w(A)\). Therefore \(R_w(A) \equiv A\). We conclude therefore that \(A\) is an algebra. From (21), which in the hypothesis of the Proposition reads \(\|X \circ Y\| \leq \|X\|\|Y\|\), we know that \(A\) is a Banach algebra. Furthermore here (22), \(\|X^* \circ X\| = \|X\|^2\), gives the \(C^*\)-property for the elements of \(A\).

\[\square\]

Remark 5.3. Of course, by (20), if \(\|X\|_\beta = \|X\|\) then also \(\|X\|_\alpha = \|X\|\).

We will now define two more norms, \(\|X\|_R\) and \(\|X\|_L\) and two subsets of \(A\), \(AR\) and \(AL\) where they are respectively finite. As we will see in Section 6, they will play a relevant role in the study of some spectral properties of a CQ*-algebra.

For \(A \in A\), we put

\[
\| A \|^2_R \equiv \sup \left\{ \frac{\Omega(AB, AB)}{\Omega(B, B)} ; \ \Omega \in \mathcal{S}(A), B \in RA, \ \Omega(B, B) \neq 0 \right\} \quad (23)
\]

\[
\| A \|_L \equiv \| A^* \|_R \quad (24)
\]

These norms are not necessarily finite for arbitrary \(A \in A\). We introduce therefore the following non empty subsets of \(A\):
\[ A_R \equiv \{ A \in A : \| A \|_R < \infty \} \]

and

\[ A_L \equiv \{ A \in A : \| A \|_L < \infty \}. \]

It is easy to prove that \( \| \|_R \) and \( \| \|_L \) are really norms and also that they can be expressed in equivalent forms

\[ \| A \|_R^2 = \sup_{\Omega \in S(A)} \frac{\Omega(A, A)}{\Omega(I, I)}, \quad (25) \]

\[ \| A \|_L^2 \equiv \sup_{\Omega \in S(A), B \in L_A, \Omega^*(B, B) \neq 0} \frac{\Omega^*(BA, BA)}{\Omega^*(B, B)} = \sup_{\Omega \in S(A)} \frac{\Omega^*(A, A)}{\Omega^*(I, I)}, \quad (26) \]

The above norms satisfy the following inequalities:

\[ \| A \|_R \geq \| A \|_\alpha \quad \forall A \in A_R \quad (27) \]

and

\[ \| A \|_L \geq \| A^* \|_\alpha \quad \forall A \in A_L, \quad (28) \]

which are used to prove the following

**Proposition 5.4.** Both \( A_R \) and \( A_L \) are linear normed spaces containing \( A_0 \) as a subspace. Moreover \( A \in A_R \) if, and only if, \( A^* \in A_L \). Finally, if \( A \) is regular, then both \( A_R \) and \( A_L \) are Banach spaces.

**Proof.** We start proving that \( A_0 \) belongs to both \( A_R \) and \( A_L \). This follows from the fact that the weak product of \( X \in A_0 \) with \( X^* \) exists (together with all its powers) and coincides with the usual product. Therefore the following estimate, obtained using \( k \) times the Schwarz inequality and property (iii) of the set \( S(A) \), holds true:

\[ \Omega(X, X) \leq \| (X^*X)^{2^{k-1}} \|^{1/2^{k-1}} \Omega(I, I)^{(\frac{1}{k^p} + \frac{1}{k^{p+1}} + \cdots + \frac{1}{k^q})} \]

Taking the limit \( k \to \infty \) and recalling that \( \| B \| \leq \| B \|_0 \), see [4], we get \( \Omega(X, X) \leq \Omega(I, I) \| X^*X \|_k \) and therefore

\[ \| X \|_R \leq \| X \|_0 < \infty \quad (29) \]
This inequality shows that if $X$ belongs to $A_0$ then $X \in A_R$. Being $A_0$ closed with respect to the involution $\ast$, $X$ also belongs to $A_L$.

In order to prove that, if $\mathcal{A}$ is regular, $A_R$ and $A_L$ are Banach spaces, we only have to show the completeness of, say, $A_R$. Let us consider a sequence $\{A_n\} \subset A_R$ which is $|||_R -$Cauchy. We need to verify that it is also $|||_R -$converging to an element $B \in A_R$.

Inequality (27) for regular algebras becomes $|| X ||_R \geq || X || \forall X \in A_R$. Therefore, if $\{A_n\}$ is $|||_R -$Cauchy it is also $||| -$Cauchy. Using the $||| -$completeness of $\mathcal{A}$ we conclude that there exists an element $B \in \mathcal{A}$ which is the $||| -$limit of $A_n$.

To prove that $B$ belongs to $A_R$ we observe that

$$|| B ||^2_R = \sup_{\Omega \in \mathcal{S}(\mathcal{A})} \frac{\Omega(B,B)}{\Omega(\mathbb{1},\mathbb{1})} = \sup_{\Omega \in \mathcal{S}(\mathcal{A})} \lim_{n \to \infty} \frac{\Omega(A_n,A_n)}{\Omega(\mathbb{1},\mathbb{1})} = \lim_{n \to \infty} \sup_{\Omega \in \mathcal{S}(\mathcal{A})} \frac{\Omega(A_n,A_n)}{\Omega(\mathbb{1},\mathbb{1})}$$

so that $|| B ||_R = \lim_{n \to \infty} || A_n ||_R$. This limit is finite since, being $\{A_n\} |||_R -$Cauchy, then the sequence $\{|| A_n ||_R\}$ is convergent. It is worthwhile to observe that the interchange of lim and sup above is possible due to the uniformity of $\Omega(A_n,A_n)/\Omega(\mathbb{1},\mathbb{1})$ in $n$.

Finally, using the uniqueness of the limit, we also prove that $B =|||_R -$lim$_{n \to \infty} A_n$.

A completely analogous proof can be set on to prove completeness of $A_L$.

□

In principle, we are not sure that $A_0$ is really a proper subset of, say, $A_R$. The following proposition, however, implies the proper nature of this inclusion.

We need first to introduce the notion of weak length of an element, see [10, 21]:

**Definition 5.5.** We say that $X \in \mathcal{A}$ has weak length $N$ if all the weak product $X^{(k)} \circ X^{(l)}$, with $k + l = n$, $n = 1, 2, \ldots N$ exist and coincide for fixed $n$. In this case we write $l_w(X) = N$.

**Proposition 5.6.** Let $(\mathcal{A}, \ast, RA, \triangleright)$ be a $*$-semisimple $\text{CQ}^*$-algebra and $X \in \mathcal{A}$ be such that:

(i) $X^* \circ X$ is well-defined;
(ii) \( l_w(X^* \circ X) = \infty; \)
(iii) \( \liminf_{k \to \infty} \| (X^* \circ X)^{2^k} \|^{1/2^k} < \infty. \)

Then \( X \in \mathcal{A}_R. \) In particular, if \( X = X^* \) then \( X \in \mathcal{A}_R \cap \mathcal{A}_L. \)

Remark 5.7. Obviously condition (ii) is satisfied if there exists a positive constant \( M \) such that \( \| (X^* \circ X)^{2^k} \| < M^{2^k}, \forall k \in \mathbb{N}. \) It is also worth remarking that if \( X^* \) satisfies the assumptions of Proposition 5.6 then \( X \in \mathcal{A}_L \) and this implies the second part of the statement. Furthermore, for a self-adjoint \( X \) the conditions (i) and (ii) can be replaced with the unique requirement \( l_w(X) = \infty. \)

Instead of giving the proof, which is very similar to the one of Proposition 5.4, we observe that any element of \( \mathcal{A}_0 \) satisfies the conditions of the above Proposition, but these are not the only ones. In \( (L^p([0,1], d\mu), C[0,1]) \) any step function \( s(x) \) defined on \([0,1]\) is in \( L^p([0,1], d\mu) \) but not in \( C([0,1]) \). It is immediate to verify that \( s(x) \) satisfies the above hypothesis. Therefore, in general, \( \mathcal{A}_0 \) is properly included in \( \mathcal{A}_R \) and \( \mathcal{A}_L. \)

The set \( \mathcal{A}_L \) contains, as we shall see in a while, the Banach algebra \( \mathcal{A}_\lambda \) of the \( \lambda \)-bounded elements:

**Definition 5.8.** Let \((\mathcal{A}, *, RA, b)\) be a \( \ast \)-semisimple \( CQ \ast \)-algebra and \( X \in \mathcal{A}. \) We say that \( X \) is \( \lambda \)-bounded if \( D(L_X) = \mathcal{A}. \)

The terminology is motivated by the fact that, in this case, the map
\[
A \in \mathcal{A} \mapsto L_X(A) = X \cdot A \in \mathcal{A}
\]
is an everywhere defined and closed, therefore bounded, linear map of \( \mathcal{A} \) into itself; therefore, there exists \( C > 0 \) such that
\[
\| X \cdot A \| \leq C \| A \|.
\]
We put
\[
\| X \|_\sharp = \sup_{\| A \| \leq 1} \| X \cdot A \|. \tag{30}
\]
The fact that \( \| X \|_\sharp = \| X \|_z \) for \( X \in LA \) motivates the notation we used. Of course, we can consider as well the set \( \mathcal{A}_\rho \) of all \( \rho \)-bounded elements which are defined analogously to \( \lambda \)-bounded elements. All the
Proposition 5.9. The set $\mathcal{A}_\lambda$ of all the $\lambda$-bounded elements is a Banach algebra with respect to the strong multiplication $\cdot$ and the norm $\|\|_\sharp$. If $X \in \mathcal{A}_\lambda$ then $X^* \in \mathcal{A}_\rho$. Moreover $L\mathcal{A} \subseteq \mathcal{A}_\lambda \subseteq \mathcal{A}_L$.

Proof. It is easy to show that if $X, Y \in \mathcal{A}_\lambda$ and $\mu \in \mathbb{C}$ then $X + Y$, $\mu X$ and $X \cdot Y$ all belong to $\mathcal{A}_\lambda$. Then $\mathcal{A}_\lambda$ is an algebra, since it is isomorphic to a subalgebra of the algebra $\mathcal{L}(\mathcal{A})$ of bounded operators in the Banach space $\mathcal{A}$. We will show now that $\mathcal{A}_\lambda$ is, in fact, isomorphic to a closed subalgebra of $\mathcal{L}(\mathcal{A})$. First notice that for $X \in \mathcal{A}_\lambda$, $\|X\|_\sharp$ coincides with the norm of $\overline{L}_X$ as a bounded operator in $\mathcal{A}$. Let now $\{X_n\}$ be a sequence in $\mathcal{A}_\lambda$ such that $\overline{L}_{X_n}$ converges to $L \in \mathcal{L}(\mathcal{A})$ with respect to the natural norm of $\mathcal{L}(\mathcal{A})$.

Since

$$\| (X_n - X_m) \cdot A \| \leq \| X_n - X_m \|_\sharp \| A \|,$$

taking $A = I$, there exists $X \in \mathcal{A}$ such that $\|X_n - X\| \to 0$ and for each $A \in \mathcal{A}$, there exists $Y_A \in \mathcal{A}$ such that $\|X_n \cdot A - Y_A\| \to 0$. By Proposition 4.10 the right multiplication by $A$ is a closed linear map in $\mathcal{A}$, then it follows that $X \in D(\overline{R}_A)$ and $X_n \cdot A \to X \cdot A = \overline{L}_X(A)$ in the norm of $\mathcal{A}$. This implies that $\overline{L}_X = L$ and so $X$ is $\lambda$-bounded.

If $X \in \mathcal{A}_\lambda$, it is immediate to prove that the map $A \in \mathcal{A} \mapsto A \cdot X^* \in \mathcal{A}$ is everywhere defined.

The inclusion $L\mathcal{A} \subseteq \mathcal{A}_\lambda$ is obvious, whereas the second inclusion can be deduced from Proposition 5.6, taking into account the above identification of $\mathcal{A}_\lambda$ (and of $\mathcal{A}_\rho$) with a subalgebra of $\mathcal{L}(\mathcal{A})$ and the inequality (30).

The $\lambda$-bounded elements will be useful in deriving some aspects of the functional calculus in a $*$-semisimple CQ*-algebra. This will be discussed to some extent in the next Section.

6. Basics for a functional calculus

Proposition 5.6 suggests that the norms $\|\|_R$ and $\|\|_L$ should play a role similar to that of the spectral radius in the theory of C*-algebras.
In this Section we will deepen this question which is apparently closely linked to the possibility of generalizing to CQ*-algebras some aspects of the functional calculus for C*-algebras.

First we need to introduce the notion of inverse of an element of a CQ*-algebra.

The main problem which arises when one tries to define the inverse of an element in a partial *-algebra consists in its non-uniqueness. This fact depends on the possible lack of associativity in a partial *-algebra.

However, Proposition 3.21 provides a possible way to overcome the problem.

**Definition 6.1.** An element $X \in \mathcal{A}$ has a (strong-) inverse in $\mathcal{A}$ if there exists $X^{-1} \in R_s(X) \cap L_s(X)$ such that

$$X \cdot X^{-1} = X^{-1} \cdot X = \mathbb{1}$$

Due to Proposition 3.21, the inverse, when it exists, is unique.

**Remark 6.2.** In spite of the fact that the weak-multiplication makes of any CQ*-algebra, a partial *-algebra, the inverse will be always considered in the above strong sense because of its uniqueness. For this reason, in what follows, we will systematically omit (as far as no ambiguity arises) the adjective strong speaking of the inverse.

It is worth mentioning that for some elements of $\mathcal{A}$ the inverse may exists in an even stronger sense: this is defined coming back to the original lattice of multipliers $\{\mathcal{A}, RA, LA, A_0\}$. We do not enter in the details because this definition is of little use.

We list in the next proposition, without proving them, some elementary properties of the inverse.

**Proposition 6.3.** Let $X, Y$ be invertible elements of $\mathcal{A}$. Then

(i) $(X^{-1})^{-1} = X$;

(ii) If $X \in L_w(Y)$ and $Y^{-1} \in L_w(X^{-1})$ then $(XY)^{-1}$ exists and $(XY)^{-1} = Y^{-1} \circ X^{-1}$;

(iii) $(X^*)^{-1} = (X^{-1})^*$
Definition 6.4. Let $X \in \mathcal{A}$. The domain of regularity $\Delta(X)$ of $X$ is the following subset of $\mathbb{C}$

$$\Delta(X) = \left\{ z \in \mathbb{C} : (X - z)^{-1} \text{ exists in } \mathcal{A} \right\}$$

The resolvent $\rho(X)$ of $X$ is the largest open subset of $\Delta(X)$ where the function $z \to f(z) = (X - z)^{-1}$ is analytic with respect to the norm of $\mathcal{A}$.

The set $\sigma(X) = \mathbb{C} \setminus \rho(X)$ is called the spectrum of $X$.

In general, $\rho(X) \subset \Delta(X)$, in contrast with the Banach algebra case.

Example 6.5. In $L^2(0, 1)$, let us consider the function $u(x) = x^{\frac{1}{4}}$ which is continuous in $[0, 1]$. It is readily seen that the spectrum of $u$ in the $C^*$-algebra $C[0, 1]$ is exactly the closed interval $[0, 1]$. Since $u^{-1}(x) = x^{-\frac{4}{3}}$ is in $L^2(0, 1)$, then $0 \in \Delta(u)$. Nevertheless, $0 \notin \rho(u)$. Indeed, setting $f(z) = (u - z)^{-1}$, we have $f'(0) = x^{-\frac{3}{2}} \notin L^2(0, 1)$. In conclusion, $ho(u) = \Delta(u) \setminus \{0\}$.

As is clear, the function $f(z) = (X - z)^{-1}$ has the power series expansion $f(z) = \sum_{n=0}^{\infty} T_n(z - z')^n$ throughout the largest open disk with center $z'$ contained in $\rho(X)$, for each $z' \in \rho(X)$. The coefficients $T_n$ (which belong to $\mathcal{A}$) are given by

$$T_n = \frac{1}{2\pi i} \int_C \frac{f(w)dw}{(w - z')^{n+1}} = \frac{f^{(n)}(z')}{n!} \quad (31)$$

for each closed curve $C$ surrounding $z'$, with $C \subset \rho(X)$ (the integral is, clearly, understood to converge with respect to the norm of $\mathcal{A}$).

Lemma 6.6. Let $z, z' \in \Delta(X)$; then $(X - z)^{-1} \circ (X - z')^{-1}$ is well defined and

$$(X - z)^{-1} - (X - z')^{-1} = (z - z')(X - z)^{-1} \circ (X - z')^{-1} \quad (32)$$

and therefore,

$$(X - z)^{-1} \circ (X - z')^{-1} = (X - z')^{-1} \circ (X - z)^{-1}, \quad \forall z, z' \in \Delta(X)$$
Proof. The strong product \((X - z) \cdot (X - z')^{-1}\) is, as is easily seen, well defined and one has

\[(X - z) \cdot (X - z')^{-1} = ((X - z') - (z - z')) \cdot (X - z')^{-1} = \mathbb{I} - (x - f') (X - f')^{-1}\]

where we made use of the distributivity of the weak multiplication.

Now, using Proposition 3.21 we get

\[(X - z)^{-1} \circ ((X - z) \cdot (X - z')^{-1}) = (X - z')^{-1}\]

and thus

\[(X - z')^{-1} = (X - z)^{-1} \circ (\mathbb{I} - (x - f') (X - f')^{-1})\]

So if \(\Omega \in \mathcal{S}(\mathcal{A}), \mathcal{B}_\infty, \mathcal{B}_\in \in \mathcal{R}(\mathcal{A})\) we have

\[\Omega((X - z')^{-1} B_1, B_2) = \Omega((\mathbb{I} - (x - f') (X - f')^{-1}) \mathcal{B}_\infty, (X^* - x)^{-1} \mathcal{B}_\in)\]

\[= \Omega(B_1, (X^* - x)^{-1} B_2) - (z - z') \Omega((X - z')^{-1} B_1, (X^* - x)^{-1} B_2)\]

This implies that \((X - z)^{-1} \circ (X - z')^{-1}\) is well defined and

\[(X - z)^{-1} - (X - z')^{-1} = (z - z')(X - z)^{-1} \circ (X - z')^{-1}\]

\(\square\)

The first statement of the next Proposition is concerned with a very elementary aspect of the functional calculus. However in the framework of partial *-algebras it needs a non-trivial proof which puts in evidence how far partial *-algebras are from the ordinary *-algebras.

**Proposition 6.7.** Let \(X \in \mathcal{A}\) with \(\rho(X) \neq \emptyset\). Then the following statements hold:

(i) If \(z \in \rho(X)\), all weak powers \((X - z)^{-n}\) exist in \(\mathcal{A}\) and, setting \(f(z) = (X - z)^{-1}\), one has:

\[f^{(n)}(z) = n!(X - z)^{-(n+1)}\]  \(\quad (33)\)

(ii) If \(X = X^*\) and \(\sigma(X) \neq \emptyset\), then \((X - z)^{-1} \in \mathcal{A}_R \cap \mathcal{A}_L\), \(\forall z \in \rho(X) \cap \mathbb{R}\).

**Proof.** (i) We proceed by induction on \(n\). Let \(n = 1\). The function \(f(z)\) is, clearly \(||||\)-continuous; so, if \(\Omega \in \mathcal{S}(\mathcal{A}); \mathcal{B}_\infty, \mathcal{B}_\in \in \mathcal{R}(\mathcal{A})\) making
use of Lemma 6.6, we have

\[
\lim_{z' \to z} \Omega \left( \frac{f(z') - f(z)}{z' - z}B_1, B_2 \right) = \lim_{z' \to z} \Omega \left( (X - z')^{-1}B_1, (X^* - z)^{-1}B_2 \right)
\]

But

\[
|\Omega \left( ((X - z')^{-1} - (X - z)^{-1})B_1, (X^* - z)^{-1}B_2 \right)| \\
\leq \| (X - z')^{-1} - (X - z)^{-1} \| \| B_1 \| \| (X^* - z)^{-1} \| \| B_2 \|. 
\]

And therefore \( \Omega(f'(z)B_1, B_2) = \Omega((X - z)^{-1}B_1, (X^* - z)^{-1}B_2) \).

Let \( n \in \mathbb{N} \) and assume that \( \forall r \leq n \) \((X - z)^{-r}\) exists and that

\[
f^{(r-1)}(z) = (r - 1)! (X - z)^{-r}
\]

Then, for any \( k \geq 1 \) we have

\[
\Omega(f^{(n)}(z)B_1, B_2) = \lim_{z' \to z} \Omega \left( \frac{f^{(n-1)}(z') - f^{(n-1)}(z)}{z' - z}B_1, B_2 \right)
\]

\[
= (n - 1)! \lim_{z' \to z} \Omega \left( \frac{(X - z')^{-n} - (X - z)^{-n}}{z' - z}B_1, B_2 \right)
\]

\[
= (n - 1)! \lim_{z' \to z} \Omega \left( (X - z')^{-(n-k)}B_1, \frac{(X^* - z')^{-k} - (X^* - z)^{-k}}{z' - z}B_2 \right)
\]

\[
+ (n - 1)! \lim_{z' \to z} \Omega \left( (X - z')^{-(n-k)} - (X - z)^{-(n-k)}B_1, (X^* - z)^{-k}B_2 \right)
\]

\[
= (n - 1)! k \Omega \left( (X - z)^{-(n-k)}B_1, (X^* - z)^{-(k+1)}B_2 \right)
\]

\[
+ (n - 1)! (n - k) \Omega \left( (X - z)^{-(n-k+1)}B_1, (X^* - z)^{-k}B_2 \right)
\]

This implies that \((X - z)^{-k} \circ (X - z)^{-(n-k+1)}\) is well-defined if, and only if, \((X - z)^{-(k+1)} \circ (X - z)^{-(n-k)}\) is well-defined. We will now show that if \((X - z)^{-n}\) is well-defined then \((X - z)^{-k} \circ (X - z)^{-l}\) exists for any \( k, l \) such that \( k + l = n + 1 \), hence the weak-power \((X - z)^{-(n+1)}\) is well-defined. Indeed, by the hypothesis of induction, we get

\[
(X - z)^{-k} = \frac{f^{(k-1)}(z)}{(k-1)!} = \frac{1}{2\pi i} \int_{|\xi - z| = r_1} \frac{f(\xi)}{(\xi - z)^k} d\xi, \quad 1 \leq k \leq n - 1
\]
and thus, applying the calculus of residues, we obtain for $\Omega \in S(A)$, $B_1, B_2 \in RA$

$$
\Omega \left( \frac{1}{2\pi i} \int_{|\eta-z|=r_1} \frac{f(\eta) \, d\eta}{(\eta-z)^l} B_1, - \frac{1}{2\pi i} \int_{|\xi-z|=r_1} \frac{f^*(\xi) \, d\xi}{(\xi-z)^k} B_2 \right) =
$$

$$
= - \frac{1}{4\pi^2} \Omega \left( \int_{|\eta-z|=r_1} \int_{|\eta-z|=r_2} \frac{f(\xi) \circ f(\eta)}{(\xi-z)^k(\eta-z)^l} \, d\xi \, d\eta B_1, B_2 \right)
$$

$$
= - \frac{1}{4\pi^2} \Omega \left( \int_{|\eta-z|=r_1} \int_{|\eta-z|=r_2} \frac{f(\xi) - f(\eta)}{(\xi-\eta)(\xi-z)^k(\eta-z)^l} \, d\xi \, d\eta B_1, B_2 \right)
$$

$$
= \frac{1}{2\pi i} \Omega \left( \int_{|\xi-z|=r_1} \frac{f(\xi) \, d\xi}{(\xi-z)^{k+l}} B_1, B_2 \right) = \Omega \left( f^{(n)}(z) B_1, B_2 \right)
$$

where we also made use of Lemma 6.6. Therefore, since the right hand side exists due to the analiticity of $f(z)$, $(X - z)^{-k} \circ (X - z)^{-l}$ exists for any $k, l$ such that $k + l = n + 1$; from this it follows

$$
f^{(n)}(z) = (n-1)!k(X - z)^{-(k+1)} \circ (X - z)^{-(n-k)}
$$

$$
+ (n-1)!(n-k)(X - z)^{-k} \circ (X - z)^{-(n-k+1)}
$$

(34)

and then

$$
f^{(n)}(z) = n!(X - z)^{-(n+1)}
$$

(35)

(ii) Since $\sigma(X) \neq \emptyset$, for $z \in \rho(X)$ and $r \leq d(z, \partial\Omega) \neq 0$, we get the inequality

$$
\| (X - z)^{-k} \| = \left\| \frac{1}{2\pi i} \int_{|\xi-z|=r} \frac{f(\xi) \, d\xi}{(\xi-z)^k} \right\|
$$

$$
\leq \frac{1}{2\pi} \int_{|\xi-z|=r} \frac{\| (X - \xi)^{-1} \| \, |d\xi|}{|\xi-z|^k}
$$

$$
\leq \frac{1}{2\pi} \frac{1}{r^k} \int_{|\xi-z|=r} \| (X - \xi)^{-1} \| \, |d\xi|.
$$

If $z \in \mathbb{R}$ then Proposition 5.6 can be applied. \hfill \Box

Remark 6.8. Notice that the analiticity of $f(z) = (X - z)^{-1}$, $z \in \rho(X)$ implies the existence of $(X - z)^{-n}$, $\forall n \in \mathbb{N}$ but not the existence of $(X - z)^n$ for $n > 1$. As an example, let us consider the CQ*-algebra $(L^2(X, \mu), C(X))$ where $X = [0, 1]$ and $\mu$ is the Lebesgue measure on $X$. The function $v(x) = x^{-\frac{1}{2}}$ is in $L^2(X, \mu)$; obviously, $0 \in \Delta(v)$ since $v^{-1}(x) = x^{\frac{1}{2}} \in L^2(X, \mu)$ and an easy computation shows that
actually $0 \in \rho(v)$. We have $v^{-n}(x) = x^{\frac{n}{2}} \in L^2(X, \mu)$, $\forall n \in \mathbb{N}$, but $v^{2}(x) = x^{-\frac{1}{2}} \notin L^2(X, \mu)$.

We will now prove that if the absolute value of a complex number $z$ is bigger than both $\| X \|_R$ and $\| X \|_L$, for a certain fixed $X \in A_R \cap A_L$, then $z$ belongs to a the domain of regularity of $X$.

In what follows we will set $\| A \|_\alpha \equiv \| A^* \|_\alpha \forall A \in \mathcal{A}$.

**Lemma 6.9.** Let $X \in A_R$, $C \in D(\overline{L_X})$, and let $z \in \mathbb{C}$ such that $|z| > \| X \|_R$. Therefore

$$\| (X - z) \cdot C \|_\alpha \geq \| C \|_\alpha (|z| - \| X \|_R).$$

Analogously, if $X \in A_L$, $B \in D(\overline{R_X})$, and let $\mu \in \mathbb{C}$ such that $|\mu| > \| X \|_L$ then

$$\| B \cdot (X - \mu) \|_\pi \geq \| B \|_\pi (|\mu| - \| X \|_L).$$

**Proof.** We start by proving the first statement for an element $C \in RA$.

In our hypothesis, taking an $\Omega \in \mathcal{S}(\mathcal{A})$ we can easily prove the following inequality:

$$\Omega((X - z)C, (X - z)C) \geq \Omega(C, C)(|z| - \| X \|_R)^2$$

Taking the supremum over the family $\mathcal{S}(\mathcal{A})$ we get the statement for elements in $RA$. The general result is obtained with a simple limit argument, using the fact that $\| \|_\pi$–convergence implies $\| \|_\alpha$–convergence.

The left counterpart of the lemma is proved in the very same way, starting with the state $\Omega^*$, for a given $\Omega \in \mathcal{S}(\mathcal{A})$. $\Box$

**Lemma 6.10.** Let $(\mathcal{A}, *, RA, b)$ be a $^*$-semisimple CQ*-algebra and $X \in \mathcal{A}$.

i) If $\| X \|_R < \infty$ and if $|z| > \| X \|_R$ then the set

$$\text{Ran } L_{X - z} \equiv \{ (X - z)B : B \in RA \}$$

is $\| \|_\alpha$–dense in $\mathcal{A}$.

ii) If $\| X \|_L < \infty$ and if $|z| > \| X \|_L$ then the set

$$\text{Ran } R_{X - z} \equiv \{ B(X - z) : B \in LA \}$$
is \(\|\|_{\pi}\)-dense in \(A\).

**Proof.** Were it not so, then there would exist a non zero \(\|\|_{\alpha}\)-continuous functional \(F\) on \(A\) such that \(F((X - z)B) = 0\) \(\forall B \in RA\). Therefore we should have \(F(XB) = zF(B)\) \(\forall B \in RA\). From the \(\|\|_{\alpha}\)-continuity of \(F\) we get
\[
|F(XB)| \leq \|F\|_{\alpha}|XB|_{\alpha}.
\]
Using the definition of \(\|\|_R\) we deduce that \(|XB|_{\alpha} \leq \|X\|_R\|B\|_{\alpha}\), so that
\[
|F(XB)| \leq \|F\|_{\alpha}\|X\|_R\|B\|_{\alpha}.
\]
Defining \(F_X(B) \equiv F(XB), \forall B \in RA\) and computing \(\|F\|_0\), that is the norm of the functional restricted to \(RA\), we find the following contradictory inequality: \(|z| \leq \|X\|_R\). In finding this result one also has to use that \(F_X = zF\).

This proof can also be adapted with minor modifications to prove the left counterpart of the statement. \(\square\)

**Proposition 6.11.** Let \((A,*,RA,b)\) be a *-semisimple and regular CQ*-algebra. Let \(X \in A \cap \mathcal{A}_L\) and \(z \in \mathbb{C}\) such that \(|z| > \max\{\|X\|_R, \|X\|_L\}\). Then the inverse \((X - z)^{-1}\) exists. Moreover, \((X - z)^{-1} \in \mathcal{A}_\lambda \cap \mathcal{A}_\rho\) and
\[
\{ z \in \mathbb{C} : |z| > \max\{\|X\|_R, \|X\|_L\}\} \subseteq \rho(X).
\]

**Proof.** Using the previous Lemmas we can prove that the following sets
\[
\text{Ran}L_{X - z} \equiv \{(X - z) \bullet B : B \in D(L_X)\}
\]
and
\[
\text{Ran}R_{X - z} \equiv \{B \bullet (X - z) : B \in D(R_X)\}
\]
both coincide with the whole space \(A\). The reason is that \(\text{Ran}L_{X - z} \supseteq \text{Ran}L_{X - z}\) and \(\text{Ran}R_{X - z} \supseteq \text{Ran}R_{X - z}\), so they are both dense in \(A\) due to Proposition 6.10. Using Lemma 6.9 one can prove that both the sets are \(\|\|\)-closed, so that \(\text{Ran}L_{X - z} \equiv \text{Ran}R_{X - z} \equiv A\). In this last step the regularity of the algebra plays a crucial role.

Since our CQ*-algebra contains the unity \(I\) we deduce that there exist \(B_1 \in D(L_X)\) and \(B_2 \in D(R_X)\) such that \((X - z) \bullet B_1 = I\) and \(B_2 \bullet (X - z) = I\). In this way we have defined a left and a right inverse.
Due to Proposition 2.6, which ensures the associativity of the product in this situation, we have
\[ B_2 = B_2 \cdot I = B_\nu \cdot ((X - \nu) \cdot B_\nu) = (B_\nu \cdot (X - \nu)) \cdot B_\nu = I \cdot B_\nu = B_\nu. \]
Therefore the inverse of \( (X - z) \) exists.
The maps
\[ \Lambda_{(X-z)^{-1}} : A \in \mathcal{A} \mapsto (X - z)^{-1} \cdot A \in \mathcal{A} \]
and
\[ P_{(X-z)^{-1}} : A \in \mathcal{A} \mapsto A \cdot (X - z)^{-1} \in \mathcal{A} \]
are, therefore, everywhere defined and as is easily seen, closed in \( \mathcal{A} \). Hence \( (X - z)^{-1} \in \mathcal{A}_\lambda \cap \mathcal{A}_\rho \).
Let now \( z_0 \in \mathbb{C} \) satisfy \( |z_0| > \max\{\|X\|_R, \|X\|_L\} \) and \( z \in \mathbb{C} \) such that \( |z - z_0| \leq \| (X - z_0)^{-1} \|_2 \) then, by Proposition 5.9, the power (in strong sense) series
\[
(X - z_0)^{-1} \left\{ I + \sum_{k=1}^{\infty} (X - \nu)^k (X - \nu)^{-k} \right\}
\]
converges with respect to \( \| \|_2 \) to an element \( Y \) of \( \mathcal{A}_\lambda \). It is easily checked that \( Y = (X - z)^{-1} \). Hence the function \( f(z) = (X - z)^{-1} \) admits a Taylor expansion at \( z_0 \) and is, therefore, analytic. \( \square \)

There are yet several aspects of the functional calculus on a CQ*-algebra that should be investigated in details: first of all the spectral characterization of positive elements. We hope to discuss them in a further paper.

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