Applications of Topological *-Algebras of Unbounded Operators to Modified Quons

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Abstract

In this paper we discuss some applications of topological *-algebras of unbounded operators to what we call Modified Quons (MQ). In particular, the existence of the thermodynamical limit for some models of free and interacting mq is proved in the same framework proposed by the author in a recent paper for ordinary bosons.

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I Introduction

In a recent work, Ref. [1], the author has discussed some physical applications of topological quasi *-algebras, see Ref. [2], to free bosons and to bosons interacting with the matter. The need for using quasi *-algebras was related to the unbounded nature of the boson operators $a, a^\dagger$ and $N = a^\dagger a$. We have shown how to introduce a reasonable cut-off, related to the number of bosons, which regularizes all the above operators. The removal of this cut-off can be controlled and an algebraic dynamics can be rigorously defined.

In a series of papers, Mohapatra, Fivel and Greenberg, Refs. [3, 4, 5], introduced and discussed new excitations which, in a natural way, interpolate between bosons and fermions. These particles are known as quons. Mathematical aspects of quons have been considered by a number of authors, like Bozejko, Speicher and Dubin among the others, [6].

They are defined essentially by their q-mutator relation

$$aa^\dagger - qa^\dagger a = 1, \quad q \in [-1, 1],$$

(1.1)

between the creation and the annihilation operators $a^\dagger$ and $a$, which reduces to the CCR for $q = 1$ and to the CAR for $q = -1$. For $q$ in the interval $[-1, 1]$, equation (1.1) describes particles which are neither bosons nor fermions.

It is natural to ask, first of all, if this is the only way to interpolate between bosons and fermions. Moreover, also in view of constructing hamiltonian examples in which topological quasi *-algebras are relevant, we may also ask whether it is possible to extend the analysis of Ref. [1] to models with quons.

These two questions are strongly related and, in fact, contain the core of the results of this paper. First of all, it is evident that one can go from $q = -1$ to $q = 1$ also going out of the strip $[-1, 1]$: instead of a linear function $f(q) = q$, we could interpolate between fermions and bosons by making use of a different function $f(q)$ which, for instance, reaches the value +1 only asymptotically from above. This different approach, which will generate what we call modified quons (mq), makes the second question above relevant. In fact, proceeding in this way, we get creation and annihilation operators which are unbounded. Therefore, in order to study mathematical (more than physical) models of free or interacting mq, it is useful to extend the approach of Ref. [1]. This is exactly what we will do in Section III, after having introduced the mq in Section II. Section IV and Section V are respectively devoted to the analysis of free and interacting mq, while Section VI contains our conclusions and plans for the future. The paper ends with two Appendices which contain the proofs of some mathematical results.
II Modified Quons

In this Section we introduce the mq by means of a modified q-mutator, and we deduce many of their mathematical properties. After that we give our (mathematical) motivations for dealing with these particles. We begin with a short review on the standard results on quons, which are taken mainly from Refs. [3, 4, 5].

The standard commutation rule for a single mode quon is given by Mohapatra, Fivel et al. as

\[ aa^\dagger - qa^\dagger a = 1, \quad q \in [-1, 1]. \]

This relation interpolates between bosons \((q = 1)\) and fermions \((q = -1)\) and was originally introduced to analyze small difference from the Fermi \((q = -1 + \epsilon)\) and Bose statistics \((q = 1 - \epsilon)\).

If \(\Phi_0\) is the vector of the Hilbert space \(\mathcal{H}\) annihilated by the annihilation quon operator \(a, a\Phi_0 = 0\), then the set of the vectors defined recursively by

\[ \Phi_{n+1} = \frac{1}{\beta_n} a\Phi_n, \]

is an orthonormal basis for \(\mathcal{H}\). The value of the normalization constant \(\beta_n\) depends on \(q\) and \(n\) through the expression

\[ \beta_n^2 = \begin{cases} \frac{1-q^{n+1}}{1-q}, & \text{if } q \neq 1, \\ n + 1, & \text{if } q = 1. \end{cases} \]

Defining the self-adjoint operator

\[ N_0 = a^\dagger a, \]

it is easy to see that

\[ N_0\Phi_n = \beta_{n-1}^2 \Phi_n, \]

where \(\beta_{-1} := 0\). It is worthwhile to notice that the operator \(N_0\) is not a number operator for general \(q\), since \(\beta_{n-1}^2\) is different from \(n\) for any \(q\) different from 1, that is, whenever we are not dealing with bosons. Anyway, at least if \(q > 0\), it is possible to introduce what is really a number operator \(\tilde{N}\), see Ref. [3], as

\[ \tilde{N} = \frac{1}{\log q} \log(1 - (1 - q)N_0). \]

Indeed, for this operator, we get \(\tilde{N}\Phi_n = n\Phi_n\).

As far as single mode quons are considered there is no reason to force \(q\) to belong to the interval \([-1, 1]\). This requirement is crucial, by the way, if we have more than one quon mode, since we always require to any Hilbert space to have a positive definite norm. Examples of vectors with negative norm for \(q \notin [-1, 1]\) are contained in Ref. [4].
Another peculiarity of the quons is the following one: in Ref. [5] it is shown that it is not possible to introduce q-mutators different from the ordinary CCR or CAR containing, for instance, only the quons annihilation operators. In fact, the relation \( a_k a_l - q a_l a_k = 0 \) can hold if and only if \( q = 1 \) or \( q = -1 \). Nevertheless, in the same reference it is also discussed in detail the fact that this relation is not really needed to compute matrix elements in the Fock representation, or even to normal order a string of annihilation and creation operators.

The reason why we want to modify the original definition of quons is contained in the following lines, which show that the operator \( a \), as well as \( a^\dagger \), is bounded for any \( q \in [-1,1] \):

\[
\|a\|^2 = \sup_{\Psi \in \mathcal{H}, \|\Psi\| \leq 1} \langle a\Psi, a\Psi \rangle = \sup_{\Psi \in \mathcal{H}, \|\Psi\| \leq 1} \langle \Psi, N_0\Psi \rangle = \sup_{d \in l^2, \|d\| \leq 1} \sum_k |d_k|^2 \beta_{k-1}^2.
\]

Here we have written \( \Psi = \sum_k d_k \Phi_k \), using the fact that the vectors \( \Phi_k \)'s form an orthonormal basis of \( \mathcal{H} \). With \( l^2 \) we indicate the usual Hilbert space of the square-integrable sequences. Therefore we have:

- \( q = 1 \Rightarrow \beta_{k-1}^2 = k \). This implies that \( \|a\| = \infty \).
- \( q = -1 \Rightarrow \beta_{k-1}^2 = 0,1 \) depending on whether the index \( n \) is even or odd. However, in both cases, \( \beta_{k-1}^2 \leq 1 \), and therefore

\[
\|a\|^2 \leq \sup_{d \in l^2, \|d\| \leq 1} \sum_k |d_k|^2 = 1.
\]

(Of course for fermions we know that the strict equality \( \|a\| = 1 \) holds).

- for general \( q \in [-1,1] \) we see that \( \beta_{k-1}^2 \leq \frac{2}{1-q^2} \), independently of \( n \). As a consequence, it can be shown that \( \|a\| \) is bounded by \( \frac{2}{1-q} \).

**Remarks.**— These considerations suggest a possibility for ‘regularizing’ a quantum model containing bosons. For instance, if we consider what we have done in Ref. [1], we see that the operator \( a \) has been replaced by a bounded operator \( a_L \) which, in a sense, converges to \( a \) (in a given topology \( \tau \)), when the cut-off \( L \) is removed. It is clear now that we could consider, instead of \( a_L \), the operator \( a_q \) satisfying, together with its adjoint, the q-mutation relation (2.1). Now, \( a_q \) should be the bounded version of the boson operator \( a \), and the cut-off can be removed simply by taking the limit \( q \to +1^- \). In this way the free bosons hamiltonian \( H = a^\dagger a \) could be regularized using the bounded operator \( H_q = a_q^\dagger a_q \) instead of the \( H_L = a_L^\dagger a_L \). Of course, with this regularization, the problem of the removal of the cut-off in the dynamics could, in principle, be much harder to be solved.
These results show that, for standard quons, there is no need of using algebra of unbounded operators, but for $q = 1$. However, there is no reason a priori, if we confine our analysis to single mode quons, to interpolate between fermions and bosons keeping $|q| \leq 1$. On the contrary, we could obtain Fermi-Dirac statistics starting from the Bose-Einstein one following a different path, which is the one we will describe in a while, and which forces us to deal with unbounded operators.

The first ingredient is the following modified form of $q$-mutator:

$$aa^\dagger - f(q)a^\dagger a = 1,$$  \hspace{1cm} (2.4)

where $f(q)$ is a function, with domain $D(f) \subset \mathbb{R}$, satisfying the following properties:

- (p1) $f(q) \geq -1$, $\forall q \in D(f)$;
- (p2) $\exists D_+ \subset D(f)$, set of non-zero measure, such that $f(q) > 1$, $\forall q \in D_+$;
- (p3) $\exists q_f, q_b \in D(f)$, with in case $q_f, q_b = \pm \infty$, such that

$$\lim_{q \to q_f} f(q) = -1, \quad \lim_{q \to q_b} f(q) = 1.$$  

Furthermore, we have to consider a vector $\Phi_0$, called the ground vector of the theory, which is normalized and annihilated by the operator $a$:

$$\|\Phi_0\| = 1, \quad a\Phi_0 = 0.$$  \hspace{1cm} (2.5)

We can carry on computations analogous to those for the 'standard' quons: first of all, we define recursively the vectors $\Phi_k$ of an orthonormal basis of $\mathcal{H}$ starting from $\Phi_0$:

$$\Phi_{n+1} = \frac{1}{b_n}a^\dagger \Phi_n.$$  \hspace{1cm} (2.6)

The normalization constant turns out to be

$$b_n^2 = \begin{cases} \frac{1-f(q)^{n+1}}{1-f(q)}, & \text{if } f(q) \neq 1, \\ n+1, & \text{if } f(q) = 1. \end{cases}$$  \hspace{1cm} (2.7)

Sometimes we write also

$$\Phi_n = B_n (a^\dagger)^n \Phi_0,$$  \hspace{1cm} (2.8)

where we have introduced $B_n = (b_0 b_1...b_{n-1})^{-1}$. With these definitions it is easily checked that

$$\langle \Phi_k, \Phi_n \rangle = \delta_{kn}.$$  \hspace{1cm} (2.9)
Moreover, defining as before the self-adjoint operator

\[ N_0 = a^\dagger a, \]  

one can also check that

\[ N_0 \Phi_n = b_{n-1}^2 \Phi_n. \]  

No complication arises, at least for these single mode \( mq \), from the existence of vectors with negative norms. In fact we have

\[ A_n = \| (a^\dagger)^n \Phi_0 \|^2 = \left( \frac{1}{B_n} \right)^2 \| \Phi_n \|^2 = \frac{1}{B_n^2} = (b_0 b_1 ... b_{n-1})^2, \]  

which is strictly positive. From this positivity also follows positivity of arbitrary vectors \( \Psi \) in \( \mathcal{H} \), since any such \( \Psi \) is of the form \( \sum_n c_n \Phi_n \). The situation is different for more modes, where an example of a vector with negative norm can be constructed following the same ideas of Fivel, [4].

As for the standard quons, while \( a^\dagger \) creates an excitation, \( a \) annihilates it; in fact we have \( a^\dagger \Phi_n = b_n \Phi_{n+1} \) and \( a \Phi_{n+1} = b_n \Phi_n \).

We show now that the modified quons may be unbounded operators, depending on the value of \( f(q) \). In fact, following the same steps as before, we have

\[ \| a \|^2_q = \sup_{d \in \mathbb{C}, \sum |d_k|^2 \leq 1} \sum_k |d_k|^2 b_{k-1}^2(q), \]  

where we have written explicitly that \( b_k \), as well as the norm of \( a \), depends on \( q \). Let us consider a value of \( q \) in \( D_+ \). For such a \( q, \tilde{q} \), we know from assumption (p2) above that \( f(\tilde{q}) > 1 \). Therefore we have

\[ b_{k-1}^2(q) = \frac{\overline{f}^k - 1}{\overline{f} - 1}, \]

where \( \overline{f} = f(\tilde{q}) \). It is clear therefore that \( \| a \|_{\tilde{q}} = \infty \). We conclude that for any \( q \in D_+ \) the \( mq \) are unbounded. Of course, they are also unbounded for any other value of \( q, \tilde{q} \), satisfying \( f(\tilde{q}) = 1 \).

We end this Section with the following remark: in the rest of the paper we will use only \( \overline{f} \) instead of the whole function \( f(q) \). For this reason, the reader might wonder why such a function has been introduced. This was done because \( f(q) \) allows again to interpolate between fermions and bosons, even if this interpolation requires the use of unbounded operators.
III The Mathematical Framework

In this Section we introduce the algebras and the topologies relevant for the models discussed in the rest of the paper. In particular, we will adapt to mq the strategy introduced in Ref. [1] by the author. As it can be seen by a comparative analysis, there are not many substantial differences between the two strategies.

Let \( N_0 \equiv a^\dagger a \). This is a symmetric operator on \( D_0 = \{(a^\dagger)^n \Phi_0, n \in \mathbb{N}\} \) which can be uniquely extended to a self-adjoint operator \( N \), not to be confused with the number operator \( \tilde{N} \) introduced in the last Section. Let \( D(N^k) \) be the domain of the operator \( N^k \), \( k \in \mathbb{N} \), and \( D \) the domain of all the powers of \( N \):

\[
D \equiv D^\infty(N) = \cap_{k \geq 0} D(N^k).
\] (3.1)

This set is dense in \( \mathcal{H} \). Starting from \( D \) we can define, following Lassner, the \(*\)-algebra \( \mathcal{L}^+(D) \) of all the closable operators defined on \( D \) which, together with their adjoints, map \( D \) into itself, Ref.[2]. It is clear that all the powers of \( a \) and \( a^\dagger \) belong to this set.

In Ref. [2], the topological structures of both \( D \) and \( \mathcal{L}^+(D) \) are discussed in details; in particular, in \( D \) the topology is defined by the seminorms

\[
\phi \in D \rightarrow \| \phi \|_n \equiv \| N^n \phi \|,
\] (3.2)

which, as it is evident, depend on \( N \) and on the natural integer \( n \). Here \( \| \| \) is the norm of \( \mathcal{H} \). The topology in \( \mathcal{L}^+(D) \) is introduced in the following way. We start defining the set \( \mathcal{C} \) of all the positive, bounded and continuous functions \( f(x) \) on \( \mathbb{R}_+ \), which are decreasing faster than any inverse power of \( x \). The seminorms on \( \mathcal{L}^+(D) \) are labeled by functions in \( \mathcal{C} \) and by the integers \( N \). We have

\[
X \in \mathcal{L}^+(D) \rightarrow \| X \|^{f,k} \equiv \max \left\{ \| f(N)XN^k \|, \| N^kXf(N) \| \right\}.
\] (3.3)

Here \( \| \| \) is the usual norm in \( B(\mathcal{H}) \), \( f \in \mathcal{C} \) and \( k \in \mathbb{N} \). We use for this norm the same symbol as in equation (3.2) since there is no possibility of confusion. We observe that definition (3.3), together with the self-adjointness of \( N \), implies that for all \( X \in \mathcal{L}^+(D) \), \( \| X \|^{f,k} = \| X^\dagger \|^{f,k} \). We call \( \tau \) the topology on \( \mathcal{L}^+(D) \) defined by the seminorms (3.3). In Ref. [2] it has been proved that \( \mathcal{L}^+(D)[\tau] \) is a complete locally convex topological \(*\)-algebra.

As in Ref. [1], we define the one-dimensional spaces \( \mathcal{E}_l, l \in \mathbb{N} \), as the set of all the vectors of \( \mathcal{H} \) which are proportional to \( (a^\dagger)^l \Phi_0 \) and the \((L+1)\)-dimensional space \( \mathcal{F}_L \) as the direct sum of the first \( L+1 \) spaces \( \mathcal{E}_l, \mathcal{F}_L \equiv \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_L \).
The spectral decomposition of the operator $N$ is a bit different from the one in Ref. [1]. In fact we have now $N = \sum_{l=0}^{\infty} c_l \Pi_l$. Here $c_l = b_l^2 - 1$ and coincides with the decomposition of the boson number operator if $c_l = l$, that is if $f = 1$. The following properties are obvious:

\[ \Pi_k \Pi_l = \delta_{kl} \Pi_l, \quad \Pi_l^\dagger = \Pi_k, \quad (3.4) \]

and

\[ Q_L Q_M = Q_L, \quad \text{if } L \leq M, \quad Q_L^\dagger = Q_L. \quad (3.5) \]

We see that $\Pi_k : \mathcal{H} \to \mathcal{E}_k$, while $Q_L : \mathcal{H} \to \mathcal{F}_L$. From equations (3.4) and (3.5) we can check that the spaces $\mathcal{E}_k$ are mutually orthogonal, while the $\mathcal{F}_L$ are not. Of course, the following inclusions of spaces hold:

\[ \mathcal{E}_l \subset \mathcal{F}_L \subset \mathcal{F}_{L+1} \subset \mathcal{D} \subset \mathcal{H}, \quad (3.6) \]

whenever $l \leq L$.

Using the same argument discussed in Ref. [1], that is, the same computational nature of the two contributions $\|f(N)XN^k\|$ and $\|N^kXf(N)\|$, from now on we will identify $\|X\|^{f,k}$ simply with $\|f(N)XN^k\|$ which, as a consequence of the spectral decomposition for $N$, can be written as follows:

\[ X \in \mathcal{L}^+(\mathcal{D}) \longrightarrow \|X\|^{f,k} = \sum_{l,s=0}^{\infty} f(c_l)c_s^k\|\Pi_l\Pi_s\|, \quad (3.7) \]

see Ref. [2] for bosons. We will call physical the topology $\tau$ generated by these seminorms.

Now that the mathematical structure used to describe the mq has been discussed, we focus our attention on the spin operators. We refer to Ref. [7] for further details.

Let $\mathbb{Z}^d$ be a $d$-dimensional infinite lattice, $V \subset \mathbb{Z}^d$, and $|V|$ the number of the points of $V$. We call $\mathcal{A}_V$ the $C^*$-algebra generated by the spin operators $\sigma_i^\alpha$, $i \in V$ and $\alpha = x, y, z$, and $\mathcal{A}_o$ the norm closure of $\bigcup_{V} \mathcal{A}_V$.

We call relevant a state $\omega$ over $\mathcal{A}_o$ if, denoting by $\mathcal{H}_\omega$ the Hilbert space defined by the GNS construction on $\mathcal{A}_o$ and $\omega$, and by $\Psi_\omega$ the vector which represents $\omega$ in $\mathcal{H}_\omega$ ($\omega(A) = \langle \Psi_\omega, \pi_\omega(A)\Psi_\omega \rangle$, for all $A \in \mathcal{A}_o$), then $\Psi_\omega$ belongs to the following set of vectors $\mathcal{F}$:

\[ \mathcal{F} = \left\{ \Psi \in \mathcal{H}_\omega : \lim_{|V|,|V|^\infty} \frac{1}{|V|} \sum_{p \in V} \pi_\omega(\sigma_3^p)\Psi = \pi_\omega(\sigma_3^\infty)\Psi, \quad \|\sigma_3^\infty\| \leq 1 \right\}. \quad (3.8) \]

Here $\sigma_3^\infty$ belongs to the center of the algebra $\mathcal{A}_o$ and $\pi_\omega$ is the canonical representation for the spin algebra, Ref. [7].
In Ref. [7], it has been proven, among other things, that all the powers of $\sigma_3^V$ converge in the $\mathcal{F}$-strong topology (the strong topology ‘restricted’ to those vectors which belong to $\mathcal{F}$), as well as all the analytic functions of $\sigma_3^V = \frac{1}{|V|} \sum_{p \in V} \sigma_3^p$. The topology is therefore defined by the following seminorms:

$$X \in \mathcal{A}_o \simeq B(\mathcal{H}_{\text{spin}}) \to \|X\|^\Psi \equiv \|X\Psi\|,$$

where $\Psi \in \mathcal{F}$ and $X$ is identified with its ‘canonical’ representative in $B(\mathcal{H}_{\text{spin}})$, $\mathcal{H}_{\text{spin}}$ being the infinite tensor product of two-dimensional complex spaces $\mathbb{C}_2^i$, $i \in \mathbb{Z}^d$, see Ref.[8]. In this way $\mathcal{A}_o$ is canonically identified with its representative $B(\mathcal{H}_{\text{spin}})$. This identification will be used all throughout this paper in order to simplify the notation.

In Section V we will consider an interacting toy model of spin and mq for which some extra considerations are needed: in particular, we will need to define the complete algebra $\mathcal{A}$ for such a model and its related topology. As it is reasonable, we will take $\mathcal{A} = \mathcal{L}^+(\mathcal{D}) \otimes B(\mathcal{H}_{\text{spin}})$ and the topology as the ‘composition’ of the two topologies introduced by equations (3.3) and (3.9).

### IV The Free Modified Quons

In this Section we propose a rigorous algebraic treatment of the existence problem for the time evolution of a single mode MQ described by the following free hamiltonian:

$$H = a^\dagger a,$$

where $a$ and $a^\dagger$ are the unbounded mq operators satisfying equation (2.4). We are considering a modified q-mutator related to some value of $\mathcal{q}$ in $\mathcal{D}_+$, so that the value of $f(q)$, $f$, is bigger than one. As in Ref. [1], and almost everywhere in the literature, to deal properly with this problem it is convenient to use the following general strategy: first we regularize the hamiltonian introducing a certain cutoff; then we obtain the equations of motion which are solved keeping this cutoff fixed, and finally we remove the cutoff. If all these steps can be performed, we define the dynamics for the ‘infinite’ model as the limit of these cutoff depending dynamics.

We have already discussed in Ref. [1] a convenient way of implementing the cutoff. This is defined starting from the mq operators themselves and making use of the projection operators $Q_L$. Explicitly, we perform the following substitutions:

$$a \rightarrow a_L \equiv Q_L a Q_L, \quad a^\dagger \rightarrow a^\dagger_L \equiv Q_L a^\dagger Q_L,$$

where $\mathcal{A}_o$ is canonically identified with its representative $B(\mathcal{H}_{\text{spin}})$. This identification will be used all throughout this paper in order to simplify the notation.

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so that, for the model in (4.1), we are led to consider the operator

\[ H_L = a_L^\dagger a_L, \quad (4.3) \]

which is, of course, still self-adjoint. A generalization of this method for hamiltonians containing bosonic operators is given in Ref. [1]. That approach can be applied also to \( mQ \) and, mutatis mutandis, to any problem involving unbounded operators \( X \) for which the ‘free’ hamiltonian operator \( X^\dagger X \) is defined on a dense domain.

We give now a Lemma stating various properties of the operators \( a, a^\dagger, a_L, a_L^\dagger, N, Q_L, \Pi_l \), properties which are used everywhere in the rest of the paper. The proof of this Lemma is contained in Appendix A.

**Lemma 4.1.**– The operators \( a, a^\dagger, a_L = Q_L a Q_L, a_L^\dagger, N, Q_L, \Pi_l \) satisfy the following properties:

\[
\begin{align*}
a \Pi_l &= \Pi_{l-1} a, & a \Pi_{l-1} &= \Pi_l a^\dagger, & l &= 1, 2, 3, \ldots \quad (4.4) \\
Q_L a &= a Q_{L+1}, & Q_{L+1} a^\dagger &= a^\dagger Q_L, & L &= 1, 2, 3, \ldots \quad (4.5) \\
[Q_L, a] &= \Pi_L a, & [Q_L, a^\dagger] &= -a^\dagger \Pi_L, & L &= 1, 2, 3, \ldots \quad (4.6) \\
[a, a^\dagger] &= \I + FN_0, & [N_0, a] &= -(\I + FN_0)a \quad (4.7) \\
[Q_L, N_0] &= [N_0, \Pi_l] = 0 \quad (4.8) \\
[a_L, a] &= \Pi_{l-1} a^2 = a^2 \Pi_{l+1}, \quad [a^\dagger_L, a] &= -Q_{L-1} + Q_L N_0 - FN_{L-1} N_0 \quad (4.9) \\
\|\Pi_l a s\|_2^2 &= c_s \delta_{l,s-1}, \quad \|a_L\|_2^2 \leq L c_L, \quad (4.10)
\end{align*}
\]

where \( F := f - 1 \).

**Remarks.**– (a) We see that many of these properties are significantly different from the ones in Ref. [1]. However, they appear to be the generalizations of those for bosons, which, in fact, can be recovered for \( f = 1 \).

(b) Using these results it is easy to show that for the free hamiltonian \( H = N \) the regularization (4.3) coincides with \( Q_L N Q_L \). Indeed, by equations (4.5) and (3.5), we have

\[
a_L^\dagger a_L = Q_L a_L^\dagger Q_L a Q_L = Q_L a_L^\dagger Q_L a Q_L = Q_L a_L^\dagger a Q_{L+1} Q_L = Q_L N Q_L.
\]

Once we have regularized the hamiltonian, we come to the main problem of this Section, that is the proof of the existence of the limit, for \( L \) going to infinity, of the following algebraic dynamics:

\[
a_L^t(X) = e^{iH_L t} X e^{-iH_L t},
\]
being an element of \( L^+(D) \). The topology in which this limit will be considered is the physical one defined in the previous Section, which makes of \( L^+(D) \) a complete topological *-algebra. The result is stated in the following

**Proposition 4.2.** – The limits of \( \alpha_t^L(a^n) \) and \( \alpha_t^L((a^\dagger)^n) \) exist in \( L^+(D)[\tau] \) for all natural \( n \).

**Proof**
We prove the Proposition only for the quon annihilation operator \( a \). The statement for \( a^\dagger \) follows from the \( \dagger \)-invariance of \( \tau \). We proceed by induction on \( n \). We start therefore by proving the statement for \( n = 1 \).

The first step consists in observing that the multiple commutators \( [H_L, a]_m \), defined by recurrence in the usual way \( ([H_L, a]_0 = a \text{ and } [H_L, a]_m = [H_L, [H_L, a]_{m-1}] ) \), can be written as

\[
[H_L, a]_m = (-Q_{L-1} + Q_L N - \overline{f}Q_{L-1} N)^m a = \left( (-1)^m (\overline{F} N + 1)^m Q_{L-1} + N^m \Pi_L \right) a. \tag{4.11}
\]

Again, the proof of these commutation rules goes on by induction on \( m \), and is a simple consequence of the commutation relations given in Lemma 4.1. In particular, one can easily find that \( [H_L, a]_1 = (-Q_{L-1} + Q_L N - \overline{f}Q_{L-1} N) a \). The result for \( [H_L, a]_m \), for \( m > 1 \), follows from the commutativity between \( H_L \) and \( -Q_{L-1} + Q_L N - \overline{f}Q_{L-1} N \). The second equality above is simply a consequence of the properties of the projection operators \( \Pi_l \) and \( Q_L \).

By means of (4.11) we can prove that

\[
\alpha_t^L(a) = e^{iH_L t} a e^{-iH_L t} = \left( e^{-i(\overline{F} N+1) Q_{L-1} + \Pi_L} \right) a. \tag{4.12}
\]

In fact, since \( H_L \) is bounded, we have

\[
\alpha_t^L(a) = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} [H_L, a]_m = \sum_{m=0}^{\infty} \frac{((-1)^m (\overline{F} N+1)^m Q_{L-1} + N^m \Pi_L)}{m!} a,
\]

so that equation (4.12) follows by simple resummation. Incidentally, we notice that, when \( \overline{F} = 0 \), that is for bosons, the above formula coincides with the one obtained in Ref. [1], recalling that in this case we have \( c_L = L \).

Now we can prove the \( \tau \)-Cauchy nature of \( \alpha_t^L(a) \). Using the seminorms (3.7) we have

\[
Q_{L,M}(t) \equiv \| \alpha_t^L(a) - \alpha_t^M(a) \|_{h,k}^{h,k} = \sum_{l,s=0}^{\infty} h(c_l)c_s^k \| \Pi_l(\alpha_t^L(a) - \alpha_t^M(a)) \Pi_s \|.
\]
As it is evident, we are not explicitly writing here the dependence of \( Q_{L,M}(t) \) on the seminorm, that is from \( h \) and \( k \), since this dependence does not play any role. Using the expression (4.12) for \( \alpha^t_L(a) \) we get the following inequality:

\[
Q_{L,M}(t) \leq \sum_{l,s=0}^{\infty} h(c_l)c_s^k(\|\Pi_{\alpha}(e^{-it(FN+1)}(Q_{L,-1} - Q_{M,-1})a\Pi_s\| + \\
+ \|\Pi_{\alpha}(e^{it\epsilon(t,\Pi_L - e^{it\epsilon(t,\Pi_M)}a\Pi_s})\|) \leq \sum_{l,s=0}^{\infty} h(c_l)c_s^k(2\delta_{l,M} + \delta_{l,M+1} + ... + \delta_{l,L})||\Pi_{\alpha}a\Pi_s|| \leq
\]

\[
\leq h(c_M)c_{M+1}^{k+1/2} + \sum_{l=M}^{L} h(c_l)c_{l+1}^{k+1/2}.
\]

Here we have used the first formula in (4.10). Recalling now that, as far as \( F > 1 \) and \( M \gg 1 \), then \( c_{M+1} = 1 + c_MF \simeq c_MF \), we see that, apart from an irrelevant quantity, for \( M \) and \( L \) large enough,

\[
Q_{L,M}(t) \leq F^{k+1/2} \left( h(c_M)c_{M+1}^{k+1/2} + \sum_{l=M}^{L} h(c_l)c_{l+1}^{k+1/2} \right).
\]

This estimate proves the \( \tau \)-Cauchy nature of \( \alpha^t_L(a) \), and, therefore, its convergence to an element of \( \mathcal{L}^+(\mathcal{D})[\tau] \) which will be called \( \alpha^t(a) \).

**Remark:** As already mentioned in Section II, we have focused our attention to the seminorm \( \|f(N)N^k\| \). From the above computations, it appears clear how to reproduce the analogous estimates for \( \|N^k.f(N)\| \) and it is evident that it behaves essentially in the same way. This implies, in turn, that due the \( \hat{t} \)-invariance of the 'complete' physical topology, given by the seminorms (3.3), \( \alpha^t_L(a^t) \) is \( \tau \)-Cauchy and therefore \( \tau \)-convergent in \( \mathcal{L}^+(\mathcal{D})[\tau] \).

At this point we can prove the second step of the induction. Hence, we suppose that \( \alpha^t_L(a^{n-1}) \) is \( \tau \)-convergent (and, therefore, \( \tau \)-Cauchy) and we prove that also \( \alpha^t_L(a^n) \) is \( \tau \)-Cauchy. The proof goes as follows:

\[
Q_{L,M}^n(t) \equiv \|\alpha^t_L(a^n) - \alpha^t_M(a^n)\|^{h,k} \leq \|\alpha^t_L(a)((\alpha^t_L(a))^{n-1} - (\alpha^t_M(a))^{n-1})\|^{h,k} + \\
+ \|((\alpha^t_L(a) - \alpha^t_M(a))(\alpha^t_M(a))^{n-1})^{h,k} \|^{h,k}. \quad (4.13)
\]

It is possible to see that the first contribution is bounded above by the seminorm \( \|((\alpha^t_L(a^{n-1}) - \alpha^t_M(a^{n-1}))\|^{\bar{h},k} \), where \( \bar{h} \) is a function, related to \( h \), which is still in \( \mathcal{C} \). We know then, by the induction hypothesis, that this quantity converges to zero. The proof is a bit long: first we introduce the operator \( V_L(t) = (e^{-it(FN+1)}Q_{L-1} + e^{it\epsilon(t,\Pi_L}) \), which obviously commutes with \( N, \Pi_{t} \) and \( Q_L \). Moreover, the norm of \( V_L(t) \) is less or equal to 2. Using these
properties we estimate the first contribution above with
\[ 2 \sum_{l,s=0}^{\infty} h(c_l)c_s^{k}\sqrt{c_{l+1}}\Pi_{l+1}((\alpha_L^t(a))^{n-1} - (\alpha_M^t(a))^{n-1})\Pi_s\].

At this stage we can perform resummations over \( l \) and \( s \). Before doing this, we introduce a function \( \tilde{h} \) which is linked to the original \( h \) by the requirements: (1) \( \tilde{h}(c_l) = h(c_{l-1}) \) for all \( l \geq 0 \), where \( c_{-1} := 0 \), and (2) \( \tilde{h} \) still belongs to \( \mathcal{C} \). Therefore, we get
\[ \|\alpha_L^t(a)((\alpha_L^t(a))^{n-1} - (\alpha_M^t(a))^{n-1})\|^h,k \leq 2\sqrt{N}\tilde{h}(N)((\alpha_L^t(a))^{n-1} - (\alpha_M^t(a))^{n-1})N^k, \]
which goes to zero since \( \sqrt{N}\tilde{h} \) still belongs to \( \mathcal{C} \), and since, by assumption, \( \alpha_L^t(a))^{n-1} \) is \( \tau \)-Cauchy.

The second contribution in (4.13) can be estimated with similar techniques. Since the main steps are very similar to those in Ref. [1] we do not repeat them here. We only give the result, which is a bit different:
\[ \|(\alpha_L^t(a) - \alpha_M^t(a))(\alpha_M^t(a))^{n-1}\|^h,k \leq 2^{n-1} \left( \sum_{i=M}^{L} h(c_i)c_i^{k+n/2} + h(c_M)c_{M+n}^{k+n/2} \right). \]

At this point we observe that, as before, for \( T > 1 \) and \( M \) large enough, for any integer \( n \) we have \( c_{i+n} \approx Tc_{i+n-1} \approx T^2c_{i+n-2} \approx T^n c_i \), so that
\[ \|(\alpha_L^t(a) - \alpha_M^t(a))(\alpha_M^t(a))^{n-1}\|^h,k \leq 2^{n-1}T^n \left( \sum_{i=M}^{L} h(c_i)c_i^{k+n/2} + h(c_M)c_{M}^{k+n/2} \right). \]

Also this quantity converges to zero for \( L \) and \( M \) diverging. This concludes the proof.

We end this Section showing a difference with what has been done in Ref. [1]: in that paper we have introduced two simple models related to the hamiltonian of the free bosons, which could be treated in the same way as the bosons themselves simply because we could consider unitary transformations on the boson operators preserving the commutation rules. This is no longer the situation here: if we consider a perturbed hamiltonian of the form \( H = a^\dagger a + \gamma(a + a^\dagger) \), \( a \) being a (modified) quon operator, the operator \( b = a + \gamma \) does not obey a q-mutator relation, the reason being that \( b \) is not unitarily equivalent to \( a \).

The situation changes drastically if we consider a model with more than a type of quons, since, as we know, in this case the positivity of the norm of all the vectors in \( \mathcal{H} \) require \( f(q) \) in the q-mutator (2.4) to take values in the interval \([-1, 1]\), which, in turns, implies that the mq are really ordinary quons and, therefore, always bounded but for \( f(q) = 1 \).
An Interacting Model

In this Section we will show how to use the framework introduced in this paper to prove the existence of the algebraic dynamics for the model described by the following finite-volume Hamiltonian:

\[ H_V = \frac{J}{|V|} \sum_{i,j \in V} \sigma_3^i \sigma_3^j + \gamma \sigma_3^V a^\dagger a. \]  

Here, as before, \( \sigma_3^V = \frac{1}{|V|} \sum_{i \in V} \sigma_3^i \), while \( \gamma \) is a real parameter. This is simply a toy model of an ‘interaction’ between the matter, considered as a family of two levels atoms, and conveniently mimicked using spin matrices, with single mode \( m q \). It is worthwhile to mention that the interest of this model, at least in the author’s knowledge, is essentially mathematical, and for this reason we will consider only the mathematical difficulties arising from such an Hamiltonian. It is also worth remarking that a volume cutoff appears in \( H_V \). The reason for this volume cutoff has been widely discussed, for instance, in Ref. [8].

We remind the reader that the Pauli spin matrices satisfy the following commutation rules:

\[ [\sigma^i_\alpha, \sigma^j_\beta] = 2i\epsilon_{\alpha\beta\gamma} \delta_{i,j} \sigma^i_\gamma. \]  

A consequence of this commutation relation, together with the definition of \( \sigma_3^V \), is that

\[ [\sigma_3^V, \sigma_3^j] = \frac{2i\epsilon_{3,3\gamma}}{|V|} \sigma_3^i \sigma_3^\gamma, \]  

which shows already that, in the limit \( |V| \to \infty \), \( \sigma_3^V \) commutes with all the other elements of the \( C^* \)-algebra of the bounded spin operators, \( B(\mathcal{H}_{\text{spin}}) \).

The way of proving the existence of the thermodynamical limit of this model is suggested by the results contained in Section III and in Ref. [7]: the algebra \( \mathcal{A} \) should reflect the composite nature of the model, and is therefore reasonable to take \( \mathcal{A} \) as the tensor product of \( B(\mathcal{H}_{\text{spin}}) \), and the Lassner mq algebra \( \mathcal{L}^+(\mathcal{D}) \): \( \mathcal{A} = B(\mathcal{H}_{\text{spin}}) \otimes \mathcal{L}^+(\mathcal{D}) \). The topology on \( \mathcal{A}, \tau \), is generated by seminorms which deal separately with the spin and the quon variables. Obviously, in view of the results of Section III for the free mq and what is known about the mean field spin models, Ref. [7], we take these seminorms essentially as the ones in (3.7) for the quon operators and as the \( F \)-strong ones in (3.9) for the spin observables. We give here the definition of these seminorms which will be discussed in some more details in Appendix B. First of all, following [7, 9, 1], we give the definition of what we will call relevant vectors in the present contest. We call relevant any vector of
the following set:

$$F = \left\{ \Psi \in H_{\text{spin}} : \lim_{|V|,|L| \to \infty} \sum_{p \in V} \sigma_3^p \Psi = \sigma_3^\infty \Psi, \quad \|\sigma_3^\infty\| \leq 1 \right\},$$

(5.4)

$$\sigma_3^\infty$$ being an element of the center of the C*-spin algebra, which is slightly different from the one introduced in Section II. The utility of this set relies on the fact that $$\sigma_3^V$$ (together with its powers and its analytical functions) converges strongly only on the vectors of $$F$$ and not on general vectors of $$H_{\text{spin}}$$ (and, of course, not in norm).

As we have done in Ref. [1], we introduce here the unbounded operator

$$M \equiv N + 1 = a^\dagger a + \mathbb{1} = \sum_{l=1}^\infty c_l \Pi_l.$$  

(5.5)

The only difference between the operators $$M$$ and $$N$$ is in the lower value of $$l$$ in their spectral decompositions. We just want to mention that, while in Ref. [1] the introduction of $$M$$ was really necessary, here it could be avoided. With these considerations in mind, we define, for each $$X \in \mathcal{L}^+(\mathcal{D})$$ and for each $$A \in \mathcal{B}(H_{\text{spin}}),$$

$$\|XA\|^{f,k}_\Psi = \|X\|^{f,k}_\Psi \|A\|_\Psi = \|f(M)XM^k\| \|A\Psi\|,$$

(5.6)

where $$f \in C, k \in \mathbb{N}, \Psi \in F,$$ and where we identify $$XA$$ with the tensor product $$X \otimes A.$$ We stress once more that, again, we are considering only one contribution in the Lassner seminorm; a complete definition should include also $$\|M^kXf(M)\|,$$ see (3.3).

Now we proceed to a complete regularization of the Hamiltonian $$H_V.$$ The approach we follow is the natural extension of the one already discussed in Section IV. Therefore, we consider the following operator

$$H_{V,L} = \frac{J}{|V|} \sum_{i,j \in V} \sigma_3^i \sigma_3^j + \gamma \sigma_3^V a_L^\dagger a_L, $$

(5.7)

where, as before, $$a_L = Q_L a Q_L = a Q_L,$$ see (4.5). We observe that, from Lemma 4.1, we could also write the second term in $$H_{V,L}$$ as $$\gamma \sigma_3^V Q_L N$$ or, yet, $$\gamma \sigma_3^V Q_L N Q_L.$$ Then we define the algebraic dynamics $$\alpha_{V,L}$$ with the 'complete' cutoff in canonical way:

$$\alpha_{V,L}^t(X) = e^{iH_{V,L}t} X e^{-iH_{V,L}t},$$

(5.8)

where $$X \in \mathcal{A}.$$

We can now state the main result of this Section:

**Proposition 5.1**— The limit of $$\alpha_{V,L}^t(a)$$ for $$|V|$$ and $$L$$ both divergent exists in $$\mathcal{A}[\tau].$$ Furthermore, if the cutoffs are removed in such a way that $$\frac{L_{\text{cut}}}{|V|} \to 0,$$ then also the limit of $$\alpha_{V,L}^t(\sigma_a^i)$$ for $$|V|$$ and $$L$$ divergent exists in $$\mathcal{A}[\tau].$$
Proof

We begin with the proof of the convergence of \( \alpha_{V,L}^t(a) \). Since \( [J|V|(\sigma_3^Y)^2, \gamma \sigma_3^Y a_L^\dagger a_L] = 0 \), we can write

\[
\alpha_{V,L}^t(a) = e^{i\gamma \sigma_3^Y a_L^\dagger a_L t} (e^{i J|V|(\sigma_3^Y)^2 t} a e^{-i J|V|(\sigma_3^Y)^2 t}) e^{-i \gamma \sigma_3^Y a_L^\dagger a_L t} = 
\]

\[
e^{i \gamma \sigma_3^Y a_L^\dagger a_L t} a e^{-i \gamma \sigma_3^Y a_L^\dagger a_L t} = \sum_{n=0}^{\infty} \frac{(i \gamma t \sigma_3^Y)^n}{n!} [a_L^\dagger a_L, a]_n, \tag{5.9}
\]

due to the commutation relation \([a, \sigma_i^a] = 0\). Here \([a_L^\dagger a_L, a]_n\) is the usual multiple commutator

\[
[a_L^\dagger a_L, a]_0 = a, \quad [a_L^\dagger a_L, a]_n = [a_L^\dagger a_L, [a_L^\dagger a_L, a]_{n-1}], \quad n \geq 1.
\]

This multiple commutators can be easily computed using induction on \(n\) and we get

\[
[a_L^\dagger a_L, a]_n = (-Q_L(\mathbb{1} + F(q)N) + c_{L+1} \Pi_L) a. \tag{5.10}
\]

Therefore we obtain

\[
\alpha_{V,L}^t(a) = e^{it \gamma \sigma_3^Y Z_L a}, \tag{5.11}
\]

which can be written, to simplify the notation, as

\[
\alpha_{V,L}^t(a) = e^{it \gamma \sigma_3^Y Z_L a}, \tag{5.12}
\]

where \( Z_L = -Q_L(\mathbb{1} + F(q)N) + c_{L+1} \Pi_L \).

Of course, it appears evident that the problem of studying the existence of the thermodynamical limit for this model is by far simpler than the analogous problem for the interacting models in Ref. [1], since now we have at our disposal the explicit expression of \( \alpha_{V,L}^t(a) \). This essential simplification arises because of the differences between the two hamiltonians which define the different models.

In order to show that \( \alpha_{V,L}^t(a) \) is \( \tau \)-Cauchy in \( \mathcal{A} \) we will often refer to the properties of the seminorms defining \( \tau \), see Appendix B. We proceed in the following way:

(I) first we observe that, for all \( h \in \mathcal{C}, k \in \mathbb{N} \) and \( \Psi \in \mathcal{F} \),

\[
I_{V,W,L,N} \equiv \|\alpha_{V,L}^t(a) - \alpha_{W,N}^t(a)\|_{h,k;\Psi} \leq \|\alpha_{V,L}^t(a) - \alpha_{V,N}^t(a)\|_{h,k;\Psi} + \|\alpha_{V,N}^t(a) - \alpha_{W,N}^t(a)\|_{h,k;\Psi} =: I_{V,L,N}^{(1)} + I_{V,W,N}^{(2)}, \tag{5.14}
\]

so that the \( \tau \)-convergence of \( \alpha_{V,L}^t(a) \) is controlled considering separately the volume and the occupation number cutoffs.
(II) Now we prove that \( I_{V,L,N}^{(1)} \to 0 \) when \( N, L \to \infty \) uniformly in the volume \( V \). In the following we will assume that \( N > L \). The proof of this convergence is a bit long. We will only sketch the main steps.

\[
I_{V,L,N}^{(1)} = \| e^{it\gamma_3 V} Z_L (1 - e^{it\gamma_3 V} (Z_N - Z_L)) a \|^{h,k,\Psi} \leq \sum_{l,s=1}^{\infty} h(c_l) c_s^k \| \Pi_l e^{it\gamma_3 V} Z_L (1 - e^{it\gamma_3 V} (Z_N - Z_L)) a \|^{\Psi} \leq \sum_{l,s=1}^{\infty} h(c_l) c_s^k \| \Pi_l a \| \| e^{it\gamma_3 V} \xi_{L,l} (1 - e^{it\gamma_3 V} (\xi_{N,l} - \xi_{L,l})) \Psi \|.
\]

Here we have used the following equality:

\[
Z_L \Pi_l = \xi_{L,l} \Pi_l, \quad \xi_{L,l} = c_{L+1} \delta_{L,l} - (1 + F(q) c_l)(\delta_{l,0} + \delta_{l,1} + ... + \delta_{l,L}), \quad (5.15)
\]

which allows us to split in the estimate above the dependence on the spin and on the quon variables. Using now the unitarity of the operator \( e^{it\gamma_3 V} \xi_{L,l} \) for all \( l, L \) and \( V \), and the equation (4.10), we end up with the following upper bound

\[
I_{V,L,N}^{(1)} \leq \sum_{l=1}^{\infty} h(c_l) c_{l+1}^{k+1/2} \| (1 - e^{it\gamma_3 V} (\xi_{N,l} - \xi_{L,l})) \Psi \|.
\]

To conclude, we use the spectral decomposition for the bounded and self-adjoint operator \( \sigma_3^V \). We can write, for any fixed volume \( V \) and \( \forall \varphi_1, \varphi_2 \in \mathcal{H}_{spin} \),

\[
< \sigma_3^V \varphi_1, \varphi_2 > = \int \lambda d < E_\lambda^V \varphi_1, \varphi_2 >,
\]

which implies, after some easy estimates on trigonometrical functions, that

\[
\| (1 - e^{it\gamma_3 V} (\xi_{N,l} - \xi_{L,l})) \Psi \|^2 \leq (t\gamma (\xi_{N,l} - \xi_{L,l}))^2 \| (\sigma_3^V)^2 \Psi \| \leq (t\gamma (\xi_{N,l} - \xi_{L,l}))^2.
\]

Therefore we have

\[
I_{V,L,N}^{(1)} \leq (t\gamma)^2 \sum_{l=1}^{\infty} h(c_l) c_{l+1}^{k+1/2} |\xi_{N,l} - \xi_{L,l}|,
\]

which can be shown to go to zero when \( L \) and \( N \) both diverge, uniformly with respect to \( V \), recalling the definition of \( \xi_{N,l} \) in equation (5.15) and the properties of the functions in \( \mathcal{C} \).

(III) Now we prove that \( I_{V,W,N}^{(2)} \to 0 \) when \( V, W \to \infty \) uniformly in \( N \). In the following we will assume that \( W \supset V \). By making use of equations (5.12), (5.15), (4.10), and of
the properties of the seminorms, we get

\[
I_{V,W,N}^{(2)} = \| \alpha_{V,N}^t(a) - \alpha_{W,N}^t(a) \|_{h,k} \Psi \leq \sum_{l,s=1}^\infty h(c_l)c_s^k \Pi_l e^{it\gamma\sigma_3^V} Z_N (1 - e^{it\gamma(\sigma_3^W - \sigma_3^V)} Z_N) a \Pi_s \| \Psi \leq \sum_{l=1}^\infty h(c_l)c_{l+1}^{k+1/2} \left( 1 - e^{it\gamma(\sigma_3^W - \sigma_3^V)\xi_{N,l+1}} \right) \| \Psi \|. 
\]

The contribution above can be estimated again using the spectral decomposition of the self-adjoint operator \( \sigma_3^W - \sigma_3^V \), in a way similar to that used above. After some easy steps we get

\[
I_{V,W,N}^{(2)} \leq 2t\gamma \| (\sigma_3^W - \sigma_3^V) \Psi \| \sum_{l=1}^\infty h(c_l)c_{l+1}^{k+1/2} |\xi_{N,l+1}|. 
\]

Using the definition of \( \xi_{N,l+1} \), together with the nature of the function \( h \), we see that the sum above remains, for all \( N \in \mathbb{N} \), bounded by a certain positive constant \( \tilde{J} \). Therefore

\[
I_{V,W,N}^{(2)} \leq 2t\gamma \tilde{J} \| (\sigma_3^W - \sigma_3^V) \Psi \| \to 0, 
\]

evertheless \( |V| \) and \( |W| \) both diverge, since \( \Psi \) belongs to \( \mathcal{F} \).

(IV) At this point we can conclude that \( I_{V,W,L,N} \) goes to zero for \( |V|, |W|, L, N \) all diverging, which is true for any \( h, k \) and \( \Psi \) defining the seminorms of the topology \( \tau \). This implies, therefore, that \( \alpha_{V,L}^t(a) \) is \( \tau \)-Cauchy and, being \( \mathcal{A} \) \( \tau \)-complete, \( \alpha_{V,L}^t(a) \) is \( \tau \)-convergent to an element of \( \mathcal{A} \) which will be called \( \alpha^t(a) \) or \( a^t(t) \).

**Remarks.**— (1) It is clear from the above procedure that the order of the limits (\( |V| \to \infty \) or \( L \to \infty \)) has no importance.

(2) It is worthwhile to observe that \( a(t) \) can be written explicitly and it has the following form

\[
a(t) = e^{-it\gamma\sigma_3^\infty(1+F(q)N)}a, 
\]

which shows that \( a(t) \) is obtained via the action of the operator \( e^{-it\gamma\sigma_3^\infty(1+F(q)N)} \) on the operator \( a \). This result coincides with the one we would get starting with the formal operator \( H = \gamma N\sigma_3^\infty \) and then considering \( e^{iHt}ae^{-iHt} \). This is just what we expected since \( H \) is, but for the irrelevant matter contribution, exactly the formal limit of \( H_{V,L} \).

Let us now discuss the convergence of \( \alpha_{V,L}^t(\sigma_3^a) \). Using the commutativity of the two pieces of the hamiltonian, formula (2.3) of Ref. [9], and the obvious commutation relation
\[ [\sigma_3^V, \sigma_3^t] = 0, \text{ we have} \]

\[
\alpha_{V,L}^i(\sigma_\alpha) = e^{iH_{V,L}t} \sigma_\alpha e^{-iH_{V,L}t} = e^{i\gamma a^i_L a_L \sigma_3^V t} \left( e^{iJ|V|(\sigma_\gamma^V)^2 t} \sigma_\alpha e^{-iJ|V|(\sigma_\gamma^V)^2 t} \right) e^{-i\gamma a^i_L a_L \sigma_3^V t} = \beta_{V,L}^i(\sigma_\alpha^V) \cos^2(S_3^V) - 2\epsilon_{3\alpha\beta}\beta_{V,L}^i(\sigma_\beta^V) \sin(S_3^V) \cos(S_3^V) + \sigma_3^i \beta_{V,L}^i(\sigma_\alpha^V) \sigma_3^i \sin^2(S_3^V) + O(|V|^{-1}), \tag{5.16}
\]

where \( S_3^V = 2Jt\sigma_3^V \), \( O(|V|^{-1}) \) is norm convergent to zero and we have introduced the notation

\[
\beta_{V,L}^i(\sigma_\alpha^V) = \exp\{i\gamma a^i_L a_L \sigma_3^V t\} \sigma_\alpha^i \exp\{-i\gamma a^i_L a_L \sigma_3^V t\}. \tag{5.17}
\]

It is reasonable to expect that, in the limit \( |V| \to \infty \), the difference between \( \beta_{V,L}^i(\sigma_\alpha^V) \) and \( \sigma_\alpha^i \) tends to disappear, the reason being that in this limit \( \sigma_3^V \) converges (strongly on \( \mathcal{F} \)) to an element of the center of \( B(\mathcal{H}_\text{spin}) \). In fact we can show that

\[
K_{V,L} \equiv |||\beta_{V,L}^i(\sigma_\alpha^V) - \sigma_\alpha^i|||_{h,k;\Psi} \to 0, \text{ for } |V|, L \to \infty, \tag{5.18}
\]

for any \( h \in \mathcal{C} \), \( k \in \mathbb{N} \) and \( \Psi \in \mathcal{F} \). Since \( \beta_{V,L}^i(\sigma_\alpha^V) - \sigma_\alpha^i = \sum_{n=1}^{\infty} \frac{(i\gamma Q_L N)^n}{n!} [\sigma_3^V, \sigma_\alpha^i]_n \), we get

\[
K_{V,L} \leq \sum_{n=1}^{\infty} \frac{|\gamma|^n}{n!} \|h(M)(Q_L N)^n M^k\| \|[[\sigma_3^V, \sigma_\alpha^i]]_n \Psi\|.
\]

Now we recall the estimate \( ||[\sigma_3^V, \sigma_\alpha^i]_n \Psi|| \leq \left( \frac{2}{|V|} \right)^n \), Ref. [1], and we observe that,

\[
\|h(M)(Q_L N)^n M^k\| = \|h(M)Q_L N^n M^k\| \leq \|Q_L N^n\| \|h(M)M^k\| \leq \Gamma_{h,k}(L_{CL})^n,
\]

where we have defined \( \Gamma_{h,k} := \|h(M)M^k\| \) and we have used the bound \( ||Q_L N|| \leq L_{CL} \).

Putting everything together, we conclude that

\[
K_{V,L} \leq \Gamma_{h,k} \sum_{n=1}^{\infty} \frac{(2\gamma L_{CL}/|V|)^n}{n!} = \Gamma_{h,k} \left( e^{2\gamma L_{CL}/|V|} - 1 \right),
\]

which goes to zero for \( |V| \) and \( L \) divergent, as far as \( \frac{L_{CL}}{|V|} \to 0 \). Therefore, but for contributions which go to zero for \( |V| \) going to infinity, we can write

\[
\alpha_{V,L}^i(\sigma_\alpha^V) = \sigma_\alpha^i \cos^2(S_3^V) - 2\epsilon_{3\alpha\beta}\sigma_\beta^i \sin(S_3^V) \cos(S_3^V) + \sigma_3^i \sigma_3^i \sigma_3^i \sin^2(S_3^V), \tag{5.19}
\]

Using now the expression above and since the spin part of the seminorms \( ||||_{h,k;\Psi} \) are exactly the ones of the \( \mathcal{F} \)-strong topology, Ref. [7], we conclude that the sequence \( \alpha_{V,L}^i(\sigma_\alpha^V) \) is \( \tau \)-Cauchy, so that its limit exists in the algebra \( \mathcal{A} \). \( \square \)

Remarks:
(a) The condition on the two cutoffs can be considered as a sort of “non-uniformity” in the model: it is impossible to consider separately the two cutoffs when the thermodynamical limit is concerned: on the contrary, \(|V|\) and \(L\) must be related as described above.

(b) As already remarked in Ref. [1], we could substitute in the description of the model, \(B(\mathcal{H}_{\text{spin}})\) with the algebra \(\mathcal{L}^+(\mathcal{D})\), \(\mathcal{D}\) being the domain of a certain 'number' operator for the spin variables, dense in \(\mathcal{H}\). However, since \(\sigma_3^V\) and all the other relevant spin variables are bounded, it is not worthwhile to proceed in this direction here.

(c) With the same kind of estimates considered above, or with some simple argument based again on the results of Ref. [7], it is also possible to prove the existence of the limits of the following quantities: \(\alpha_{V,L}^V(a\sigma_i^j)\), \(\alpha_{V,L}^V(\sigma_{a_1}^{i_1}......\sigma_{a_n}^{i_n})\), \(\alpha_{V,L}^V(a^n)\), \(\alpha_{V,L}^V((a^\dagger)^n)\). Here \(n\) is an arbitrary integer.

We can conclude, therefore, that the algebraic dynamics for the model (5.1) can be rigorously defined using the regularization (5.7) and that a natural topological \(*\)-algebra where to consider the model is \(\mathcal{A}[\tau] = B(\mathcal{H}_{\text{spin}}) \otimes \mathcal{L}^+(\mathcal{D})\), \(B(\mathcal{H}_{\text{spin}})\) taken with its \((\mathcal{F}_-)\)strong topology and \(\mathcal{L}^+(\mathcal{D})\) with the physical one.

**VI Outcome and Future Projects**

In this paper some mathematically, if not physically, interesting models have been analyzed making use of the algebras of unbounded operators originally introduced by Lassner. In particular, we have shown that for the free mq it is possible to introduce a mq 'quasi occupation number' cutoff, whose removal can be performed in the complete topological \(*\)-algebra \(\mathcal{L}^+(\mathcal{D})\), \(\mathcal{D}\) being the domain of all powers of the mq 'quasi number operator' \(N\).

Moreover, for an interacting model of spins and mq, the same kind of regularization allows a rigorous definition of the cutoffed hamiltonian. Again, this cutoff can be removed, together with its other volume cutoff, working in a large topological \(*\)-algebra \(\mathcal{A}[\tau] = B(\mathcal{H}_{\text{spin}}) \otimes \mathcal{L}^+(\mathcal{D})\), where \(B(\mathcal{H}_{\text{spin}})\) and \(\mathcal{L}^+(\mathcal{D})\) are endowed respectively with the strong and with the physical topologies.

For what concerns our future projects, our main goal is to analyze, within the mathematical structure proposed here and in Ref. [1], some more physically realistic models like the original conservative Dicke model, Ref. [10], or its non-conservative generalization proposed by Alli and Sewell, Ref. [11]. We also are interested in a deeper analysis of the possibility of regularizing a boson model mapping it in a (ordinary) quon one, as we have
briefly discussed in the Remark of Section II.

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A Appendix: Proof of Lemma 4.1.

In this Appendix we will prove those statements which are not already proven in Ref. [1]. In particular, we will not prove equations (4.4), (4.5), (4.6) as well as (4.8) which is trivial.

The commutation rules in (4.7) are easily proven: since $aa^\dagger = 1 + f a^\dagger a = 1 + f N_0$ then $[a, a^\dagger] = (1 + f N_0) - N_0 = 1 + f N_0$. From this also follows $[N_0, a] = [a^\dagger, a] a = -(1 + f N_0) a$.

The proof of commutation rules in (4.9) goes like follows:

$$[a_L, a] = [Q_L a Q_L, a] = [a Q_L, a] = a [Q_L, a] = a \Pi_L a = a^2 \Pi_{L+1} = \Pi_{L-1} a^2,$$

where we have used equations (4.4), (4.5) and (4.6). Moreover, by means of the same relations, together with (4.7), we get

$$[a_L^\dagger, a] = [a^\dagger Q_{L-1}, a] = [a^\dagger, a] Q_{L-1} + a^\dagger [Q_{L-1}, a] = -Q_{L-1} - f Q_{L-1} N_0 + Q_L N_0.$$

We end with the proof of the estimates in (4.10). The equality $\|\Pi_L a \Pi_s\| = \sqrt{c_s \delta_{l,s-1}}$ is a consequence of the following equalities:

$$\|\Pi_L a \Pi_s\|^2 = \|a \Pi_{L+1} \Pi_s\|^2 = \delta_{l,s-1} \|a \Pi_s\|^2 = \delta_{l,s-1} \sup_{\|\varphi\| \leq 1} < a \Pi_s \varphi, a \Pi_s \varphi > = \delta_{l,s-1} \sup_{\|\varphi\| \leq 1} < \Pi_s \varphi, N \Pi_s \varphi > = c_s \delta_{l,s-1},$$

while $\|a_L\|^2 \leq L c_L$ follows from:

$$\|a_L\|^2 = \|a Q_L\|^2 = \sup_{\|\varphi\| \leq 1} < Q_L \varphi, N Q_L \varphi > = \sup_{\|\varphi\| \leq 1} < (\Pi_0 + \cdots \Pi_L) \varphi, (c_0 \Pi_0 + c_1 \Pi_1 + \cdots c_L \Pi_L) \varphi > = \sup_{\|\varphi\| \leq 1} \left( c_0 \|\Pi_0 \varphi\|^2 + c_1 \|\Pi_1 \varphi\|^2 + \cdots + c_L \|\Pi_L \varphi\|^2 \right).$$

Since $f > 1$ the sequence $\{c_l\}$ is increasing. Moreover $c_0 = 0$. Therefore we get $\|a_L\|^2 \leq \sup_{\|\varphi\| \leq 1} c_L (\|\Pi_1 \varphi\|^2 + \cdots \|\Pi_L \varphi\|^2) \leq L c_L.$
Appendix: The topology $\tau$ for the Interacting Models

Let $\mathcal{H}$ and $\mathcal{H}_{spin}$ be the Hilbert spaces respectively of the mq and the spin. $\mathcal{D}$ is the subset of $\mathcal{H}$ defined as in Section II and $\mathcal{A} = B(\mathcal{H}_{spin}) \otimes \mathcal{L}^+(\mathcal{D})$ is the relevant $*$-algebra for the model in Section IV.

We start defining, for $X = X_1 \otimes X_2 \in B(\mathcal{H}_{spin}) \otimes B(\mathcal{H})$, and for $\Psi \in \mathcal{F}$ given in (5.4), the following 'strong' seminorms:

$$
\|X\|_\Psi = \sup_{\|\chi_1\| \leq 1, \|\chi_2\| \leq 1, \|\phi_2\| \leq 1} |< \chi_1 \otimes \chi_2, (X_1 \otimes X_2)\Psi \otimes \phi_2 > | = \|X_1\|_\Psi \|X_2\|, \quad (B.1)
$$

(obviously in $\|X_1\|_\Psi$ the norm is the one in $\mathcal{H}_{spin}$, while in $\|X_2\|$ is the one in $\mathcal{H}$). It is straightforwardly proven that these are really seminorms. It is also evident the meaning of the above definition: we are considering the $(\mathcal{F}-)$strong topology on $B(\mathcal{H}_{spin})$ and the usual norm topology on $B(\mathcal{H})$. At this point, since if $A \in \mathcal{L}^+(\mathcal{D})$ then both $f(M)AM^k$ and $M^kAf(M)$ are bounded operators on $\mathcal{H}$, the following definition appears as the most natural: for any $X \in \mathcal{A}$ we define the following seminorms

$$
\|\|X\|_{f,k,\Psi} = \max\left\{\|f(M)XM^k\|_\Psi, \|M^kXf(M)\|_\Psi\right\}. \quad (B.2)
$$

Again, it is not difficult to prove that $\{|\|\|_{f,k,\Psi}\}$ is a system of seminorms which define a topology $\tau$ and this topology makes of $\mathcal{A}$ a complete topological $*$-algebra.

We conclude this Appendix with the following estimate

$$
\|f(M)XM^k\|_\Psi \leq \sum_{l,s=1}^{\infty} f(c_l)c_s^k\|\Pi_l\Pi_s\|_\Psi,
$$

which is an easy consequence of the spectral decomposition of the operator $M$, and which is used many times in Section V.
References


