Multiresolution analysis and $QM_{\infty}$

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A multiresolution analysis of $\mathcal{L}^2(\mathbb{R})$ is an increasing sequence of closed subspaces

\[
\ldots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots \subset \mathcal{L}^2(\mathbb{R}), \tag{1}
\]

with $\bigcup_{j \in \mathbb{Z}} V_j$ dense in $\mathcal{L}^2(\mathbb{R})$ and $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$, and such that

(1) $f(x) \in V_j \iff f(2x) \in V_{j+1}$

(2) There exists a function $\phi \in V_0$, called a scaling function, such that $\{\phi(x - k), k \in \mathbb{Z}\}$ is an o.n. basis of $V_0$.

(1) + (2) $\Rightarrow \{\phi_{j,k}(x) \equiv 2^{j/2} \phi(2^j x - k), k \in \mathbb{Z}\}$ is an o.n. basis of $V_j$.

Each $V_j$ can be interpreted as an approximation space: the approximation of $f \in \mathcal{L}^2(\mathbb{R})$ at the resolution $2^j$ is defined by its projection onto $V_j$. The additional details needed for increasing the resolution from $2^j$ to $2^{j+1}$ are given by the projection of $f$ onto the orthogonal complement $W_j$ of $V_j$ in $V_{j+1}$:

\[
V_j \oplus W_j = V_{j+1}, \tag{2}
\]
and we have:

$$\bigoplus_{j \in \mathbb{Z}} W_j = L^2(\mathbb{R}).$$

(3)

Now, there exists a function $\psi$, the mother wavelet, explicitly computable from $\phi$, such that \( \{\psi_{j,k}(x) \equiv 2^{j/2}\psi(2^j x - k), j, k \in \mathbb{Z}\} \) constitutes an orthonormal basis of $L^2(\mathbb{R})$.

The construction of $\psi$ proceeds as follows. First, the inclusion $V_0 \subset V_1$ yields the relation

$$\phi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} h_n \phi(2x - n), \quad h_n = \langle \phi_1^n | \phi \rangle. \quad (4)$$

Then one uses these coefficients to define the function $\psi$ as

$$\psi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} (-1)^{n-1} h_{n-1} \phi(2x - n). \quad (5)$$

$\psi(x)$ can be used to generate the wavelet transform of a given $L^2$-function $f(x)$ as

$$(T_{\text{wav}} f)(a, b) = |a|^{-1/2} \int dx f(x) \psi\left( \frac{x - b}{a} \right),$$

where $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$. $a$ and $b$ are the zooming and the translation parameter.
EXAMPLES: (1) Haar MRA:

\[ h(x) = \begin{cases} 
1, & \text{if } 0 \leq x < 1/2 \\
-1, & \text{if } 1/2 \leq x < 1 \\
0, & \text{otherwise} 
\end{cases} \quad (6) \]

This function gives an explicit example of a wavelet o.n. basis in \( L^2(\mathbb{R}) \), obtained via the usual formula

\[ h_{mn}(x) = 2^{-m/2}h(2^{-m}x - n). \quad (7) \]

(2) Littlewood Paley

\[ \Psi(x) = (\pi x)^{-1}(\sin 2\pi x - \sin \pi x). \quad (8) \]

The behavior of this function is, in a sense, complementary to that of the Haar wavelet: it is very well localized in frequency space (it has a compact support)

\[ \hat{\Psi}(\omega) = \begin{cases} 
(2\pi)^{-1/2}, & \text{if } \pi \leq |\omega| \leq 2\pi \\
0, & \text{otherwise,} 
\end{cases} \]
(3) Journé basis

\[ \hat{\Psi}(\omega) = \begin{cases} 
(2\pi)^{-1/2} & \text{if } \frac{4\pi}{7} \leq |\omega| \leq \pi \text{ and } 4\pi \leq |\omega| \leq \frac{32\pi}{7} \\
0 & \text{otherwise} 
\end{cases} \]  

(9)

(3) More Examples:

- splines (linear, quadratic, cubic,...),
- Daubechies’ wavelets,
- sinc wavelets,...

**Problem:** how to find new scaling functions \( \phi \) and MR?
Definition: We call **relevant** any sequence $h = \{h_n, n \in \mathbb{Z}\}$ which satisfies the following properties:

1. **(r1)** $\sum_{n \in \mathbb{Z}} h_n h_{n+2l} = \delta_{l,0}$;
2. **(r2)** $h_n = O\left(\frac{1}{1+|n|^2}\right)$, $n \gg 1$;
3. **(r3)** $\sum_{n \in \mathbb{Z}} h_n = \sqrt{2}$;
4. **(r4)** $H(\omega) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} h_n e^{-i\omega n} \neq 0 \quad \forall \omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Relevant sequences $\Rightarrow$ Scaling functions and Mother Wavelets (Mallat’s algorithm).

**Main question:** how to produce relevant sequences?
MRA $\leftrightarrow$ FQHE

The old results: FQHE

Full hamiltonian:

\[
H^{(N)} = H_0^{(N)} + \lambda (H_C^{(N)} + H_B^{(N)}),
\]

\[
H_0^{(N)} = \sum_{i=1}^{N} H_0(i).
\]

Here \(H_0(i)\) describes the minimal coupling of the \(i\)-th electron with the magnetic field:

\[
H_0 = \frac{1}{2} \left( p + A(r) \right)^2 = \frac{1}{2} \left( p_x - \frac{y}{2} \right)^2 + \frac{1}{2} \left( p_y + \frac{x}{2} \right)^2.
\]

\(H_C^{(N)}\) is the Coulomb interaction between charged particles, and \(H_B^{(N)}\) is the interaction of the charges with the background. Both are small compared with \(H_0^{(N)}\)!

Let us introduce the following operators:

\[
P' = p_x - y/2, \quad Q' = p_y + x/2, \quad P = p_y - x/2, \quad Q = p_x + y/2.
\]

Then we have

\[
H_0 = \frac{1}{2} (Q'^2 + P'^2).
\]
and

\[ [Q, P] = [Q', P'] = i, \quad [Q, P'] = [Q', P] = [Q, Q'] = [P, P'] = 0. \] (15)

Therefore, defining the magnetic translation operators \( T(\vec{a}_i) \) for a square lattice with basis \( \vec{a}_1 = a(1, 0), \vec{a}_2 = a(0, 1), a^2 = 2\pi \) by

\[ T_1 := T(\vec{a}_1) = e^{iaQ}, \quad T_2 := T(\vec{a}_2) = e^{iaP}, \] (16)

we find

\[ [T(\vec{a}_1), T(\vec{a}_2)] = [T(\vec{a}_1), H_0] = [T(\vec{a}_2), H_0] = 0, \] (17)

and, given a generic function \( f(x, y) \in L^2(\mathbb{R}^2) \),

\[ f_{m,n}(x, y) := T_1^m T_2^n f(x, y) = (-1)^{mn} e^{i\frac{a}{2}(my-nx)} f(x+ma, y+na). \] (18)

The wave function in the \((x, y)\)-space is related to its \( PP'\)-expression by the formula

\[ \Psi(x, y) = \frac{e^{ixy/2}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(xP'+yP+PP')} \Psi(P, P') dPdP', \] (19)
which can be easily inverted:

\[
\Psi(P, P') = \frac{e^{-iPP'}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(xP'+yP+xy/2)} \Psi(x, y) \, dx \, dy.
\]

(20)

The ground state of \( H_0 \) must have the form \( f_0(P')h(P) \), where

\[
f_0(P') = \pi^{-1/4} e^{-P'^2/2},
\]

(21)

while the function \( h(P) \) is arbitrary, which manifests the degeneracy of the LLL, and should be fixed by the interaction. With \( f_0 \) as above formula (19) becomes

\[
\Psi(x, y) = \frac{e^{ixy/2}}{\sqrt{2\pi^{3/4}}} \int_{-\infty}^{\infty} e^{iyP} e^{-(x+P)^2/2} h(P) \, dP,
\]

(22)

which produces the lattice as \( \psi_{m,n}(x, y) = T_m^1 T_n^2 \Psi(x, y) \).

Our main result is that, if the following ONC holds,

\[
< \psi_{l_1,2l_2}, \psi_{0,0} > = \int_{-\infty}^{\infty} dx e^{-2il_2ax} \hat{h}(x - l_1a)h(x) =
\]

\[
= \int_{-\infty}^{\infty} dp e^{il_1ap} \hat{h}(p - 2l_2a)\hat{h}(p) = \delta_{l_1,0}\delta_{l_2,0},
\]

(23)

for all \( l_1, l_2 \in \mathbb{Z} \), where \( \hat{h}(p) \) is the Fourier transform of \( h(x) \),
then, defining

\[ h_n = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} dpe^{-inx} h(x), \]  

(24)

it follows automatically that

\[ \sum_{n \in \mathbb{Z}} h_n \overline{h_{n+2l}} = \delta_{l,0}. \]  

(25)

We also have proved the opposite implication: given any relevant sequence we can produces an o.n. set in the LLL.

**Therefore:** At that stage we believed in the existence of a deep relationship between FQHE and MRA. We also analyzed the other requirements of a relevant sequence in that perspective.....
New results

.....but this is not strictly necessary: the procedure is only related to the unitary map between two pairs of conjugate operators. All the results deduced are model independent!

Consider the operators \(((\hat{x}, \hat{p}_x), (\hat{y}, \hat{p}_y))\) and \(((\hat{x}_1, \hat{p}_1), (\hat{x}_2, \hat{p}_2))\), satisfying

\[
[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i, \quad [\hat{x}_1, \hat{p}_1] = [\hat{x}_2, \hat{p}_2] = i
\]

Let \(\xi_x\) and \(\eta_y\) be (generalized) eigenstates of \(\hat{x}\) and \(\hat{y}\): \(\hat{x}\xi_x = x\xi_x\), \(\hat{y}\eta_y = y\eta_y\), and \(\xi'_{x_1}\) and \(\eta'_{x_2}\) eigenstates of the new position operators \(\hat{x}_1\) and \(\hat{x}_2\). We have

\[
\int dx \int dy |\xi_{x,y}\rangle \langle \xi_{x,y}| = \int dx_1 \int dx_2 |\xi'_{x_1,x_2}\rangle \langle \xi'_{x_1,x_2}| = 1,
\]

where \(\xi_{x,y} = \xi_x \otimes \eta_y\) and \(\xi'_{s,t} = \xi'_s \otimes \eta'_t\). Any \(\Psi \in \mathcal{H}\) can be written in the \((x, y)\)-coordinates or in the \((x_1, x_2)\)-coordinates:

\[
\Psi(x, y) = \langle \xi_{x,y} | \Psi > \quad \text{and} \quad \Psi'(x_1, x_2) = \langle \xi'_{x_1,x_2} | \Psi >,
\]
which are related as

$$\Psi(x, y) = \int dx_1 \int dx_2 \langle \xi_{x,y} | \xi'_{x_1,x_2} \rangle \Psi'(x_1, x_2)$$

$$\Psi'(x_1, x_2) = \int dx \int dy \langle \xi'_{x_1,x_2} | \xi_{x,y} \rangle \Psi(x, y)$$

**Remark:** $\Psi(x, y)$ and $\Psi'(x_1, x_2)$ are different representations of the same element of $\mathcal{H}$.

Let us choose $\Psi'(x_1, x_2) = \varphi(x_1) h(x_2)$ and let us define three commuting operators: $H = H(\hat{x}_1, \hat{p}_1) = H^\dagger$, $T_1 = e^{ia\hat{x}_2}$ and $T_2 = e^{ia\hat{p}_2}$. Here $a^2 = 4\pi$. We still call the unitary operators *magnetic translations* and $H$ the *hamiltonian*.

**But there is no physical system behind, now.**

As for the FQHE we require orthonormality of the functions $\Psi_{h,\ell}(x, y) = T_1^{\ell_1} T_2^{\ell_2} \Psi_{h}(x, y)$, which generate a lattice with cell area $= 4\pi$.

**Remark:** In the FQHE the function $\varphi(x_1)$ was the ground state of $H$. We will show that this is unessential.
We find that, independently of the model, that is, of the explicit expressions for $H$, $T_1$ and $T_2$, as well as of the choice of $((\hat{x}_1, \hat{p}_1), (\hat{x}_2, \hat{p}_2))$, (mainly) due to the resolution of the identity above, if $\varphi$ has $L^2(\mathbb{R})$-norm 1, then:

$$< \Psi_{h,(l_1,l_2)}, \Psi_{h,(0,0)}> = \int_{\mathbb{R}} ds \, h(s) \overline{h(s + al_2)} e^{-isal_1} =$$

$$= \int_{\mathbb{R}} dp \, \hat{h}(p) \overline{\hat{h}(p - al_1)} e^{-ipal_2} =: S_{l_1,l_2}^{(h)}$$

which does not depend on the choice of $\varphi$, [B, JMP, 2001], [B, JPA, 1996]. We call ONC the equation

$$S_{l_1,l_2}^{(h)} = \delta_{l_0}.$$

Solutions of the ONC can be easily found using a relevant sequence $h = \{h_n, n \in \mathbb{Z}\}$ \((\sum_{n \in \mathbb{Z}} h_n \overline{h_{n+2l}} = \delta_{l_0})$$:

$$h(s) = \begin{cases} \frac{1}{\sqrt{a}} \sum_{n \in \mathbb{Z}} h_n e^{-isna/2}, & s \in [0, a[, \\ 0 & \text{otherwise} \end{cases} \quad \text{(26)}$$

**Remark 1:** any MRA produces an o.n. basis in the LLL (as well as in all the other Landau levels!).

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**Remark 2:** If we have a d-MRA, we still find solutions of the ONC if $a^2 = 2\pi d$.

**Remark 3:** Since $H, T_1, T_2$ may have no physical meaning, it is not very relevant to use MRA to produce o.n. sets in the Landau levels. We do the opposite: we use the ONC to produce MRA!!!!

**The recipe:**

(1) Find a solution $h(s)$ of the ONC.

(2) Put

$$h_n = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} h(s) e^{i s n a/2} ds = \sqrt{\frac{2\pi}{a}} \hat{h} \left( -\frac{na}{2} \right)$$

Then, using the Poisson summation formula, we can easily see that

$$\sum_{n \in \mathbb{Z}} h_n h_{n+2l} = \delta_{l0}.$$ 

**Remark 4:** It is much simpler to find solutions of the ONC than, directly, sequences satisfying (r1).
Examples: solutions of the ONC and related sequences

Obvious solutions of the ONC are the ones arising from a MRA. Other solutions are:

(1) the trivial ones:

\[ h(s) = \begin{cases} \frac{1}{\sqrt{a}}, & s \in [0, a[, \\ 0 & \text{otherwise} \end{cases}, \quad \hat{h}(p) = \begin{cases} \sqrt{\frac{2}{a}}, & p \in [0, a/2[, \\ 0 & \text{otherwise} \end{cases} \]

then \( h_n = \delta_{n0} \). Trivially we have \( \sum_{n \in \mathbb{Z}} h_n \overline{h_{n+2l}} = \delta_{l0} \).

(2) the effects of a phase:

\[ h(s) = \begin{cases} \frac{e^{-is}}{\sqrt{a}}, & s \in [0, a[, \\ 0 & \text{otherwise} \end{cases} \]

then \( h_n = i(1 - e^{-ia}) \frac{1}{2\pi n - a} \). \( \sum_{n \in \mathbb{Z}} h_n \overline{h_{n+2l}} = \delta_{l0} \) is not trivial!

(3) close to the sinc wavelets:

\[ h(s) = \begin{cases} \sqrt{\frac{2}{a}}, & s \in [0, a/2[, \\ 0 & \text{otherwise}, \end{cases} \]
\[ h(s) = \begin{cases} 
\frac{1}{\sqrt{2a}}, & s \in [0, a/2] \cup [2a, 3a[, \\
-\frac{1}{\sqrt{2a}}, & s \in [a/2, a[, \\
0 & \text{otherwise}
\end{cases} \]

and

\[ h(s) = \begin{cases} 
\frac{1}{\sqrt{2a}}, & s \in [0, a/2] \cup [a, 2a[, \\
-\frac{1}{\sqrt{2a}}, & s \in [a/2, a[, \\
0 & \text{otherwise}
\end{cases} \]

they all satisfy the ONC and produce the following sequence:

\[ h_0 = \frac{1}{\sqrt{2}}, \ h_{2n} = 0 \text{ if } n \neq 0, \ \text{and} \ h_{2n+1} = \frac{i\sqrt{2}}{\pi(2n+1)}, \] which satisfies (r1).

(4) the Haar wavelet:

\[ \hat{h}(p) = \begin{cases} 
\frac{1}{\sqrt{a}}, & p \in [0, a[, \\
0 & \text{otherwise}
\end{cases} \]

then \( h_0 = h_{-1} = \frac{1}{\sqrt{2}} \), which are the Haar coefficients.

(4) more compactly supported solutions (in \( p \))
\[ \hat{h}(p) = \begin{cases} \frac{1}{\sqrt{2a}}, & p \in [0, a/2] \cup [2a, 3a], \\ -\frac{1}{\sqrt{2a}}, & s \in [a/2, a], \\ 0 & \text{otherwise} \end{cases} \]

produces \( h_0 = h_{-4} = h_{-5} = -h_{-1} = \frac{1}{2} \), while

\[ \hat{h}(p) = \begin{cases} \frac{1}{\sqrt{2a}}, & s \in [0, a/2] \cup [a, 2a], \\ -\frac{1}{\sqrt{2a}}, & s \in [a/2, a], \\ 0 & \text{otherwise} \end{cases} \]

produces \( h_0 = h_{-2} = h_{-3} = -h_{-1} = \frac{1}{2} \).

**Remark:** The ONC satisfies an useful symmetry property: if \( h(s) \) is a solution of the ONC, another solution is obtained simply by taking the inverse Fourier transform of a \( \hat{g}(p) \equiv h(p) \).
More solutions: the orthonormalization trick

More solutions of the ONC, and more relevant sequences as a consequence, can be found using this ONT:

let \( h(s) \) be a generic \( L^2 \) function and let us compute \( S^{(h)}_{l_1,l_2} = \langle \psi_{l_1,l_2}^{(h)}, \psi_{0,0}^{(h)} \rangle \). If \( S^{(h)}_{l_1,l_2} \neq \delta_{l_1,0} \delta_{l_2,0} \) then the \( \psi_{l_1,l_2}^{(h)}(\vec{r}) \) generate a non o.n. lattice (in \( L^2(\mathbb{R}^2) \)). However, [BMS, PRB (1993)], we can find an o.n. set in \( L^2(\mathbb{R}^2) \), invariant under magnetic translations, simply by considering

\[
\psi_{l_1,l_2}^{(H)}(\vec{r}) = T_1^{l_1} T_2^{l_2} \psi_{0,0}^{(H)}(\vec{r}), \quad \text{where} \quad \psi_{0,0}^{(H)}(\vec{r}) = \sum_{\vec{n} \in \mathbb{Z}^2} f_{\vec{n}} \psi_{\vec{n}}^{(h)}(\vec{r}),
\]

and the coefficients \( f_{\vec{n}} \) are fixed by requiring that

\[
S^{(H)}_{l_1,l_2} = \langle \psi_{l_1,l_2}^{(H)}, \psi_{0,0}^{(H)} \rangle = \delta_{l_1,0} \delta_{l_2,0}
\]

In this way we deduce that \( H(s) = \sum_{\vec{n} \in \mathbb{Z}^2} f_{\vec{n}} e^{i\alpha n_1 h(s + an_2)} \), and the related coefficients are

\[
H_n = \sqrt{\frac{2\pi}{a}} \hat{H} \left( -\frac{na}{2} \right) = \ldots = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \frac{ds}{\sqrt{S^{(h)}(as,0)}} e^{i\alpha s n/2},
\]
where

\[ S^{(h)}(\vec{p}) = \sum_{\vec{n} \in \mathbb{Z}^2} S^{(h)}_{\vec{n}} e^{i\vec{p} \cdot \vec{n}}. \]

Then \( \sum_{n \in \mathbb{Z}} H_n \overline{H_{n+2l}} = \delta_{l,0}. \)

Examples of this construction are contained in [B., JMP, 2003]

**Remarks:**

1. if \( < \psi_{l_1,l_2}^{(h)}, \psi_{0,0}^{(h)} > = \delta_{l_1,0}\delta_{l_2,0} \) then \( S^{(h)}(\vec{p}) = 1 \)

and we go back to the original result: \( H_n = h_n \) and no o.n. trick is needed!

2. again the procedure is *model independent!* It only depends on the unitary map from \( ((\hat{x}, \hat{p}_x), (\hat{y}, \hat{p}_y)) \) to \( ((\hat{x}_1, \hat{p}_1), (\hat{x}_2, \hat{p}_2)) \)

and not on the particular expressions for \( H, T_1 \) and \( T_2. \)
The other conditions

- **Condition (r2),** \( h_n = O\left(\frac{1}{1+|n|^2}\right), \ n \gg 1 \)

For this to be satisfied we have a very easy criterion: since
\[
h_n = \sqrt{\frac{2\pi}{a}} \hat{h} \left( -\frac{na}{2} a \right),
\]
it is enough to look for solutions of the ONC such that \( \hat{h}(p) \) is compactly supported.

It would be enough to find solutions \( h(s) \) of the ONC regular enough: the more regular \( h(s) \) is, the more rapidly \( h_n \) goes to zero when \( n \to \infty \).

If we adopt the o.n. trick, then we can repeat the same considerations with \( h(s) \) replaced by \( \tilde{h}(s) = \frac{h(s)}{\sqrt{S(h)(as,0)}}. \)

- **Condition (r3),** \( \sum_{n \in \mathbb{Z}} h_n = \sqrt{2} \)

A necessary condition for (r3) to hold \( h(s) \) must satisfies the ONC and one of the following equalities: \( \sum_{n \in \mathbb{Z}} h(na) = \sqrt{\frac{2}{a}} \) or \( \sum_{n \in \mathbb{Z}} \hat{h}\left(\frac{na}{2}\right) = \sqrt{\frac{\pi}{a}}. \) If we adopt the o.n. trick then we can relax the assumption that \( h(s) \) satisfies the ONC. It
is enough to have one of the following equalities satisfied:

\[
\sum_{n \in \mathbb{Z}} h(an) = \sqrt{\sum_{\bar{l} \in \mathbb{Z}^2} \hat{h} \left( \frac{a l_1}{2} \right) \hat{h} \left( \frac{a l_1}{2} + a l_2 \right)}
\]

or

\[
\sum_{n \in \mathbb{Z}} \hat{h} \left( \frac{an}{2} \right) = \sqrt{2 \sum_{\bar{l} \in \mathbb{Z}^2} \hat{h} \left( \frac{a l_2}{2} \right) \hat{h} \left( \frac{a l_2}{2} + a l_2 1 \right)}
\]

(The Haar example above satisfies both!)

- **Condition (r4)**, \(\sum_{n \in \mathbb{Z}} h_n e^{-i\omega n} \neq 0\) for all \(\omega \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]\).

For that we find that the following condition must be satisfied by the seed function \(h(s)\):

\[
\sum_{n \in \mathbb{Z}} h \left( a \left( n + \frac{\omega}{2\pi} \right) \right) \neq 0, \quad \omega \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]
\]

whether \(h(s)\) satisfies the ONC or not.
Future Plains

1. We have to consider still the examples from the point of view of MRA: what kind of mother wavelets do we get? For that we need to find more examples: this is work in progress!

2. How to impose stronger conditions (e.g. on the support of the mother wavelets or its regularity)?

3. More physical applications

4. Explore the underlying symmetry between \((\hat{x}_1, \hat{p}_1)\) and \((\hat{x}_2, \hat{p}_2)\): this produces examples of the Tomita-Takesaki procedure for constructing, e.g., a modular operator. A first concrete application has been explored in [Ali, B., JMP, submitted] where the FQHE has been analyzed.

5. ..................