Algebras of unbounded operators and physical applications: a survey

Fabio Bagarello

Bialowieza – July 2006
I. Plan of the talk

1. few words of motivation
I. Plan of the talk

1. few words of motivation

2. few words on non relativistic ordinary quantum mechanics and quantum mechanics for systems with infinitely degrees of freedom
I. Plan of the talk

1. few words of motivation

2. few words on non relativistic ordinary quantum mechanics and quantum mechanics for systems with infinitely degrees of freedom

3. a list of (physically relevant) results and open points...
1. few words of motivation

2. few words on non relativistic ordinary quantum mechanics and quantum mechanics for systems with infinitely degrees of freedom

3. a list of (physically relevant) results and open points...

4. algebras of unbounded operators and their uses in $QM_\infty$
I. Plan of the talk

1. few words of motivation

2. few words on non relativistic ordinary quantum mechanics and quantum mechanics for systems with infinitely degrees of freedom

3. a list of (physically relevant) results and open points...

4. algebras of unbounded operators and their uses in $QM_\infty$

5. future projects
II. Motivations

Main goal: description of systems with a very large ($10^{24}$) number of degrees of freedom. We cannot deal with these kind of systems considering separately their constituents, otherwise we cannot explain their collective effects. We also cannot try to solve $10^{24}$ coupled Schrödinger equations and, after that, read the solution!
II. Motivations

Main goal: description of systems with a very large \((10^{24})\) number of degrees of freedom. We cannot deal with these kind of systems considering separately their constituents, otherwise we cannot explain their collective effects. We also cannot try to solve \(10^{24}\) coupled Schrödinger equations and, after that, read the solution!

Better to consider systems with infinite degrees of freedom: this allows a simpler analysis of, e.g., phase transitions.

Price to pay: the mathematical apparatus is rather sophisticated.
III. Ordinary quantum mechanics [NR]

Possible descriptions:

*Hilbert space description*

Observable $A$ of the physical system $\leftrightarrow$ self-adjoint operator $\hat{A}$ in some Hilbert space $\mathcal{H}$;
III. Ordinary quantum mechanics [NR]

Possible descriptions:

_Hilbert space description_

Observable $A$ of the physical system $\leftrightarrow$ self-adjoint operator $\hat{A}$ in some Hilbert space $\mathcal{H}$; 

pure states of the physical system $\leftrightarrow$ normalized vectors of $\mathcal{H}$;
III. Ordinary quantum mechanics [NR]

Possible descriptions:

\textit{Hilbert space description}

Observable $A$ of the physical system $\leftrightarrow$ self-adjoint operator $\hat{A}$ in some Hilbert space $\mathcal{H}$; pure states of the physical system $\leftrightarrow$ normalized vectors of $\mathcal{H}$; expectation values of $A$ $\leftrightarrow$ $\langle \psi, \hat{A}\psi \rangle = \rho_{\psi}(\hat{A}) = tr(P_{\psi} \hat{A})$;
III. Ordinary quantum mechanics [NR]

Possible descriptions:

Hilbert space description

Observable $A$ of the physical system $\leftrightarrow$ self-adjoint operator $\hat{A}$ in some Hilbert space $\mathcal{H}$;

pure states of the physical system $\leftrightarrow$ normalized vectors of $\mathcal{H}$;

expectation values of $A$ $\leftrightarrow$ $\langle \psi, \hat{A} \psi \rangle = \rho_{\psi}(\hat{A}) = \text{tr}(P_{\psi} \hat{A})$;

mixed states: $\hat{\rho} = \sum_j w_j \rho_{\psi_n}$, with $\sum_j w_j = 1$;
III. Ordinary quantum mechanics [NR]

Possible descriptions:

Hilbert space description

Observable $A$ of the physical system $\longleftrightarrow$ self-adjoint operator $\hat{A}$ in some Hilbert space $\mathcal{H}$; 

pure states of the physical system $\longleftrightarrow$ normalized vectors of $\mathcal{H}$; 

expectation values of $A$ $\longleftrightarrow$ $\langle \psi, \hat{A} \psi \rangle = \rho_\psi(\hat{A}) = \text{tr}(P_\psi \hat{A})$; 

mixed states: $\hat{\rho} = \sum_j w_j \rho_\psi$, with $\sum_j w_j = 1$; 

dynamics (Schrödinger representation): $U_t := e^{iHt/\hbar}$ then $\hat{\rho} \rightarrow \hat{\rho}_t = U_t^* \hat{\rho} U_t$, $H$ being the hamiltonian.
III. Ordinary quantum mechanics [NR]

Possible descriptions:

**Hilbert space description**

**Observable** $A$ of the physical system $\leftrightarrow$ self-adjoint operator $\hat{A}$ in some Hilbert space $\mathcal{H}$;
**pure states** of the physical system $\leftrightarrow$ normalized vectors of $\mathcal{H}$;
**expectation values** of $A$ $\leftrightarrow$ $\langle \psi, \hat{A}\psi \rangle = \rho_\psi(\hat{A}) = \text{tr}(P_\psi \hat{A})$;
**mixed states**: $\hat{\rho} = \sum_j w_j \rho_{\psi_n}$, with $\sum_j w_j = 1$;
**dynamics** *(Schrödinger representation)*: $U_t := e^{iHt/\hbar}$ then $\hat{\rho} \rightarrow \hat{\rho}_t = U_t^* \hat{\rho} U_t$, $H$ being the *hamiltonian*.
**dynamics** *(Heisenberg representation)*: $\hat{A} \rightarrow \hat{A}_t = U_t \hat{A} U_t^*$: $\frac{d}{dt} \hat{A}_t = i\frac{\hbar}{\hbar} [H, \hat{A}_t]$. 
III. Ordinary quantum mechanics [NR]

Possible descriptions:

**Hilbert space description**

Observable $A$ of the physical system $\leftrightarrow$ self-adjoint operator $\hat{A}$ in some Hilbert space $\mathcal{H}$;

pure states of the physical system $\leftrightarrow$ normalized vectors of $\mathcal{H}$;

expectation values of $A$ $\leftrightarrow$ $\langle \psi, \hat{A}\psi \rangle = \rho_{\psi}(\hat{A}) = tr(P_{\psi}\hat{A})$;

mixed states: $\hat{\rho} = \sum_j w_j \rho_{\psi_j}$, with $\sum_j w_j = 1$;

dynamics (Schrödinger representation): $U_t := e^{iHt/\hbar}$

then $\hat{\rho} \rightarrow \hat{\rho}_t = U_t^*\hat{\rho}U_t$, $H$ being the *hamiltonian*.

dynamics (Heisenberg representation): $\hat{A} \rightarrow \hat{A}_t = U_t\hat{A}U_t^*$: $\frac{d}{dt}\hat{A}_t = \frac{i}{\hbar}[H, \hat{A}_t]$.

They have the same physical content: $\hat{\rho}(\hat{A}_t) = \hat{\rho}_t(\hat{A})$. 
Algebraic description

The observables are elements of the C*-algebra $\mathcal{A}(=B(\mathcal{H}))$:

$$A, B \in \mathcal{A} \Rightarrow AB, A + B, \alpha A \in \mathcal{A},$$

there exists a norm $\| . \| : \mathcal{A} \rightarrow \mathbb{R}_+$ such that:

$$\|A + B\| \leq \|A\| + \|B\|; \quad \|\lambda A\| = |\lambda| \|A\|;$$

$$\|A^* A\| = \|A\|^2; \quad \|AB\| \leq \|A\| \|B\|,$$

and $\mathcal{A}$ is norm complete.
Algebraic description

The observables are elements of the C*-algebra $\mathfrak{A}(=B(\mathcal{H}))$:

$$A, B \in \mathfrak{A} \Rightarrow AB, A + B, \alpha A \in \mathfrak{A},$$

there exists a norm $\| \cdot \| : \mathfrak{A} \to \mathbb{R}^+$ such that:

$$\|A + B\| \leq \|A\| + \|B\|; \quad \|\lambda A\| = |\lambda| \|A\|;$$

$$\|A^* A\| = \|A\|^2; \quad \|AB\| \leq \|A\| \|B\|,$$

and $\mathfrak{A}$ is norm complete.

The states are linear, positive and normalized functional on $\mathfrak{A}$, $\rho(\hat{A})(= tr(\hat{\rho}A)$, where $\hat{\rho}$ is a trace-class operator):

$$\rho(\alpha_1 A + \alpha_2 B) = \alpha_1 \rho(A) + \alpha_2 \rho(B);$$

$$\rho(A^* A) \geq 0; \quad \rho(\mathbb{1}) = 1. \quad (\Rightarrow |\rho(A)| \leq \|A\|)$$
Dynamics in HR (for conservative systems):

$$\mathcal{A} \ni A \rightarrow \alpha^t(A) = U_t A U_t^* \in \mathcal{A}, \ \forall t.$$ 

$$\alpha^t$$ is a 1-parameter group of *-automorphisms of $$\mathcal{A}$$:

$$\alpha^t(\lambda A) = \lambda \alpha^t(A),$$

$$\alpha^t(A + B) = \alpha^t(A) + \alpha^t(B),$$

$$\alpha^t(AB) = \alpha^t(A) \alpha^t(B),$$

$$\|\alpha^t(A)\| = \|A\|,$$

$$\alpha^{t+s} = \alpha^t \alpha^s.$$ 

Remark:– in SR the time evolution is $$\hat{\rho} \rightarrow \hat{\rho}_t = \alpha^{t*} \hat{\rho}.$$
von Neumann theorem (1931):

for finite quantum mechanical systems there exists only one irreducible representation (but for unitary equivalence):
von Neumann theorem (1931):

for finite quantum mechanical systems there exists only one irreducible representation (but for unitary equivalence):

\[ [Q, P] = i\hbar \mathbb{I} \]

they can be represented on \( \mathcal{H} = L^2(\mathbb{R}) \) as follows:

\[ \hat{q}f(q) = qf(q), \quad \hat{p}f(q) = -i\hbar f'(q), \quad \forall f \in \mathcal{S}(\mathbb{R}). \]

If \( \hat{q}', \hat{p}' \) is a different irreducible representation of \( Q, P \) on \( \mathcal{H}' \), then there exists an unitary map \( V : \mathcal{H} \to \mathcal{H}' \) such that

\[ \hat{q}' = V \hat{q} V^*, \quad \hat{p}' = V \hat{p} V^*. \]
von Neumann theorem (1931):

for finite quantum mechanical systems there exists only one irreducible representation (but for unitary equivalence):

let \([Q, P] = i\hbar I\) they can be represented on \(\mathcal{H} = L^2(\mathbb{R})\) as follows: \(\hat{q}f(q) = qf(q), \hat{p}f(q) = -i\hbar f'(q), \forall f \in S(\mathbb{R}).\)

If \(\hat{q}', \hat{p}'\) is a different irreducible representation of \(Q, P\) on \(\mathcal{H}'\), then there exists an unitary map \(V : \mathcal{H} \to \mathcal{H}'\) such that \(\hat{q}' = V\hat{q}V^*, \hat{p}' = V\hat{p}V^*\).

There is no difference between the Hilbert space and the algebraic descriptions of QM
IV. $\text{QM}_\infty [T = 0]$

Radical difference: We may have inequivalent representations of the same physical system.
Radical difference: *We may have inequivalent representations of the same physical system.*

**Example:** infinite spin chain (Ising model): we have two inequivalent representations ($T = 0$):

$$\psi_0^{(+1)} = \ldots \otimes \uparrow \otimes \uparrow \otimes \uparrow \otimes \uparrow \ldots$$

and

$$\psi_0^{(-1)} = \ldots \otimes \downarrow \otimes \downarrow \otimes \downarrow \otimes \downarrow \ldots$$

where $\psi_0^{\pm 1}$ are two possible ground states of the energy. They are labeled by different values of the magnetization, $m \simeq < \psi_0^{(\pm 1)}, \frac{1}{|V|} \sum_{j \in V} \pi^{(\pm 1)}(\sigma^3_j) \psi_0^{(\pm 1)} > \rightarrow \pm 1$, and, therefore, they are *unitarily inequivalent*. This is a (first) example of *spontaneously broken symmetry*: we have a symmetry of the (formal) hamiltonian $H = -J \sum_j \sigma^3_j \sigma^3_{j+1}$ ($\sigma^3_j \rightarrow -\sigma^3_j$) which is not a symmetry of the ground state.

**Remark:** local actions on $\psi_0^{(+1)}$ do not change the results.
Haag and Kastler construction (∼1964)

Let $\Sigma$ our physical system, $\mathcal{V} \subset \mathbb{R}^d$ a finite $d$-dimensional region, $\mathcal{H}_\mathcal{V}$ the related Hilbert space (whose construction depends on $\Sigma$), $\mathcal{A}_\mathcal{V} = B(\mathcal{H}_\mathcal{V})$ the associated C*-algebra and $H_\mathcal{V}$ the self-adjoint energy operator for $\Sigma_\mathcal{V}$, the restriction of $\Sigma$ in $\mathcal{V}$. 
Haag and Kastler construction (~1964)

Let $\Sigma$ our physical system, $V \subset \mathbb{R}^d$ a finite $d$-dimensional region, $\mathcal{H}_V$ the related Hilbert space (whose construction depends on $\Sigma$), $\mathfrak{A}_V = B(\mathcal{H}_V)$ the associated C*-algebra and $H_V$ the self-adjoint energy operator for $\Sigma_V$, the restriction of $\Sigma$ in $V$.

$\{\mathfrak{A}_V\}$ satisfies the following properties:

- *isotony*: if $V_1 \subset V_2$ then $\mathfrak{A}_{V_1} \subset \mathfrak{A}_{V_2}$. Moreover $\|\cdot\|_2|_{V_1} = \|\cdot\|_1 (\Rightarrow \mathfrak{A}_{V_1}, \mathfrak{A}_{V_2} \subset \mathfrak{A}_{V_1 \cup V_2})$;

- if $V_1 \cap V_2 = \emptyset$ then $[\mathfrak{A}_{V_1}, \mathfrak{A}_{V_2}] = 0$;

Then we define $\mathfrak{A} = \overline{\mathfrak{A}_0\|\cdot\|}$, where $\mathfrak{A}_0 = \bigcup_V \mathfrak{A}_V$.

$\mathfrak{A}$ is the *quasi-local C*-algebra of the bounded observables.

On $\mathfrak{A}$ we introduce the *spatial translations* $\{\gamma_x\}$, which is a group of *-automorphisms of $\mathfrak{A}$: $\gamma_x \mathfrak{A}_V = \mathfrak{A}_{V+x}$, $\gamma_{x_1} \gamma_{x_2} = \gamma_{x_1+x_2}$.
The states of $\Sigma$ are positive, normalized linear functionals on $\mathcal{A}$ which, when restricted to $\mathcal{V}$, reduces to the states over the finite system $\mathcal{A}_\mathcal{V}$: they corresponds to a family of density matrices $\rho_\mathcal{V}$: $\hat{\rho}(A) = tr_\mathcal{V}(\rho_\mathcal{V} A)$ for each $A \in \mathcal{A}_\mathcal{V}$ (Here $tr_\mathcal{V}$ is the trace in $\mathcal{H}_\mathcal{V}$) satisfying the consistency condition $tr_\mathcal{V}(\rho_\mathcal{V} A) = tr_{\mathcal{V}'}(\rho_{\mathcal{V}'} A)$ $\forall A \in \mathcal{A}_\mathcal{V}, \mathcal{V} \subset \mathcal{V}'$. 
The states of $\Sigma$ are positive, normalized linear functionals on $\mathcal{A}$ which, when restricted to $\mathcal{V}$, reduces to the states over the finite system $\mathcal{A}_\mathcal{V}$: they correspond to a family of density matrices $\rho_\mathcal{V}$: $\hat{\rho}(A) = tr_{\mathcal{V}}(\rho_\mathcal{V} A)$ for each $A \in \mathcal{A}_\mathcal{V}$ (Here $tr_{\mathcal{V}}$ is the trace in $\mathcal{H}_\mathcal{V}$) satisfying the consistency condition $tr_{\mathcal{V}}(\rho_\mathcal{V} A) = tr_{\mathcal{V}'}(\rho_{\mathcal{V}'} A)$ $\forall A \in \mathcal{A}_\mathcal{V}$, $\mathcal{V} \subset \mathcal{V}'$.

Ruelle, Dell'Antonio, Doplicher theorem (1966): these physical states have zero probability to describe an infinite number of particles in a finite region. ($\rightarrow$ locally finite states)
The states of $\Sigma$ are positive, normalized linear functionals on $\mathcal{A}$ which, when restricted to $\mathcal{V}$, reduces to the states over the finite system $\mathcal{A}_V$: they correspond to a family of density matrices $\rho_V$: $\hat{\rho}(A) = tr_V(\rho_V A)$ for each $A \in \mathcal{A}_V$ (Here $tr_V$ is the trace in $\mathcal{H}_V$) satisfying the consistency condition $tr_V(\rho_V A) = tr_{V'}(\rho_{V'} A)$ $\forall A \in \mathcal{A}_V$, $V \subset V'$.

Ruelle, Dell'Antonio, Doplicher theorem (1966): these physical states have zero probability to describe an infinite number of particles in a finite region. ($\rightarrow$ locally finite states)

Pure state: $\rho$ is pure if it is not a convex combination of other states, i.e. if there is no $\rho_1, \rho_2$ and $\lambda \in ]0, 1[$ such that $\rho = \lambda \rho_1 + (1 - \lambda) \rho_2$. 

**Example 1: discrete system**

Let $X$ be an infinite lattice, $0 \in X$, and $\mathcal{H}_0$ a finite dimensional Hilbert space (e.g. $\mathcal{H}_0 = \mathbb{C}^2$ for Pauli matrices). Let $\mathcal{H}_x$ a copy of $\mathcal{H}_0$ localized in $x \in X$ and $\mathcal{H}_V = \bigotimes_{x \in V} \mathcal{H}_x$, $\mathcal{A}_V = B(\mathcal{H}_V)$. 
**Example 1: discrete system**

Let $X$ be an infinite lattice, $0 \in X$, and $\mathcal{H}_0$ a finite dimensional Hilbert space (e.g. $\mathcal{H}_0 = \mathbb{C}^2$ for Pauli matrices). Let $\mathcal{H}_x$ a copy of $\mathcal{H}_0$ localized in $x \in X$ and $\mathcal{H}_V = \bigotimes_{x \in V} \mathcal{H}_x$, $\mathcal{A}_V = B(\mathcal{H}_V)$. If $V \subset V'$ then $\mathcal{H}_{V'} = \mathcal{H}_V \otimes \mathcal{H}_{V' \setminus V}$ and, $\forall A \in \mathcal{A}_V$, $A \otimes 1_{V' \setminus V} \in \mathcal{A}_{V'}$. 
Example 1: discrete system
Let $X$ be an infinite lattice, $0 \in X$, and $\mathcal{H}_0$ a finite dimensional Hilbert space (e.g. $\mathcal{H}_0 = \mathbb{C}^2$ for Pauli matrices). Let $\mathcal{H}_x$ a copy of $\mathcal{H}_0$ localized in $x \in X$ and $\mathcal{H}_V = \bigotimes_{x \in V} \mathcal{H}_x$, $\mathcal{A}_V = B(\mathcal{H}_V)$. If $V \subset V'$ then $\mathcal{H}_{V'} = \mathcal{H}_V \otimes \mathcal{H}_{V' \setminus V}$ and, $\forall A \in \mathcal{A}_V$, $A \otimes I_{V' \setminus V} \in \mathcal{A}_{V'}$.

The map $\gamma_k(a^{(1)}_{x_1} a^{(2)}_{x_2} \ldots a^{(n)}_{x_n}) = a^{(1)}_{x_1+k} a^{(2)}_{x_2+k} \ldots a^{(n)}_{x_n+k}$ is an automorphism for each $k$: it represents the spatial translations.
Example 1: discrete system

Let $X$ be an infinite lattice, $0 \in X$, and $\mathcal{H}_0$ a finite dimensional Hilbert space (e.g. $\mathcal{H}_0 = \mathbb{C}^2$ for Pauli matrices). Let $\mathcal{H}_x$ a copy of $\mathcal{H}_0$ localized in $x \in X$ and $\mathcal{H}_V = \bigotimes_{x \in V} \mathcal{H}_x$, $\mathcal{A}_V = B(\mathcal{H}_V)$. If $V \subset V'$ then $\mathcal{H}_{V'} = \mathcal{H}_{V} \otimes \mathcal{H}_{V\setminus V}$ and, $\forall A \in \mathcal{A}_V$, $A \otimes I_{V\setminus V} \in \mathcal{A}_{V'}$.

The map $\gamma_k(a^{(1)}_{x_1} a^{(2)}_{x_2} \ldots a^{(n)}_{x_n}) = a^{(1)}_{x_1+k} a^{(2)}_{x_2+k} \ldots a^{(n)}_{x_n+k}$ is an automorphism for each $k$: it represents the *spatial translations*.

The *local energy* is given by considering the interactions of all the particles inside $V$,

$$H_V = \sum_r \sum_{x_1, \ldots, x_r \in V} V_r(x_1, x_2, \ldots, x_r)$$

where $V_r$ is the $r$-body interaction.
Example 2: continuous system

Starting point: Fermi or Bose commutation rules:

\[[\Psi[f], \Psi^\dagger[g]]_\pm = \langle \bar{g}, f \rangle,\]

\[[\Psi[f], \Psi[g]]_\pm = [\Psi^\dagger[f], \Psi^\dagger[g]]_\pm = 0, \forall f, g \in \mathcal{L}^2(\mathbb{R}).\]

Let \(\Phi_0\) be the **vacuum** of the theory, i.e. a vector such that \(\Psi[f]\Phi_0 = 0, \forall f \in \mathcal{L}^2(\mathbb{R})\). \(\mathcal{H}_\mathcal{V}\) is the norm closure of \(\Psi^\dagger[f_1] \ldots \Psi^\dagger[f_n]\Phi_0\), where each \(f_j\) is supported in \(\mathcal{V}\). Observe that \(\text{dim}(\mathcal{H}_\mathcal{V}) = \infty!!\)

\(\mathcal{A}_\mathcal{V} = \{X \in B(\mathcal{H}_\mathcal{V}) : [X, N_\mathcal{V}] = 0\}\), where \(N_\mathcal{V} = \int_\mathcal{V} \Psi^\dagger(x)\Psi(x) \, dx\) is the **number operator**.
**Example 2: continuous system**

Starting point: Fermi or Bose commutation rules:

\[ [\Psi[f], \Psi^\dagger[g]]_+ = \langle \overline{g}, f \rangle, \]

\[ [\Psi[f], \Psi[g]]_- = [\Psi^\dagger[f], \Psi^\dagger[g]]_- = 0, \forall f, g \in L^2(\mathbb{R}). \]

Let \( \Phi_0 \) be the vacuum of the theory, i.e. a vector such that \( \Psi[f] \Phi_0 = 0, \forall f \in L^2(\mathbb{R}) \). \( \mathcal{H}_V \) is the norm closure of \( \Psi^\dagger[f_1] \ldots \Psi^\dagger[f_n] \Phi_0 \), where each \( f_j \) is supported in \( V \). Observe that \( \dim(\mathcal{H}_V) = \infty \)!

\( \mathfrak{A}_V = \{ X \in B(\mathcal{H}_V) : [X, N_V] = 0 \} \), where \( N_V = \int_V \Psi^\dagger(x)\Psi(x) \, dx \) is the number operator.

The local hamiltonian, for 2-body interactions, is:

\[
\mathcal{H}_V = \frac{\hbar^2}{2m} \int_V d\mathbf{x} \left| \nabla \Psi(\mathbf{x}) \right|^2 + \\
+ \frac{1}{2} \int_V d\mathbf{x} \int_V d\mathbf{x}' \Psi^\dagger(\mathbf{x}) \Psi^\dagger(\mathbf{x}') V(\mathbf{x}, \mathbf{x}') \Psi(\mathbf{x}') \Psi(\mathbf{x})
\]
Example 2: continuous system

Starting point: Fermi or Bose commutation rules:

\[ [\Psi[f], \Psi^\dagger[g]]_{\pm} = \langle g, f \rangle, \]

\[ [\Psi[f], \Psi[g]]_{\pm} = [\Psi^\dagger[f], \Psi^\dagger[g]]_{\pm} = 0, \forall f, g \in \mathcal{L}^2(\mathbb{R}). \]

Let \( \Phi_0 \) be the vacuum of the theory, i.e. a vector such that \( \Psi[f]\Phi_0 = 0, \forall f \in \mathcal{L}^2(\mathbb{R}) \). \( \mathcal{H}_V \) is the norm closure of \( \Psi^\dagger[f_1] \ldots \Psi^\dagger[f_n]\Phi_0 \), where each \( f_j \) is supported in \( V \). Observe that \( \dim(\mathcal{H}_V) = \infty! \)

\( \mathcal{A}_V = \{ X \in B(\mathcal{H}_V) : [X, N_V] = 0 \} \), where \( N_V = \int_V \Psi^\dagger(x)\Psi(x) \, dx \) is the number operator.

The local hamiltonian, for 2-body interactions, is:

\[ H_V = \frac{\hbar^2}{2m} \int_V dx \, |\nabla \Psi(x)|^2 + \]

\[ + \frac{1}{2} \int_V dx \int_V dx' \Psi^\dagger(x)\Psi^\dagger(x')V(x, x')\Psi(x')\Psi(x) \]

Remark:– even in \( \mathcal{H}_V \) unbounded operators appear!!
Time evolution of $\Sigma$
Time evolution of $\Sigma$

This is obtained from the dynamics of $\Sigma_V$ in HR as follows:

[step $\#1$]: $\mathcal{A} \ni A \rightarrow \alpha_V^t(A) := e^{\frac{iH_V t}{\hbar}} A e^{-\frac{iH_V t}{\hbar}}$.

[Step $\#2$]: $\alpha^t(A) = \tau - \lim_V \alpha_V^t(A)$, where $\tau$ is a reasonable topology of $\mathcal{A}$. Possible topologies are:
Time evolution of $\Sigma$

This is obtained from the dynamics of $\Sigma_V$ in HR as follows:

[Step 1]: $\mathcal{A} \ni A \rightarrow \alpha^t_v(A) := e^{iH_v t/\hbar} A e^{-iH_v t/\hbar}$.

[Step 2]: $\alpha^t(A) = \tau - \lim_V \alpha^t_v(A)$,

where $\tau$ is a *reasonable* topology of $\mathcal{A}$. Possible topologies are:

for short range interactions and discrete systems $\tau$ is usually the *uniform* topology [Haag, Hugenholtz, Winnink].
Time evolution of $\Sigma$

This is obtained from the dynamics of $\Sigma_V$ in HR as follows:

[step #1]: $\mathcal{A} \ni A \rightarrow \alpha^t_V(A) := e^{iH_V t/\hbar} A e^{-iH_V t/\hbar}$.

[Step #2]: $\alpha^t(A) = \tau - \lim_V \alpha^t_V(A)$,

where $\tau$ is a reasonable topology of $\mathcal{A}$. Possible topologies are:

for short range interactions and discrete systems $\tau$ is usually the uniform topology [Haag, Hugenholtz, Winnink];

for long range interactions $\alpha^t_V$ is not $\| . \|-$converging: $\tau$ is the strong topology (restricted to a relevant family of states) [Sewell, Thirring, Werhl, Strocchi, Morchio, B.,...]: $\rho$ is chosen in such a way that

$$\rho(\alpha^t_V(A)) \rightarrow \rho(\alpha^t(A)) =: \rho_t(A),$$

(which gives also $\rho_t$). The existence of (sufficiently many) such $\rho$’s has to be checked in each model.
An automorphism of $\mathfrak{A}$, $\gamma$, is a **symmetry** of $\Sigma$ if $\alpha^t(\gamma(A)) = \gamma(\alpha^t(A))$ and is local if $\gamma : \mathfrak{A}_\mathcal{V} \to \mathfrak{A}_\mathcal{V}$ and $\gamma(H_\mathcal{V}) = H_\mathcal{V}$.

$\gamma$ is a **symmetry of the state** $\rho$ if $\rho_\gamma(A) := \rho(\gamma(A)) = \rho(A), \forall A \in \mathfrak{A}$.
An automorphism of $\mathcal{A}$, $\gamma$, is a symmetry of $\Sigma$ if $\alpha^t(\gamma(A)) = \gamma(\alpha^t(A))$ and is local if $\gamma : \mathcal{A}_V \to \mathcal{A}_V$ and $\gamma(H_V) = H_V$.

$\gamma$ is a symmetry of the state $\rho$ if $\rho_{\gamma}(A) := \rho(\gamma(A)) = \rho(A), \forall A \in \mathcal{A}$.

A representation of a $\ast$-algebra is a map $\pi : \mathcal{A} \to B(\mathcal{H})$, for a certain $\mathcal{H}$, which preserves the algebraic structure of $\mathcal{A}$:

$$\pi(A + B) = \pi(A) + \pi(B), \quad \pi(\lambda A) = \lambda \pi(A),$$

$$\pi(AB) = \pi(A)\pi(B), \quad \pi(A^\ast) = \pi(A)^\ast.$$  

It follows that $\pi(\mathcal{A})$ is a $\ast$-algebra as well.
An automorphism of \( \mathcal{A} \), \( \gamma \), is a **symmetry** of \( \Sigma \) if \( \alpha^t(\gamma(A)) = \gamma(\alpha^t(A)) \) and is local if \( \gamma : \mathcal{A}_V \to \mathcal{A}_V \) and \( \gamma(H_V) = H_V \).

\( \gamma \) is a **symmetry of the state** \( \rho \) if \( \rho_\gamma(A) := \rho(\gamma(A)) = \rho(A), \forall A \in \mathcal{A} \).

A **representation** of a \(*\)-algebra is a map \( \pi : \mathcal{A} \to B(\mathcal{H}) \), for a certain \( \mathcal{H} \), which preserves the algebraic structure of \( \mathcal{A} \):

\[
\pi(A + B) = \pi(A) + \pi(B), \quad \pi(\lambda A) = \lambda \pi(A),
\]

\[
\pi(AB) = \pi(A)\pi(B), \quad \pi(A^*) = \pi(A)^*.
\]

It follows that \( \pi(\mathcal{A}) \) is a \(*\)-algebra as well.

**Important:** any state \( \rho \) over the abstract \( \mathbb{C}^* \)-algebra \( \mathcal{A} \) produces a unique (but for equivalence) **GNS** (Gelfand-Naimark-Segal) representation \( (\mathcal{H}_\rho, \pi_\rho, \Omega_\rho) \), in such a way that, \( \forall A \in \mathcal{A} \),

\[
\rho(A) = \langle \Omega_\rho, \pi_\rho(A)\Omega_\rho \rangle
\]

Here \( \Omega_\rho \) is cyclic, i.e. \( \pi_\rho(\mathcal{A})\Omega_\rho \) is dense in \( \mathcal{H}_\rho \), and \( \pi_\rho \) is irreducible iff \( \rho \) is pure.
Notice that:

1. GNS representations generated by different states need not be unitarily equivalent (e.g. Ising model)!
Notice that:

1. GNS representations generated by different states need not be unitarily equivalent (e.g. Ising model)!

2. Each (GNS) representation corresponds to a phase of the physical system. In particular, GNS representations generated by pure states correspond to pure phases [Ruelle].
Notice that:

1. GNS representations generated by different states need not be unitarily equivalent (e.g. Ising model)!

2. Each (GNS) representation corresponds to a phase of the physical system. In particular, GNS representations generated by pure states correspond to pure phases [Ruelle].

3. States which are only locally different are macroscopically indistinguishable: all the macroscopic observables have the same expectation values (e.g. Ising model again). [Hepp, Sewell]. Then they produce unitarily equivalent GNS representations.

Physical interpretation: equal values of the macroscopic observables (the order parameters) label unitarily equivalent GNS representations, which are interpreted as the same phase of the matter.
In other words: two different phases of the matter correspond to GNS representations in which some macroscopic observable assumes different values.
In other words: two different phases of the matter correspond to GNS representations in which some macroscopic observable assumes different values.

4. Under certain assumption on $\Sigma$, the dynamics in each representation $\pi_\rho$ is \textit{hamiltonian}: there exists a s.a. operator $\hat{H}_\rho$ such that, $\forall A \in \mathcal{A}$,

\[
\frac{d}{dt} \alpha^t_\rho(\pi_\rho(A)) = i[\hat{H}_\rho, \alpha^t_\rho(\pi_\rho(A))].
\]

(This is not obvious for $\Sigma$ at a pure algebraic level!) [Emch, Knops, Sewell]. $\hat{H}_\rho$ is what is often called in literature \textit{the effective hamiltonian}.

Physical interpretation: different phases may have different dynamical behaviors.
V. $QM_\infty [T > 0]$

We will give here only few considerations on *equilibrium states* and *phase structure*.

**Case 1: finite system**
We will give here only few considerations on *equilibrium states* and *phase structure*.

**Case 1: finite system**

The following are equivalent:

- $\rho$ is a Gibbs state corresponding to the trace class operator $\hat{\rho} = \frac{e^{-\beta H_V}}{tr_V(idem)}$, $\beta^{-1} = kT$, $\iff$
- it minimizes the free energy functional $\hat{F}_V(\rho) = tr_V(\rho H_V + \beta^{-1}\rho \log(\rho))$, $\iff$
- it is a KMS (Kubo-Martin-Schwinger) state, i.e. $\rho(A_t B) = \rho(B A_{t+i\hbar\beta})$.

**Therefore:** for each temperature there exists an unique equilibrium state $\Rightarrow$ an unique GNS representation $\Rightarrow$ a single phase of $\Sigma$. 

Case 2: infinite system

Assumptions on $H_V$ (relevant to prove the existence of thermodynamical limit of several physical quantities):

1. $H_{V_1 \cup V_2} - H_{V_1} - H_{V_2}$ is a surface effect (short range forces);

2. there exists $c > 0$ such that $\|H_V\| \leq c|V|$.

Under these assumptions we have

$$\alpha^t(A) = \| \| - \lim_{V \uparrow} \alpha^t_V(A)$$
Let
\[
\begin{align*}
E_V(\rho_V) &= \text{tr}_V(\rho_V H_V), \\
S_V(\rho_V) &= -k \text{tr}_V(\rho_V \log(\rho_V)), \\
F_V(\rho_V) &= E_V(\rho_V) - T S_V(\rho_V)
\end{align*}
\]
be the local energy, local entropy and the local free energy functionals.

Then, the assumptions for $H_V$ and the \textit{subadditivity of the entropy}, imply that the following \textit{global density functionals}
\[
e(\rho) = \lim_{V \uparrow} \frac{E_V(\rho_V)}{|V|}, \quad s(\rho) = \lim_{V \uparrow} \frac{S_V(\rho_V)}{|V|}, \quad f(\rho) = \lim_{V \uparrow} \frac{F_V(\rho_V)}{|V|}
\]
exist, as well as the following \textit{incremental functionals}
\[
\begin{align*}
\Delta E(\rho|\rho') &= \lim_{V \uparrow} \left( E_V(\rho'_V) - E_V(\rho_V) \right), \\
\Delta S(\rho|\rho') &= \lim_{V \uparrow} \left( S_V(\rho'_V) - S_V(\rho_V) \right), \\
\Delta F(\rho|\rho') &= \lim_{V \uparrow} \left( F_V(\rho'_V) - F_V(\rho_V) \right),
\end{align*}
\]
where $\rho'$ is a \textit{local} modification of $\rho$. 
A state $\tilde{\rho}$ is **globally thermodynamically stable** if it is invariant under translations and if it minimizes $f(\rho)$. It is **locally thermodynamically stable** if $\Delta F(\tilde{\rho}|\rho') \geq 0$ for all $\rho'$, local modification of $\tilde{\rho}$. Then, [Sewell, 1980 and later],

where the white arrows are direct implications while the green ones hold true if the state is invariant under translations and for **short range interactions**.
A state $\tilde{\rho}$ is *globally thermodynamically stable* if it is invariant under translations and if it minimizes $f(\rho)$. It is *locally thermodynamically stable* if $\Delta F(\tilde{\rho}|\rho') \geq 0$ for all $\rho'$, local modification of $\tilde{\rho}$. Then, [Sewell, 1980 and later],

where the white arrows are direct implications while the green ones hold true if the state is invariant under translations and for short range interactions.

Physical interpretation: a GTS state is an equilibrium state. The LTS (or KMS) states are, for systems with long range interactions, only *metastable* states. They are truly equilibrium states under the above assumptions.
Notice that an infinite system $\Sigma$ may possess more than one GTS state at the same temperature.

This is related to the analyticity properties of some thermodynamical potential and to the affine nature of $f(\rho)$, while $\hat{F}_\nu(\rho)$ is a convex functional. If this is the case, $\Sigma$ admits different thermodynamical phases under the same thermodynamical conditions. We say that the system possesses macroscopic degeneracy: these different equilibrium states (and the related phases) are labeled by the (different) values of some macroscopic observables (like the magnetization).

This fact has several related consequences:
Notice that *an infinite system* $\Sigma$ *may possess more than one GTS state at the same temperature.*

This is related to the analyticity properties of some thermodynamical potential and to the *affine* nature of $f(\rho)$, while $\hat{F}_V(\rho)$ is a convex functional.

If this is the case, $\Sigma$ admits different thermodynamical phases under the same thermodynamical conditions. We say that *the system possesses macroscopic degeneracy:* these different equilibrium states (and the related phases) are labeled by the (different) values of some *macroscopic observables* (like the magnetization).

This fact has several related consequences: first of all this algebraic approach provides a nice framework to study *phase transitions.*
A second consequence: spontaneous breaking of a symmetry

Suppose that $\Sigma$ has a local symmetry $\gamma$ and let $\Delta = \{\rho \in A' : \rho \text{ is GTS}\}$. Then, since $f(\rho) = f(\rho\gamma)$, if $\rho \in \Delta$ also $\rho\gamma \in \Delta$.

Then, if $\Delta = \{\rho_1\}$ consists in a single element, $\gamma$ is necessarily a symmetry of $\rho$: the symmetry is unbroken.

If, on the contrary, $\Delta = \{\rho_1, \ldots, \rho_n\}$, then, in general, we have $(\rho_i)\gamma = \rho_j$, with $i \neq j$: the symmetry is spontaneously broken.

Example ($T=0$ Ising model): $H_V$ is invariant under spin reversal but the two (transactionally invariant) ground states associated to $m = \pm 1$ are, clearly, no longer invariant: they are mapped into each other by the symmetry.
...and a related result: *non relativistic Goldstone’s theorem*

Suppose that the symmetry $\gamma_\lambda$ is generated by a local charge

$$Q_R(t) = \int_{|\vec{x}| \leq R} j_o(\vec{x}, t) \, d^3x,$$

$$\gamma_\lambda(A) = \| \cdot \| \lim_{R, \infty} e^{iQ_R \lambda} A e^{-iQ_R \lambda},$$

and suppose that $\gamma_\lambda$ commutes with the time translations $\alpha^t$. Then, if $\gamma_\lambda$ is spontaneously broken, i.e. if for some $A \in \mathcal{A}$ $\lim_{R, \infty} < [Q_R, A] > \psi_0 \neq 0$, then the energy spectrum cannot have a gap above the ground state.
Another result: KMS versus Tomita-Takesaki theory

All KMS states can be used to generate a modular structure in the sense of Tomita-Takesaki: let \( \rho \) be a KMS state (corresponding to \( \beta = -1 \)). It generates a GNS representation \((\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)\). Let \( U_\rho(t) \) the unitary operator which implements \( \alpha^t \) in this representation. Then \( \Omega_\rho \) is cyclic and, since \( \rho \) is KMS, is also separating for \( \pi_\rho(\mathcal{A})'' \), i.e. \( \pi_\rho(X)\Omega_\rho = 0 \) implies that \( \pi_\rho(X) = 0 \).

Then we are in the assumptions of Tomita-Takesaki’s theorem so that we can introduce a modular conjugation \( J_\rho \) and a modular operator \( \Delta_\rho \) associated to \((\pi_\rho(\mathcal{A})'', \Omega_\rho)\). Calling \( H_\rho \) the generator of \( U_\rho(t) \), we find that

\[
\Delta_\rho = e^{H_\rho}.
\]
VI. A list of problems

The above results are obtained under the requirements that

1. The norm of the local hamiltonian $H_V$ does not grow faster than $|V|$: $\|H_V\| \leq c|V|$;
VI. A list of problems

The above results are obtained under the requirements that

1. The norm of the local hamiltonian $H_V$ does not grow faster than $|V|$: $\|H_V\| \leq c|V|$;

This is not always true: actually, it is false quite often! For instance, it is violated already by a gas of free bosons with $H_V = \sum_{j \in V} a_j^* a_j$.

It is satisfied, on the contrary, by a gas of free fermions, for which, however, $\text{dim}(\mathcal{H}_V) < \infty$. 
2. The interactions are *short ranged*
2. The interactions are short ranged

But the Coulomb interaction is long ranged, as well as the mean field one. In this case we find that:

- $\alpha^t\nu$ is not norm convergent to $\alpha^t$ [Thirring, Werhl, Sewell, Strocchi, Morchio, B.]
- KMS states are not equilibrium states (i.e. GTS states) in general and they are not limits of Gibbs state [Sewell, Dubin, Morchio, Haag];
- surface effects become volume effects, so that variables at infinity (i.e. completely delocalized operators) appear. These are related to the order parameters used to describe different phases;
- $e(\rho)$ and $f(\rho)$ do not necessarily exist;
- the Goldstone’s theorem holds only in a modified form [Morchio, Strocchi].
3. Even in presence of unbounded operators some \textit{physically meaningful} limits do exist.
3. Even in presence of unbounded operators some physically meaningful limits do exist.

This usually means that we are introducing a cutoff on the spectrum of the operators. However, quite often this is not enough. As an example, we cite the Lindblad expression for the generator $L$ of a completely positive semigroup (describing the time evolution of a quantum open system). These structures play a role in the analysis of order-disorder transitions out of equilibrium. We have

$$LA = i[H, A] + \sum_j \left( V_j^* A V_j - \frac{1}{2} \{V_j^* V_j, A\} \right),$$

only if $A, H, V_j$ are all bounded in some Hilbert space.

For unbounded operators only few results exist, [Fagnola, Rebolledo, Chebotarev, Sewell, B.].

To avoid this kind of difficulties, in many models the boson reservoir is replaced by a fermionic one [e.g. see Martin and Buffet’s open BCS-model].
VII. Algebras of unbounded operators

A possible algebraic framework: let $\mathcal{A}$ be a linear space, $\mathcal{A}_0 \subset \mathcal{A}$ a $\ast$-algebra with unit $1$ (otherwise we can add it): $\mathcal{A}$ is a quasi $\ast$-algebra over $\mathcal{A}_0$ if
VII. Algebras of unbounded operators

A possible algebraic framework: let $\mathcal{A}$ be a linear space, $\mathcal{A}_0 \subset \mathcal{A}$ a $*$-algebra with unit $\mathbb{1}$ (otherwise we can add it): $\mathcal{A}$ is a quasi $*$-algebra over $\mathcal{A}_0$ if

[i] the right and left multiplications of an element of $\mathcal{A}$ and an element of $\mathcal{A}_0$ are always defined and linear;
VII. Algebras of unbounded operators

A possible algebraic framework: let $\mathcal{A}$ be a linear space, $\mathcal{A}_0 \subset \mathcal{A}$ a $^\ast$-algebra with unit $1$ (otherwise we can add it): $\mathcal{A}$ is a quasi $^\ast$-algebra over $\mathcal{A}_0$ if

[i] the right and left multiplications of an element of $\mathcal{A}$ and an element of $\mathcal{A}_0$ are always defined and linear;

[ii] $x_1(x_2a) = (x_1x_2)a$, $(ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$, for each $x_1, x_2 \in \mathcal{A}_0$ and $a \in \mathcal{A}$;
VII. Algebras of unbounded operators

*A possible algebraic framework:* let $\mathcal{A}$ be a linear space, $\mathcal{A}_0 \subset \mathcal{A}$ a $*$-algebra with unit $\mathbb{I}$ (otherwise we can add it): $\mathcal{A}$ is a *quasi $*$-algebra over $\mathcal{A}_0$ if

[i] the right and left multiplications of an element of $\mathcal{A}$ and an element of $\mathcal{A}_0$ are always defined and linear;

[ii] $x_1(x_2a) = (x_1x_2)a$, $(ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$, for each $x_1, x_2 \in \mathcal{A}_0$ and $a \in \mathcal{A}$;

[iii] an involution $*$ (which extends the involution of $\mathcal{A}_0$) is defined in $\mathcal{A}$ with the property $(ab)^* = b^*a^*$ whenever the multiplication is defined.
VII. Algebras of unbounded operators

A possible algebraic framework: let $\mathcal{A}$ be a linear space, $\mathcal{A}_0 \subset \mathcal{A}$ a $*$-algebra with unit $\mathbb{1}$ (otherwise we can add it): $\mathcal{A}$ is a quasi $*$-algebra over $\mathcal{A}_0$ if

[i] the right and left multiplications of an element of $\mathcal{A}$ and an element of $\mathcal{A}_0$ are always defined and linear;

[ii] $x_1(x_2 a) = (x_1 x_2)a$, $(a x_1)x_2 = a(x_1 x_2)$ and $x_1(a x_2) = (x_1 a)x_2$, for each $x_1, x_2 \in \mathcal{A}_0$ and $a \in \mathcal{A}$;

[iii] an involution $*$ (which extends the involution of $\mathcal{A}_0$) is defined in $\mathcal{A}$ with the property $(ab)^* = b^* a^*$ whenever the multiplication is defined.

A quasi $*$-algebra $(\mathcal{A}, \mathcal{A}_0)$ is locally convex (or topological) if in $\mathcal{A}$ a locally convex topology $\tau$ is defined such that (a) the involution is continuous and the multiplications are separately continuous; and (b) $\mathcal{A}_0$ is dense in $\mathcal{A}[\tau]$. 
Let \( \{ p_\alpha \} \) be directed set of seminorms which defines \( \tau \). We assume that \( \mathcal{A}[\tau] \) is complete.
Let \( \{p_\alpha\} \) be directed set of seminorms which defines \( \tau \). We assume that \( \mathcal{A}[\tau] \) is complete.

[G. Lassner, K. Schmüdgen, K.D. Kürstein, W. Timmermann (Leipzig), J.P. Antoine (LLN), A. Inoue, H. Ogi, I. Ikeda (Fukuoka), S.J. Bhatt (India), G.O.S. Ekhaguerre (Nigeria), C. Trapani, F.B. (Palermo), ... ]
Let \( \{ p_\alpha \} \) be directed set of seminorms which defines \( \tau \). We assume that \( \mathcal{A}[\tau] \) is complete.

[\text{G. Lassner, K. Schmüdgen, K.D. Kürstein, W. Timmermann (Leipzig), J.P. Antoine (LLN), A. Inoue, H. Ogi, I. Ikeda (Fukuoka), S.J. Bhatt (India), G.O.S. Ekhaguerre (Nigeria), C. Trapani, F.B. (Palermo), …}]

Why are these structures related to unbounded operators?
Let \( \{ p_\alpha \} \) be directed set of seminorms which defines \( \tau \). We assume that \( \mathfrak{A}[\tau] \) is complete.

[G. Lassner, K. Schmüdgen, K.D. Kürstein, W. Timmermann (Leipzig), J.P. Antoine (LLN), A. Inoue, H. Ogi, I. Ikeda (Fukuoka), S.J. Bhatt (India), G.O.S. Ekhaguere (Nigeria), C. Trapani, F.B. (Palermo), ...]

**Why are these structures related to unbounded operators?**

If \( a \) and \( b \) are unbounded, then \( ab \), does not exist in general. But if \( x \) is bounded, then \( xa, ax, bx \) and \( xb \) are all well defined. This is reflected by the fact that \( \mathfrak{A} \) is not an algebra but only a quasi \( * \)-algebra: only some elements of \( \mathfrak{A} \) can be mutually multiplied.
Let \( \{ p_\alpha \} \) be directed set of seminorms which defines \( \tau \). We assume that \( \mathcal{A}[\tau] \) is complete.

[G. Lassner, K. Schmüdgen, K.D. Kürstein, W. Timmermann (Leipzig), J.P. Antoine (LLN), A. Inoue, H. Ogi, I. Ikeda (Fukuoka), S.J. Bhatt (India), G.O.S. Ekhaguerre (Nigeria), C. Trapani, F.B. (Palermo), ... ]

**Why are these structures related to unbounded operators?**

If \( a \) and \( b \) are unbounded, then \( ab \), does not exist in general. But if \( x \) is bounded, then \( xa, ax, bx \) and \( xb \) are all well defined. This is reflected by the fact that \( \mathcal{A} \) is not an algebra but only a quasi \(*\)-algebra: only some elements of \( \mathcal{A} \) can be mutually multiplied.

The next example shows explicitly that \( (\mathcal{A}, \mathcal{A}_0) \) contains unbounded operators (e.g. the number operator for a boson gas).
Example: Let $\mathcal{H}$ be a separable Hilbert space and $N$ an unbounded, densely defined, self-adjoint operator. Let $D(N^k)$ be the domain of the operator $N^k$, $k \in \mathbb{N}$, and $\mathcal{D}$ the domain of all the powers of $N$: $\mathcal{D} \equiv D^\infty(N) = \bigcap_{k \geq 0} D(N^k)$. This set is dense in $\mathcal{H}$. Let us now introduce $\mathcal{L}^\dagger(\mathcal{D})$, the *-algebra of all the closable operators defined on $\mathcal{D}$ which, together with their adjoints ($\dagger$), map $\mathcal{D}$ into itself. Here the adjoint of $X \in \mathcal{L}^\dagger(\mathcal{D})$ is $X^\dagger = X^*_\mathcal{D}$. 
Example: Let $\mathcal{H}$ be a separable Hilbert space and $N$ an unbounded, densely defined, self-adjoint operator. Let $D(N^k)$ be the domain of the operator $N^k$, $k \in \mathbb{N}$, and $\mathcal{D}$ the domain of all the powers of $N$: $\mathcal{D} \equiv D^{\infty}(N) = \cap_{k \geq 0} D(N^k)$. This set is dense in $\mathcal{H}$. Let us now introduce $\mathcal{L}^\dagger(\mathcal{D})$, the $*$-algebra of all the closable operators defined on $\mathcal{D}$ which, together with their adjoints ($\ast$), map $\mathcal{D}$ into itself. Here the adjoint of $X \in \mathcal{L}^\dagger(\mathcal{D})$ is $X^\dagger = X^\ast|_\mathcal{D}$.

In $\mathcal{D}$ the topology is defined by the following $N$-depending seminorms: $\phi \in \mathcal{D} \rightarrow \|\phi\|_n \equiv \|N^n \phi\|$, $n \in \mathbb{N}_0$, while the topology $\tau_0$ in $\mathcal{L}^\dagger(\mathcal{D})$ is introduced by the seminorms

$$X \in \mathcal{L}^\dagger(\mathcal{D}) \rightarrow \|X\|^{f,k}_n \equiv \max \left\{ \|f(N)XN^k\|, \|N^kXf(N)\| \right\},$$

where $k \in \mathbb{N}_0$ and $f \in \mathcal{C}$, the set of all the positive, bounded and continuous functions on $\mathbb{R}_+$, which are decreasing faster than any inverse power of $x$: $\mathcal{L}^\dagger(\mathcal{D})[\tau_0]$ is a complete $*$-algebra.
It contains unbounded operators, e.g. all the positive powers of $N$, but not $e^N$. 
It contains unbounded operators, e.g. all the positive powers of $N$, but not $e^N$.

Let further $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ be the set of all continuous maps from $\mathcal{D}$ into $\mathcal{D}'$, with their topologies (in $\mathcal{D}'$ this is the strong dual topology), and let $\tau$ denotes the topology defined by the seminorms

$$X \in \mathcal{L}(\mathcal{D}, \mathcal{D}') \rightarrow \|X\|^f = \|f(N)Xf(N)\|,$$

$f \in \mathcal{C}$. Then $\mathcal{L}(\mathcal{D}, \mathcal{D}')[\tau]$ is a complete vector space and $e^N \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$.

In this case $\mathcal{L}^\dagger(\mathcal{D}) \subset \mathcal{L}(\mathcal{D}, \mathcal{D}')$ and the pair

$$(\mathcal{L}(\mathcal{D}, \mathcal{D}')[\tau], \mathcal{L}^\dagger(\mathcal{D})[\tau_0])$$

is a locally convex quasi $*$-algebra.
Other possible algebraic frameworks:

1. Partial *-algebras [J.P. Antoine, W. Karwowski (1981)]:
Other possible algebraic frameworks:

1. Partial *-algebras [J.P. Antoine, W. Karwowski (1981)]:

A $P^*A$ is a complex vector space $\mathcal{A}$ with involution * (with the usual properties) and a subset $\Gamma \subset (\mathcal{A}, \mathcal{A})$ such that

$(x, y) \in \Gamma$ iff $(y^*, x^*) \in \Gamma$;

if $(x, y), (x, z) \in \Gamma$ then $(x, \lambda y + \mu z) \in \Gamma$ for all $\lambda, \mu \in \mathbb{C}$;

if $(x, y) \in \Gamma$ then there exists an element $x \cdot y \in \mathcal{A}$.

This multiplication satisfies the following properties:

$x \cdot (y + \lambda z) = x \cdot y + \lambda x \cdot z$ and $(x \cdot y)^* = y^* \cdot x^*$,

$\forall (x, y), (x, z) \in \Gamma$. 

Other possible algebraic frameworks:

1. **Partial *-algebras** [J.P. Antoine, W. Karwowski (1981)]:

A $P^*A$ is a complex vector space $\mathcal{A}$ with involution $*$ (with the usual properties) and a subset $\Gamma \subset (\mathcal{A}, \mathcal{A})$ such that

$(x, y) \in \Gamma$ iff $(y^*, x^*) \in \Gamma$;

if $(x, y), (x, z) \in \Gamma$ then $(x, \lambda y + \mu z) \in \Gamma$ for all $\lambda, \mu \in \mathbb{C}$;

if $(x, y) \in \Gamma$ then there exists an element $x \cdot y \in \mathcal{A}$.

This multiplication satisfies the following properties:

$x \cdot (y + \lambda z) = x \cdot y + \lambda x \cdot z$ and $(x \cdot y)^* = y^* \cdot x^*$, $\forall (x, y), (x, z) \in \Gamma$.

**It is very general: too much general!!**
2. CQ*-algebras [Trapani, B. (≥ 1996)]:
2. CQ*-algebras [Trapani, B. (≥ 1996)]:

A (proper) CQ*-algebra is a quasi *-algebra such that: \( \mathcal{A}_0[\| . \|_0] \) is a C*-algebra; \( \mathcal{A}[\| . \|] \) is a Banach space in which \( \mathcal{A}_0 \) is dense; the two norms are related as follows: 
\[
\| x \|_0 = \max \left\{ \sup_{\| a \| \leq 1} \| ax \|, \sup_{\| a \| \leq 1} \| xa \| \right\}, \quad \forall x \in \mathcal{A}_0.
\]
2. CQ*-algebras [Trapani, B. (≥ 1996)]:

A (proper) CQ*-algebra is a quasi *-algebra such that: $A_0[∥.∥_0]$ is a C*-algebra; $A[∥.∥]$ is a Banach space in which $A_0$ is dense; the two norms are related as follows: 

$$∥x∥_0 = \max \{\sup_{∥a∥≤1} ∥ax∥, \sup_{∥a∥≤1} ∥xa∥\}, \quad ∀x ∈ A_0.$$ 

This is a natural generalization of C*-algebras: indeed the completion of any C*-algebra $(A_0, ||||_0)$ with respect to a weaker norm $|||.$ satisfying:

(i) $||A^*|| = ||A||, \forall A ∈ A_0$ and (ii) $||AB|| \leq ||A|| ||B||_0, \forall A, B ∈ A_0,$ is a CQ*-algebra.
2. CQ*-algebras [Trapani, B. (≥ 1996)]:

A (proper) CQ*-algebra is a quasi *-algebra such that: \( \mathcal{A}_0[\| . \|_0] \) is a C*-algebra; \( \mathcal{A}[\| . \|] \) is a Banach space in which \( \mathcal{A}_0 \) is dense; the two norms are related as follows: \( \| x \|_0 = \max \{ \sup_{\| a \| \leq 1} \| ax \|, \sup_{\| a \| \leq 1} \| xa \| \} \), \( \forall x \in \mathcal{A}_0 \).

This is a natural generalization of C*-algebras: indeed the completion of any C*-algebra \( (\mathcal{A}_0, \| . \|_0) \) with respect to a weaker norm \( \| . \| \) satisfying:

(i) \( \| A^* \| = \| A \| \), \( \forall A \in \mathcal{A}_0 \) and (ii) \( \| AB \| \leq \| A \| \| B \|_0 \), \( \forall A, B \in \mathcal{A}_0 \), is a CQ*-algebra.

**Examples:**

(1) \( (L^p(X, \mu), C_0(X)) \), \( X \) a compact space and \( C_0(X) \) the set of the continuous functions on \( X \) [Trapani and B., JMAA 1996];
2. **CQ*-algebras** [Trapani, B. (≥ 1996)]:

A **(proper) CQ*-algebra** is a quasi *-algebra such that: $\mathcal{A}_0[\| \cdot \|_0]$ is a C*-algebra; $\mathcal{A}[\| \cdot \|]$ is a Banach space in which $\mathcal{A}_0$ is dense; the two norms are related as follows: $\| x \|_0 = \max \left\{ \sup_{\| a \| \leq 1} \| ax \|, \sup_{\| a \| \leq 1} \| xa \| \right\}$, $\forall x \in \mathcal{A}_0$.

This is a **natural generalization** of C*-algebras: indeed the completion of any C*-algebra $(\mathcal{A}_0, \| \cdot \|_0)$ with respect to a weaker norm $\| \cdot \|$ satisfying:

(i) $\| A^* \| = \| A \|$, $\forall A \in \mathcal{A}_0$ and (ii) $\| AB \| \leq \| A \| \| B \|_0$, $\forall A, B \in \mathcal{A}_0$, is a CQ*-algebra.

**Examples:**

1. $(L^p(X, \mu), C_0(X))$, $X$ a compact space and $C_0(X)$ the set of the continuous functions on $X$ [Trapani and B., JMAA 1996];

2. $(L^p(X, \mu), L^\infty(X, \mu))$, where $(X, \mu)$ is a measure space with $\mu$ a Borel measure on the locally compact Hausdorff space $X$ [Trapani and B., JMAA 1996];
(3) Non commutative $L^p$ spaces. [Trapani, Triolo and B., Studia Mathematica, 2006]
(3) Non commutative $L^p$ spaces. [Trapani, Triolo and B., Studia Mathematica, 2006]

(4) Let $\mathcal{H}$ be a Hilbert space with scalar product $(.,.)$ and $S$ an unbounded selfadjoint operator, with $S \geq I$, with dense domain $D(S)$. The subspace $D(S)$ becomes a Hilbert space, denoted by $\mathcal{H}_+1$, with the scalar product $(f, g)_{+1} = (Sf, Sg)$. and let $\mathcal{H}_{-1}$ denote the conjugate dual of $\mathcal{H}_+1$. Then $\mathcal{H}_{-1}$ itself is a Hilbert space. Given further $\mathcal{A} = \{X \in B(\mathcal{H}_+, \mathcal{H}_{-1}) : X$ is compact from $\mathcal{H}_+$ into $\mathcal{H}_{-1}\}$, $\mathcal{A}_b = \{X \in B(\mathcal{H}_+) : X$ is compact in $\mathcal{H}_+\}$, then $(\mathcal{A}[\|\cdot\|], *, \mathcal{A}_b[\|\cdot\|_b], b)$ is a (non proper) CQ*-algebra of operators [Trapani and B., Publ. RIMS Kyoto University, 2000]
(3) Non commutative $L^p$ spaces. [Trapani, Triolo and B., Studia Mathematica, 2006]

(4) Let $\mathcal{H}$ be a Hilbert space with scalar product $(.,.)$ and $S$ an unbounded selfadjoint operator, with $S \geq I$, with dense domain $D(S)$. The subspace $D(S)$ becomes a Hilbert space, denoted by $\mathcal{H}_+^1$, with the scalar product $(f, g)_+^1 = (Sf, Sg)$. and let $\mathcal{H}_-^1$ denote the conjugate dual of $\mathcal{H}_+^1$. Then $\mathcal{H}_-^1$ itself is a Hilbert space. Given further $\mathcal{A} = \{ X \in B(\mathcal{H}_+^1, \mathcal{H}_-^1) : X$ is compact from $\mathcal{H}_+^1$ into $\mathcal{H}_-^1 \}$, $\mathcal{A}_b = \{ X \in B(\mathcal{H}_+^1) : X$ is compact in $\mathcal{H}_+^1 \}$, then $(\mathcal{A}[,], *, \mathcal{A}_b[||.||_b], b)$ is a (non proper) C$^*$-algebra of operators [Trapani and B., Publ. RIMS Kyoto University, 2000]

**Remark:** they are related to TT theory [Inoue, Trapani and B., Proc. Am. Math. Soc. 2001]
(3) Non commutative $L^p$ spaces. [Trapani, Triolo and B., Studia Mathematica, 2006]

(4) Let $\mathcal{H}$ be a Hilbert space with scalar product $(.,.)$ and $S$ an unbounded selfadjoint operator, with $S \geq I$, with dense domain $D(S)$. The subspace $D(S)$ becomes a Hilbert space, denoted by $\mathcal{H}_1$, with the scalar product $(f,g)_1 = (Sf,Sg)$. and let $\mathcal{H}_1$ denote the conjugate dual of $\mathcal{H}_1$. Then $\mathcal{H}_1$ itself is a Hilbert space. Given further $A = \{X \in B(\mathcal{H}_1, \mathcal{H}_1) : X \text{ is compact from } \mathcal{H}_1 \text{ into } \mathcal{H}_1\}$, $A_b = \{X \in B(\mathcal{H}_1) : X \text{ is compact in } \mathcal{H}_1\}$, then $(A[\|\|], *, A_b[\|\|], b)$ is a (non proper) CQ*-algebra of operators [Trapani and B., Publ. RIMS Kyoto University, 2000]

**Remark:** they are related to TT theory [Inoue, Trapani and B., Proc. Am. Math. Soc. 2001]

*In the rest of this talk we will only use topological quasi *-algebras*

Main (mathematical) references:
K. Schmüdgen, Akademie-Verlag, Berlin (1990);
*-Representations of a quasi *-algebra

Let now \((\mathcal{A}, \mathcal{A}_0)\) be a quasi *-algebra, \(\mathcal{D}_\pi\) a dense domain in a certain Hilbert space \(\mathcal{H}_\pi\), and \(\pi\) a linear map from \(\mathcal{A}\) into \(\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)\), where

\[
\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi) = \{X \text{ closable in } \mathcal{H}_\pi : D(X) = \mathcal{D}_\pi \text{ and } D(X^*) \supseteq \mathcal{D}_\pi\}.
\]

This is a partial *-algebra with the usual operations \(X + Y, \lambda X\), the involution \(X^\dagger = X^*\big|_{\mathcal{D}_\pi}\) and the weak product \(X \Box Y \equiv X^\dagger Y\) (whenever \(Y \mathcal{D}_\pi \subset D(X^\dagger)\) and \(X^\dagger \mathcal{D}_\pi \subset D(Y^*)\)). Let furthermore

\[
\mathcal{L}^\dagger(\mathcal{D}_\pi) = \{A \in \mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi) : A, A^\dagger : \mathcal{D}_\pi \rightarrow \mathcal{D}_\pi\}.
\]

\(\mathcal{L}^\dagger(\mathcal{D}_\pi)\) is a *-algebra and the weak multiplication reduces to the ordinary multiplication of operators.
A *-representation of \( \mathcal{A} \) is a linear map from \( \mathcal{A} \) into \( \mathcal{L}^+(D_\pi, \mathcal{H}_\pi) \) such that:

(i) \( \pi(a^*) = \pi(a)\dagger, \quad \forall a \in \mathcal{A}; \)

(ii) if \( a \in \mathcal{A}, x \in \mathcal{A}_0 \), then \( \pi(a)\square \pi(x) \) is well defined and \( \pi(ax) = \pi(a)\square \pi(x) \).

Moreover, if

(iii) \( \pi(\mathcal{A}_0) \subset \mathcal{L}^+(D_\pi), \)

then \( \pi \) is said to be a *-representation of the quasi *-algebra \( (\mathcal{A}, \mathcal{A}_0) \).

The *-representation \( \pi \) is called ultra-cyclic if there exists \( \xi_0 \in D_\pi \) such that \( \pi(\mathcal{A}_0)\xi_0 = D_\pi \).

\( \pi \) is faithful if \( \pi(x) = 0 \) implies \( x = 0 \).

Let \( \pi \) be a *-representation of \( \mathcal{A} \). The strong topology \( \tau_5 \) on \( \pi(\mathcal{A}) \) is defined by the seminorms: \( \{ p_\xi(.); \xi \in D_\pi \} \), where \( p_\xi(\pi(a)) \equiv \|\pi(a)\xi\|, \quad a \in \mathcal{A}, \xi \in D_\pi \).
A GNS-like construction

Let us assume here that $\tau = \| \cdot \|$ ($\Rightarrow A$ is a Banach space) and that $(A, A_0)$ has a unit $e$. Let $\varphi$ a sesquilinear form on $A \times A$ such that
(i) $\varphi(x, x) \geq 0$, $\forall x \in A$;
(ii) $\varphi(ax, y) = \varphi(x, a^*y)$, $\forall a \in A$, $x, y \in A_0$;
(iii) there exists $\gamma > 0$ such that $|\varphi(x, y)| \leq \gamma \|x\| \|y\|$, $\forall x, y \in A_0$.

Then there exists a $^*$-representation $\pi_\varphi^o$ of $(A, A_0)$ in $L^\dagger(\lambda_\varphi(A_0), \mathcal{H}_\varphi)$ with an ultra-cyclic vector $\lambda_\varphi(e)$ and such that

$$\varphi(x, y) = \langle \pi_\varphi^o(x)\lambda_\varphi(e), \pi_\varphi^o(y)\lambda_\varphi(e) \rangle$$

Here $\lambda_\varphi(A_0) = A_0/N_\varphi$, $N_\varphi = \{a \in A : \varphi(a, a) = 0\}$, is dense in $\mathcal{H}_\varphi$ wrt its own norm, [Trapani 2006].
Remarks: (1) several other possible extensions of the GNS representations also exist in the literature;
Remarks:— (1) several other possible extensions of the GNS representations also exist in the literature; (2) this approach uses *sesquilinear forms* instead of *linear* ones: indeed, even if $\omega(a^*b)$ were not well defined for certain $a, b \in \mathcal{A}$, $\varphi(a, b)$ surely makes sense;
Remarks:– (1) several other possible extensions of the GNS representations also exist in the literature; (2) this approach uses *sesquilinear forms* instead of *linear* ones: indeed, even if $\omega(a^*b)$ were not well defined for certain $a, b \in \mathcal{A}$, $\varphi(a, b)$ surely makes sense; (3) one can try to use a state $\omega$ over $\mathcal{A}_0$ to define a sesquilinear form on $\mathcal{A}_0 \times \mathcal{A}_0$: $\varphi(x, y) = \omega(x^*)\omega(y)$. If, now, $\varphi$ can be extended to all of $\mathcal{A} \times \mathcal{A}$, then we can try repeating the construction given above;
Remarks:– (1) several other possible extensions of the GNS representations also exist in the literature; (2) this approach uses *sesquilinear forms* instead of *linear* ones: indeed, even if $\omega(a^*b)$ were not well defined for certain $a, b \in \mathcal{A}$, $\varphi(a, b)$ surely makes sense; (3) one can try to use a state $\omega$ over $\mathcal{A}_0$ to define a sesquilinear form on $\mathcal{A}_0 \times \mathcal{A}_0$: $\varphi(x, y) = \omega(x^*)\omega(y)$. If, now, $\varphi$ can be extended to all of $\mathcal{A} \times \mathcal{A}$, then we can try repeating the construction given above; (4) a different approach: we start representing $\mathcal{A}_0$ via $\pi^0$ and then try to extend it as much as we can;
Remarks:– (1) several other possible extensions of the GNS representations also exist in the literature; (2) this approach uses *sesquilinear forms* instead of *linear* ones: indeed, even if \( \omega(a^*b) \) were not well defined for certain \( a, b \in \mathcal{A} \), \( \varphi(a, b) \) surely makes sense; (3) one can try to use a state \( \omega \) over \( \mathcal{A}_0 \) to define a sesquilinear form on \( \mathcal{A}_0 \times \mathcal{A}_0 \): \( \varphi(x, y) = \omega(x^*)\omega(y) \). If, now, \( \varphi \) can be extended to all of \( \mathcal{A} \times \mathcal{A} \), then we can try repeating the construction given above; (4) a different approach: we start representing \( \mathcal{A}_0 \) via \( \pi^0 \) and then try to extend it as much as we can; (5) the *physical interpretation* is analogous to that discussed before: different sesquilinear forms produce different representations which can still be interpreted as different phases of the matter.
VIII. A physical application

**Definition:** Let \( (\mathcal{A}[\tau], \mathcal{A}_0) \) be a quasi \(*\)-algebra. A \(*\)-derivation of \( \mathcal{A}_0 \) is a linear map \( \delta : \mathcal{A}_0 \rightarrow \mathcal{A} \) with the following properties:

(i) \( \delta(x^*) = \delta(x)^*, \forall x \in \mathcal{A}_0; \)

(ii) \( \delta(xy) = x\delta(y) + \delta(x)y, \forall x, y \in \mathcal{A}_0. \)

Let now \( \pi \) be a \(*\)-representation of \( (\mathcal{A}, \mathcal{A}_0) \) such that, whenever \( x \in \mathcal{A}_0 \) satisfies \( \pi(x) = 0 \), then \( \pi(\delta(x)) = 0 \). Under this assumption, the linear map

\[
\delta_{\pi}(\pi(x)) = \pi(\delta(x)), \quad x \in \mathcal{A}_0,
\]

is well-defined on \( \pi(\mathcal{A}_0) \) with values in \( \pi(\mathcal{A}) \) and it is a \(*\)-derivation of \( \pi(\mathcal{A}_0) \). We call \( \delta_{\pi} \) the \(*\)-derivation induced by \( \pi \).
Given such a representation $\pi$ and its dense domain $\mathcal{D}_\pi \subset \mathcal{H}_\pi$, we consider the graph topology $t_\dagger$ generated by the seminorms

$$\xi \in \mathcal{D}_\pi \rightarrow \|A\xi\|, \quad A \in \mathcal{L}^{\dagger}(\mathcal{D}_\pi).$$

Let $\mathcal{D}'_\pi$ be the conjugate dual space of $\mathcal{D}_\pi$ and $t'_\dagger$ the strong dual topology of $\mathcal{D}'_\pi$. Then we get the usual rigged Hilbert space

$$\mathcal{D}_\pi[t_\dagger] \subset \mathcal{H}_\pi \subset \mathcal{D}'_\pi[t'_\dagger].$$

Let $\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$ denote the space of all continuous linear maps from $\mathcal{D}_\pi[t_\dagger]$ into $\mathcal{D}'_\pi[t'_\dagger]$. Then one has

$$\mathcal{L}^{\dagger}(\mathcal{D}_\pi) \subset \mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi).$$
Each operator $A \in \mathcal{L}^\dagger(\mathcal{D}_\pi)$ can be extended to an operator $\hat{A}$ on the whole $\mathcal{D}'_\pi$ in the following way:

$$<\hat{A}\xi', \eta> = <\xi', A^\dagger \eta>, \quad \forall \xi' \in \mathcal{D}'_\pi, \eta \in \mathcal{D}_\pi.$$ 

Therefore the left and right multiplication of $X \in \mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$ and $A \in \mathcal{L}^\dagger(\mathcal{D}_\pi)$ can always be defined:

$$(X \circ A)\xi = X(A\xi), \text{ and } (A \circ X)\xi = \hat{A}(X\xi), \quad \forall \xi \in \mathcal{D}_\pi,$$

and for that we conclude that $(\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi), \mathcal{L}^\dagger(\mathcal{D}_\pi))$ is a quasi $*$-algebra.

Let $\delta$ be a $*$-derivation of $\mathfrak{A}_0$ and $\pi$ a $*$-representation of $(\mathfrak{A}, \mathfrak{A}_0)$. Then $\pi(\mathfrak{A}_0) \subset \mathcal{L}^\dagger(\mathcal{D}_\pi)$. We say that the $*$-derivation $\delta_\pi$ induced by $\pi$ is **spatial** if there exists $H_\pi = H^\dagger_\pi \in \mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$ such that $H_\pi \xi_0 \in \mathcal{H}_\pi$ and

$$\delta_\pi(\pi(x)) = i\{H_\pi \circ \pi(x) - \pi(x) \circ H_\pi\}, \quad \forall x \in \mathfrak{A}_0.$$
Theorem 1 \textbf{(BIT, IJMMS 2004)} Let \((\mathcal{A}[\tau], \mathcal{A}_0)\) be a locally convex quasi \(*\)-algebra with identity and \(\delta\) be a \(*\)-derivation of \(\mathcal{A}_0\).

Then the following statements are equivalent:
Theorem 1 (BIT, IJMMS 2004) Let \((\mathcal{A}[\tau], \mathcal{A}_0)\) be a locally convex quasi *-algebra with identity and \(\delta\) be a *-derivation of \(\mathcal{A}_0\).

Then the following statements are equivalent:

(i) There exists a \((\tau - \tau_s)\)-continuous, ultra-cyclic *-representation \(\pi\) of \(\mathcal{A}\), with ultra-cyclic vector \(\xi_0\), such that the *-derivation \(\delta_\pi\) induced by \(\pi\) is spatial.
Theorem 1 (BIT, IJMMS 2004) Let $(\mathcal{A}[\tau], \mathcal{A}_0)$ be a locally convex quasi *-algebra with identity and $\delta$ be a *-derivation of $\mathcal{A}_0$.

Then the following statements are equivalent:

(i) There exists a $(\tau - \tau_s)$-continuous, ultra-cyclic *-representation $\pi$ of $\mathcal{A}$, with ultra-cyclic vector $\xi_0$, such that the *-derivation $\delta_{\pi}$ induced by $\pi$ is spatial.

(ii) There exists a positive linear functional $f$ on $\mathcal{A}_0$ such that $f(x^*x) \leq p(x)^2$, $\forall x \in \mathcal{A}_0$, for some continuous seminorm $p$ of $\tau$ and, denoting with $\tilde{f}$ the continuous extension of $f$ to $\mathcal{A}$, the following inequality holds:

$$|\tilde{f}(\delta(x))| \leq C \left( \sqrt{f(x^*x)} + \sqrt{f(xx^*)} \right), \quad \forall x \in \mathcal{A}_0,$$

for some positive constant $C$. 
(iii) There exists a positive sesquilinear form $\varphi$ on $\mathcal{A} \times \mathcal{A}$ such that:

$\varphi$ is invariant, i.e. $\varphi(ax, y) = \varphi(x, a^*y)$, for all $a \in \mathcal{A}$ and $x, y \in \mathcal{A}_0$;

$\varphi$ is $\tau$-continuous, i.e. $|\varphi(a, b)| \leq p(a)p(b)$, for all $a, b \in \mathcal{A}$, for some continuous seminorm $p$ of $\tau$;

and $\varphi$ satisfies the following inequality:

$$|\varphi(\delta(x), \mathbb{1})| \leq C \left( \sqrt{\varphi(x, x)} + \sqrt{\varphi(x^*, x^*)} \right), \quad \forall x \in \mathcal{A}_0,$$

for some positive constant $C$. 
Remarks: (1) this theorem extends a similar result valid for C*-algebras (bounded hamiltonians!);
Remarks:— (1) this theorem extends a similar result valid for C*-algebras (bounded hamiltonians!); (2) even if $\delta$ cannot be written as $\delta(x) = i[H, x]$, if the above theorem can be applied, then $\delta_\pi$ is, essentially, the commutator with a certain symmetric operator. Again the dynamics depends on the phase of the matter;
Remarks:– (1) this theorem extends a similar result valid for C*-algebras (bounded hamiltonians);

(2) even if $\delta$ cannot be written as $\delta(x) = i[H,x]$, if the above theorem can be applied, then $\delta_\pi$ is, essentially, the commutator with a certain symmetric operator. Again the dynamics depends on the phase of the matter;

(3) suppose we add to a spatial *-derivation $\delta_0$ a perturbation $\delta_p$ such that $\delta = \delta_0 + \delta_p$ is again a *-derivation. Under which conditions $\delta$ is still spatial? A sufficient condition is that $|\tilde{f}(\delta_p(x))| \leq |\tilde{f}(\delta_0(x))|$, for all $x \in \mathcal{A}_0$, which is exactly what we expect since $\delta_p$ is smaller than $\delta_0$. If we call $H_\pi$, $H_{\pi,0}$ and $H_{\pi,p}$ the operators which implement $\delta$, $\delta_0$ and $\delta_p$, we can also prove that $i[H_{\pi},A]\psi = i[H_{\pi,0} + H_{\pi,p},A]\psi$, for all $A \in \mathcal{L}^\dagger(\mathcal{D}_\pi)$ and $\psi \in \mathcal{D}_\pi$. 
Physical justification: The dynamical behavior of the infinite system is obtained as a limit of its restriction to a finite volume $V_L$. At the infinitesimal level, this means that we have a family of spatial derivations $\delta_L$ but we don’t now if the limit of these derivations is still spatial.
**Plan of the talk**

- Motivations
- Ordinary quantum... 
  - $QM_\infty [T = 0]$
  - $QM_\infty [T > 0]$
- A list of problems
- Algebras of...
- A physical application
- The time evolution $\alpha^t$
- Work in progress
- Perspectives

---

**About the removal of the cutoff**

Physical justification: The dynamical behavior of the infinite system is obtained as a limit of its restriction to a finite volume $V_L$. At the infinitesimal level, this means that we have a family of spatial derivations $\delta_L$ but we don’t now if the limit of these derivations is still spatial.

Let $S$ be a physical system, $(\mathcal{A}, \mathcal{A}_0)$ a quasi *-algebra and $\{S_L = \{\mathcal{A}_L \subset \mathcal{A}_0, \Sigma, \alpha^t_L\}, L \in \Lambda\}$ a family of regularized systems, i.e. $S_L$ is the restriction of $S$ to some $V_L$. We suppose that the dynamics $\alpha^t_L$ is generated by a *-derivation $\delta_L$:

$$\alpha^t_L(x) = \tau_0 - \sum_{k=0}^{\infty} \frac{t^k}{k!} \delta_L^k(x), \quad \forall x \in \mathcal{A}_L.$$

Here $\tau_0$ is the topology on $\mathcal{A}_0$. 

Definition 2 The family \( \{S_L, L \in \Lambda\} \) is said to be \textit{c-representable} if there exists a \(*\)-representation \( \pi \) of \((\mathcal{A}, \mathcal{A}_0)\) into some \((\mathcal{L}(D_\pi, D'_\pi), \mathcal{L}^\dagger(D_\pi))\) such that:

\(\begin{align*}
(i) & \pi \text{ is } (\tau - \tau_s)\text{-continuous}; \\
(ii) & \pi \text{ is ultra-cyclic with ultra-cyclic vector } \xi_0; \\
(iii) & \text{when } \pi(x) = 0, \text{ then } \pi(\delta_L(x)) = 0, \forall L \in \Lambda.
\end{align*}\)

Any such representation \( \pi \) is a \textit{c-representation}. 
**Definition 2** The family \( \{S_L, L \in \Lambda\} \) is said to be c-representable if there exists a *-representation \( \pi \) of \((\mathcal{A}, \mathcal{A}_0)\) into some \((\mathcal{L}(D_\pi, D'_\pi), \mathcal{L}^\dagger(D_\pi))\) such that:

(i) \( \pi \) is \((\tau - \tau_s)\)-continuous;

(ii) \( \pi \) is ultra-cyclic with ultra-cyclic vector \( \xi_0 \);

(iii) when \( \pi(x) = 0 \), then \( \pi(\delta_L(x)) = 0 \), \( \forall L \in \Lambda \).

Any such representation \( \pi \) is a c-representation.

**Proposition 3** Let \( \{S_L, L \in \Lambda\} \) be a c-representable family and \( \pi \) a c-representation. Let \( h_L = h^*_L \in \mathcal{A}_L \) be the element which implements \( \delta_L: \delta_L(x) = i[h_L, x], \forall x \in \mathcal{A}_0 \). Suppose that \( \delta_L(x) \) is \( \tau \)-Cauchy \( \forall x \in \mathcal{A}_0 \) and that \( \sup_L \|\pi(h_L)\xi_0\| < \infty \).

Then, one has
Definition 2 The family \( \{S_L, L \in \Lambda\} \) is said to be **c-representable** if there exists a \(*\)-representation \( \pi \) of \((\mathcal{A}, \mathcal{A}_0)\) into some \((\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi), \mathcal{L}^\dagger(\mathcal{D}_\pi))\) such that:

(i) \( \pi \) is \((\tau - \tau_s)\)-continuous;
(ii) \( \pi \) is ultra-cyclic with ultra-cyclic vector \( \xi_0 \);
(iii) when \( \pi(x) = 0 \), then \( \pi(\delta_L(x)) = 0 \), \( \forall L \in \Lambda \).

Any such representation \( \pi \) is a **c-representation**.

Proposition 3 Let \( \{S_L, L \in \Lambda\} \) be a c-representable family and \( \pi \) a c-representation. Let \( h_L = h_L^* \in \mathcal{A}_L \) be the element which implements \( \delta_L \): \( \delta_L(x) = i[h_L, x] \), \( \forall x \in \mathcal{A}_0 \). Suppose that \( \delta_L(x) \) is \( \tau \)-Cauchy \( \forall x \in \mathcal{A}_0 \) and that \( \sup_L \|\pi(h_L)\xi_0\| < \infty \).

Then, one has
(a) \( \delta(x) = \tau - \lim_L \delta_L(x) \) exists in \( \mathcal{A} \) and is a \(*\)-derivation of \( \mathcal{A}_0 \);
Definition 2  The family \( \{S_L, L \in \Lambda\} \) is said to be \( c \)-representable if there exists a *-representation \( \pi \) of \((\mathcal{A}, \mathcal{A}_0)\) into some \((\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}_\pi'), \mathcal{L}^\dagger(\mathcal{D}_\pi))\) such that:

(i) \( \pi \) is \((\tau - \tau_s)\)-continuous;
(ii) \( \pi \) is ultra-cyclic with ultra-cyclic vector \( \xi_0 \);
(iii) when \( \pi(x) = 0 \), then \( \pi(\delta_L(x)) = 0 \), \( \forall L \in \Lambda \).

Any such representation \( \pi \) is a \( c \)-representation.

Proposition 3  Let \( \{S_L, L \in \Lambda\} \) be a \( c \)-representable family and \( \pi \) a \( c \)-representation. Let \( h_L = h^*_L \in \mathcal{A}_L \) be the element which implements \( \delta_L: \delta_L(x) = i[h_L, x], \forall x \in \mathcal{A}_0 \). Suppose that \( \delta_L(x) \) is \( \tau \)-Cauchy \( \forall x \in \mathcal{A}_0 \) and that \( \sup_L \| \pi(h_L)\xi_0 \| < \infty \).

Then, one has

(a) \( \delta(x) = \tau - \lim_L \delta_L(x) \) exists in \( \mathcal{A} \) and is a *-derivation of \( \mathcal{A}_0 \);
(b) \( \delta_\pi \), the *-derivation induced by \( \pi \), is well defined and spatial.
Remarks:– (1) It is clear that if the sequence \( \{ h_L \} \) is \( \tau \)-convergent, then \( \delta_L(x) \) is automatically \( \tau \)-Cauchy \( \forall x \in A_0 \).
Remarks:– (1) It is clear that if the sequence \( \{h_L\} \) is \( \tau \)-convergent, then \( \delta_L(x) \) is automatically \( \tau \)-Cauchy \( \forall x \in \mathcal{A}_0 \).

(2) This Proposition implies that any physical system \( S \) with a c-representable regularized family \( \{S_L, L \in \Lambda\} \) admits an effective hamiltonian in the sense of [Trapani and B., JMP, 1996], [Thirring, Werhl, Sewell, Lassner ...]
Remarks:– (1) It is clear that if the sequence \( \{ h_L \} \) is \( \tau \)-convergent, then \( \delta_L(x) \) is automatically \( \tau \)-Cauchy for all \( x \in \mathcal{A}_0 \).

(2) This Proposition implies that any physical system \( S \) with a c-representable regularized family \( \{ S_L, L \in \Lambda \} \) admits an effective hamiltonian in the sense of [Trapani and B., JMP, 1996], [Thirring, Werhl, Sewell, Lassner ...]

(3) this is the counterpart of Sewell’s result for on the existence of different effective hamiltonians in different GNS-like representations (different dynamics in different phases of the matter).
Remarks:– (1) It is clear that if the sequence \( \{ h_L \} \) is \( \tau \)-convergent, then \( \delta_L(x) \) is automatically \( \tau \)-Cauchy \( \forall x \in \mathcal{A}_0 \).

(2) This Proposition implies that any physical system \( S \) with a c-representable regularized family \( \{ S_L, L \in \Lambda \} \) admits an effective hamiltonian in the sense of [Trapani and B., JMP, 1996], [Thirring, Werhl, Sewell, Lassner . . .]

(3) this is the counterpart of Sewell’s result for on the existence of different effective hamiltonians in different GNS-like representations (different dynamics in different phases of the matter).

(4) Open problem: what if we consider a \( \pi' \) globally equivalent to \( \pi \)? Is \( \pi' \) a c-representation? Do the related effective hamiltonians coincide?
IX. The time evolution $\alpha^t$

Even if $\delta$ exists, what about $\delta^2, \delta^3, \ldots$? Moreover, even if all these maps do exist, this does not mean that $\sum_{k=0}^{\infty} \frac{t^k}{k!} \delta^k(x)$ exists, for $x \in \mathcal{A}_0$. Further, the effective hamiltonian $H_\pi$ is symmetric but not self-adjoint, and therefore we cannot use the spectral theorem to define $e^{iH_\pi t}$! How to define a time evolution?
IX. The time evolution $\alpha^t$

Even if $\delta$ exists, what about $\delta^2, \delta^3, \ldots$? Moreover, even if all these maps do exist, this does not mean that $\sum_{k=0}^{\infty} \frac{t^k}{k!} \delta^k(x)$ exists, for $x \in A_0$. Further, the effective hamiltonian $H_\pi$ is symmetric but not self-adjoint, and therefore we cannot use the spectral theorem to define $e^{iH_\pi t}$! How to define a time evolution?

IX.1. From $\delta$ to $\alpha^t$: the first way

Let us assume the $h_L$’s can be written in terms of some s.a. elements $s_\alpha^L, \alpha = 1, 2, \ldots, N$, which are $\tau$-converging to some elements $s^\alpha \in A$:

$$s^\alpha = \tau - \lim_L s_\alpha^L, \quad [s^\alpha, x] = 0, \quad \forall x \in A_0.$$  

(e.g. mean field spin models: $s^\alpha$ is the magnetization).
IX. The time evolution $\alpha^t$

Even if $\delta$ exists, what about $\delta^2$, $\delta^3$, ...? Moreover, even if all these maps do exist, this does not mean that $\sum_{k=0}^{\infty} \frac{t^k}{k!} \delta^k(x)$ exists, for $x \in \mathcal{A}_0$. Further, the effective hamiltonian $H_{\pi}$ is symmetric but not self-adjoint, and therefore we cannot use the spectral theorem to define $e^{iH_{\pi}t}$! How to define a time evolution?

IX.1. From $\delta$ to $\alpha^t$: the first way

Let us assume the $h_L$’s can be written in terms of some s.a. elements $s^\alpha_L$, $\alpha = 1, 2, .., N$, which are $\tau$-converging to some elements $s^\alpha \in \mathcal{A}$:

$$s^\alpha = \tau - \lim_{L} s^\alpha_L, \quad [s^\alpha, x] = 0, \forall x \in \mathcal{A}_0.$$ (e.g. mean field spin models: $s^\alpha$ is the magnetization).

We say that $\{s^\alpha_L\}$ is uniformly $\tau$-continuous if, for each continuous seminorm $p$ of $\tau$ and for all $\alpha = \ldots$
1, 2, ..., \( N \), there exist another continuous seminorm \( q \) of \( \tau \) and a positive constant \( c_{p,q,\alpha} \) such that

\[
p(s^{\alpha}_L a) \leq c_{p,q,\alpha} q(a), \quad \forall a \in \mathcal{A}, \quad \forall L \in \Lambda.
\]

\[
\Rightarrow p(as^{\alpha}_L) \leq c_{p,q,\alpha} q(a), \quad \forall a \in \mathcal{A},
\]

and that the same inequalities can be extended to \( s^{\alpha} \). Then we have
1, 2, ..., \( N \), there exist another continuous seminorm \( q \) of \( \tau \) and a positive constant \( c_{p,q,\alpha} \) such that

\[
p(s^\alpha_L a) \leq c_{p,q,\alpha} q(a), \quad \forall a \in A, \quad \forall L \in \Lambda.
\]

\[\Rightarrow p(as^\alpha_L) \leq c_{p,q,\alpha} q(a), \quad \forall a \in A, \quad \text{and that the same inequalities can be extended to } s^\alpha. \quad \text{Then we have}
\]

**Lemma 4** If \( \{s^\alpha_L\} \) is a uniformly \( \tau \)-continuous sequence and if \( \tau - \lim L s^\alpha_L = s^\alpha \), \( \forall \alpha \), then \( \tau - \lim L (s^\alpha_L)^k = (s^\alpha)^k \), \( \forall \alpha \) and for \( k = 1, 2, \ldots \).
1, 2, ..., N, there exist another continuous seminorm $q$ of $\tau$ and a positive constant $c_{p,q,\alpha}$ such that

$$p(s_L^\alpha a) \leq c_{p,q,\alpha} q(a), \ \forall a \in A, \ \forall L \in \Lambda.$$ 

$$\Rightarrow p(as_L^\alpha) \leq c_{p,q,\alpha} q(a), \ \forall a \in A,$$

and that the same inequalities can be extended to $s^\alpha$. Then we have

**Lemma 4** If $\{s_L^\alpha\}$ is a uniformly $\tau$-continuous sequence and if $\tau \leftarrow \lim_L s_L^\alpha = s^\alpha$, $\forall \alpha$, then $\tau \leftarrow \lim_L (s_L^\alpha)^k = (s^\alpha)^k$, $\forall \alpha$ and for $k = 1, 2, \ldots$.

**Proposition 5** Suppose that

(1) $\forall x \in A_0 [h_L, x]$ depends on $L$ only through $s_L^\alpha$;

(2) $s_L^\alpha \xrightarrow{\tau} s^\alpha$ and $\{s_L^\alpha\}$ is a uniformly $\tau$-continuous sequence.

Then, for each $k \in \mathbb{N}$, the following limit exists

$$\tau \leftarrow \lim_L i^k [h_L, x]_k = \tau \leftarrow \lim_L \delta_L^k(x), \ \forall x \in A_0,$$

and defines an element of $A$ which we call $\delta^{(k)}(x)$. 
Remark:– We use $\delta^{(k)}(x)$ instead of the classical $\delta^{k}_{\pi}(x) = i^{k}[H_{\pi}, \pi(x)]_{k}$ since this last quantity could not be well defined because of domain problems.
**Remark:**– We use \( \delta^{(k)}(x) \) instead of the *classical* \( \delta^{k}_{\pi}(x) = "i^{k}[H_{\pi}, \pi(x)]_{k}" \) since this last quantity could not be well defined because of domain problems.

**Definition 6** We say that \( x \in \mathfrak{A}_{0} \) is a generalized analytic element of \( \delta \) if, for all \( t \), the series \( \sum_{k=0}^{\infty} \frac{t^{k}}{k!} \pi(\delta^{(k)}(x)) \) is \( \tau_{s} \)-convergent. The set of all generalized analytic elements is denoted with \( \mathcal{G} \).

**Proposition 7** Let \( x_{\gamma} \) be a net of elements of \( \mathfrak{A}_{0} \) and suppose that, whenever \( \pi(x_{\gamma}) \xrightarrow{\tau_{s}} \pi(x) \) then \( x_{\gamma} \xrightarrow{\tau} x \). Then, \( \forall x \in \mathcal{G} \) and \( \forall t \in \mathbb{R} \), the series \( \sum_{k=0}^{\infty} \frac{t^{k}}{k!} \delta^{(k)}(x) \) converges in the \( \tau \)-topology to an element of \( \mathfrak{A} \) which we call \( \alpha^{t}(x) \).

Moreover, \( \alpha^{t} \) can be extended to the \( \tau \)-closure \( \overline{\mathcal{G}} \) of \( \mathcal{G} \).

**Remark:**– Rather strong assumptions: however they are satisfied by mean field spin models!
IX.2. From $\delta$ to $\alpha^t$: the second way

Let $\pi$ be a faithful $*$-representation of the quasi $*$-algebra $(\mathcal{A}, \mathcal{A}_0)$ and $\delta$ a $*$-derivation of $\mathcal{A}_0$ such that $\delta_\pi$, is well-defined on $\pi(\mathcal{A}_0)$ with values in $\pi(\mathcal{A})$. We define the following subset of $\mathcal{A}_0$ (a domain of regularity of $\delta$)

$$\mathcal{A}_0(\delta) := \{ x \in \mathcal{A}_0 : \delta^k(x) \in \mathcal{A}_0, \ \forall k \in \mathbb{N}_0 \}.$$ 

$\delta$ regular $\Rightarrow \mathcal{A}_0(\delta)$ is large. For instance, if $\delta$ is inner in $\mathcal{A}_0$ with an implementing element $h \in \mathcal{A}_0$, then $\mathcal{A}_0(\delta) = \mathcal{A}_0$. For general $\delta$, $\mathcal{A}_0(\delta)$ contains, at least, all the multiples of the identity $1$ of $\mathcal{A}_0$. 

IX.2. From $\delta$ to $\alpha^t$: the second way

Let $\pi$ be a faithful *-representation of the quasi *-algebra $(\mathcal{A}, \mathcal{A}_0)$ and $\delta$ a *-derivation of $\mathcal{A}_0$ such that $\delta_{\pi}$, is well-defined on $\pi(\mathcal{A}_0)$ with values in $\pi(\mathcal{A})$. We define the following subset of $\mathcal{A}_0$ (a *domain of regularity* of $\delta$)

$$\mathcal{A}_0(\delta) := \{ x \in \mathcal{A}_0 : \delta^k(x) \in \mathcal{A}_0, \ \forall k \in \mathbb{N}_0 \}.$$ 

$\delta$ regular $\Rightarrow \mathcal{A}_0(\delta)$ is large. For instance, if $\delta$ is inner in $\mathcal{A}_0$ with an implementing element $h \in \mathcal{A}_0$, then $\mathcal{A}_0(\delta) = \mathcal{A}_0$. For general $\delta$, $\mathcal{A}_0(\delta)$ contains, at least, all the multiples of the identity $1$ of $\mathcal{A}_0$.

$\mathcal{A}_0(\delta)$ is a *-algebra which is mapped into itself by $\delta$. Moreover $\pi(\delta^k(x)) = \delta^k_{\pi}(\pi(x))$, $\forall x \in \mathcal{A}_0(\delta)$ and $\forall k \in \mathbb{N}_0$. Therefore, for all $k \in \mathbb{N}_0$, and for all $x \in \mathcal{A}_0(\delta)$, $\delta^k_{\pi}(\pi(x)) \in \pi(\mathcal{A}_0)$. 


Let $\sigma_s$ be the topology on $\mathcal{A}$ defined via $\tau_s$ in the following way:

$$\mathcal{A} \ni a \rightarrow q_\xi(a) = p_\xi(\pi(a)) = \|\pi(a)\xi\|, \quad \xi \in \mathcal{D}_\pi.$$ 

Then we have the following
Let $\sigma_s$ be the topology on $\mathcal{A}$ defined via $\tau_s$ in the following way:

$$\mathcal{A} \ni a \rightarrow q_\xi(a) = p_\xi(\pi(a)) = \|\pi(a)\xi\|, \quad \xi \in D_\pi.$$ 

Then we have the following

**Theorem 8** Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi *-$\ast$-algebra with identity, $\delta$ a *-$\ast$-derivation on $\mathcal{A}_0$ and $\pi$ a faithful *-$\ast$-representation of $(\mathcal{A}, \mathcal{A}_0)$ such that the induced derivation $\delta_{\pi}$ is well defined. Then, we have:
Let $\sigma_s$ be the topology on $\mathcal{A}$ defined via $\tau_s$ in the following way:

$$\mathcal{A} \ni a \rightarrow q_\xi(a) = p_\xi(\pi(a)) = \|\pi(a)\xi\|, \quad \xi \in \mathcal{D}_\pi.$$ 

Then we have the following

**Theorem 8** Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi $*$-algebra with identity, $\delta$ a $*$-derivation on $\mathcal{A}_0$ and $\pi$ a faithful $*$-representation of $(\mathcal{A}, \mathcal{A}_0)$ such that the induced derivation $\delta_\pi$ is well defined. Then, we have:

(1) if the following inequality holds

$$\forall \eta \in \mathcal{D}_\pi \exists c_\eta > 0 : p_\eta(\delta_\pi(\pi(x))) \leq c_\eta p_\eta(\pi(x)), \quad \forall x \in \mathcal{A}_0(\delta),$$

then $\sum_{k=0}^{\infty} \frac{t^k}{k!} \delta^k(x)$ converges for all $t$ in the topology $\sigma_s$ to an element of $\mathcal{A}_0(\delta)^{\sigma_s}$ which we call $\alpha^t(x)$; $\alpha^t$ can be extended to $\mathcal{A}_0(\delta)^{\sigma_s}$. Moreover $\alpha^t : \mathcal{A}_0(\delta)^{\sigma_s} \rightarrow \mathcal{A}_0(\delta)^{\sigma_s}$ and

$$\alpha^{t+\tau}(x) = \alpha^t(\alpha^\tau(x)), \quad \forall t, \tau, \forall x \in \mathcal{A}_0(\delta);$$
(2) Suppose that the following inequality holds

\[ \exists c > 0 : \forall \eta_1 \in D_\pi \exists A_{\eta_1} > 0, \ n \in \mathbb{N} \text{ and } \eta_2 \in D_\pi : \]

\[ p_{\eta_1}(\delta^k_\pi(\pi(x))) \leq A_{\eta_1}c^k k! n^p_{\eta_2}(\pi(x)), \ \forall x \in A_0(\delta), \forall k \in \mathbb{N}, \]

then \[ \sum_{k=0}^{\infty} \frac{t^k}{k!} \delta^k(x) \] converges, for \( t < \frac{1}{c} \) in the topology \( \sigma_s \) to an element of \( \overline{A_0(\delta)}^{\sigma_s} \) which we call \( \alpha^t(x) \); \( \alpha^t \) can be extended to \( \overline{A_0(\delta)}^{\sigma_s} \). Moreover \( \alpha^t \) maps \( \overline{A_0(\delta)}^{\sigma_s} \) into itself for \( t < \frac{1}{c} \) and, \( \forall x \in A_0(\delta), \)

\[ \alpha^{t+\tau}(x) = \alpha^t(\alpha^\tau(x)), \ \forall t, \tau, \text{ with } t + \tau < \frac{1}{c}. \]
(2) Suppose that the following inequality holds

$$\exists c > 0 : \forall \eta_1 \in D_\pi \exists A_{\eta_1} > 0, n \in \mathbb{N} \text{ and } \eta_2 \in D_\pi :$$

$$p_{\eta_1}(\delta^k_\pi(\pi(x))) \leq A_{\eta_1} c^k k! k^n p_{\eta_2}(\pi(x)), \quad \forall x \in A_0(\delta), \forall k \in \mathbb{N}_0,$$

then $$\sum_{k=0}^{\infty} \frac{t^k}{k!} \delta^k(x)$$ converges, for $$t < \frac{1}{c}$$ in the topology $$\sigma_s$$ to an element of $$\overline{A_0(\delta)}^{\sigma_s}$$ which we call $$\alpha^t(x)$$; $$\alpha^t$$ can be extended to $$\overline{A_0(\delta)}^{\sigma_s}$$. Moreover $$\alpha^t$$ maps $$\overline{A_0(\delta)}^{\sigma_s}$$ into itself for $$t < \frac{1}{c}$$ and, $$\forall x \in A_0(\delta),$$

$$\alpha^{t+\tau}(x) = \alpha^t(\alpha^\tau(x)), \quad \forall t, \tau, \text{ with } t + \tau < \frac{1}{c}.$$

**Remark:** Here the spatiality of the derivation is not required. When $$H_\pi$$ exists as a **self-adjoint** operator, we could use the spectral theorem to define

$$\pi(\alpha^t(x)) = e^{iH_\pi t} \pi(x) e^{-iH_\pi t};$$

[Inoue, Trapani, B. 2005]
IX.3. a different point of view [Trapani, B. JMP 2002]

Given a self-adjoint unbounded and densely defined operator $H_0 \geq 1$, the operator $e^{iH_0 t}$ can be defined via the spectral theorem. Let $\mathcal{L}^+(\mathcal{D})$, $\mathcal{D} = D^\infty(H_0)$, then in 2002 we gave an alternative proof, useful for physical applications: let $H_0 = \int_1^\infty \lambda dE(\lambda)$. We define, for $L \geq 1$, the projectors $Q^0_L = \int_1^L dE(\lambda)$ and we introduce the \emph{regularized hamiltonian} $H_L = Q^0_L H_0 Q^0_L$. 
IX.3. a different point of view [Trapani, B. JMP 2002]

Given a self-adjoint unbounded and densely defined operator $H_0 \geq \mathbb{1}$, the operator $e^{iH_0 t}$ can be defined via the spectral theorem.

Let $\mathcal{L}^\dagger(\mathcal{D})$, $\mathcal{D} = D^\infty(H_0)$, then in 2002 we gave an alternative proof, useful for physical applications: let $H_0 = \int_1^\infty \lambda dE(\lambda)$. We define, for $L \geq 1$, the projectors $Q_0^L = \int_1^L dE(\lambda)$ and we introduce the \textit{regularized hamiltonian} $H_L = Q_0^L H_0 Q_0^L$.

For each $L$, $Q_0^L, H_L \in B(\mathcal{H}) \cap \mathcal{L}^\dagger(\mathcal{D})$. Further, $[Q_0^L, H_L] = [Q_0^L, H_0] = [H_0, H_L] = 0$.

If $\tau_0$ is the topology on $\mathcal{L}^\dagger(\mathcal{D})$ generated by the seminorms

$$\mathcal{L}^\dagger(\mathcal{D}) \ni A \mapsto \|A\|_{f,k}^\dagger = \max\{\|H_0^k Af(H_0)\|, \|f(H_0)AH_0^k\|\}$$

then we have:
(i) $H_L \rightarrow H_0$ with respect to the topology $\tau_0$;
(i) $H_L \to H_0$ with respect to the topology $\tau_0$;

(ii) $\{ e^{itH_L} \}$ is $\tau_0$-Cauchy in $L^\dagger(\mathcal{D})$ and
(i) $H_L \to H_0$ with respect to the topology $\tau_0$;

(ii) $\{e^{itH_L}\}$ is $\tau_0$-Cauchy in $L^\dagger(\mathcal{D})$ and

(iii) $\forall A \in L^\dagger(\mathcal{D}), \{e^{itH_L}Ae^{-itH_L}\}$ is $\tau_0$-Cauchy in $L^\dagger(\mathcal{D})$. 
(i) $H_L \rightarrow H_0$ with respect to the topology $\tau_0$;

(ii) $\{e^{itH_L}\}$ is $\tau_0$-Cauchy in $\mathcal{L}^+(\mathcal{D})$ and

(iii) $\forall A \in \mathcal{L}^+(\mathcal{D})$, $\{e^{itH_L}Ae^{-itH_L}\}$ is $\tau_0$-Cauchy in $\mathcal{L}^+(\mathcal{D})$.

**Remarks:**

- (1) Then $H_0$, $e^{iH_0 t}$, $\alpha^t(A) := e^{itH_0}Ae^{-itH_0}$ all belong to $\mathcal{L}^+(\mathcal{D})$, $\forall A \in \mathcal{L}^+(\mathcal{D})$;
- (2) point (ii) shows how to define $e^{iH_0 t}$;
- (3) points (ii) and (iii) show that

$$
\alpha^t(A) = \tau_0 - \lim_{L} e^{itH_L}Ae^{-itH_L} = \left(\tau_0 - \lim_{L} e^{itH_L}\right) A \left(\tau_0 - \lim_{L} e^{-itH_L}\right)
$$

- (4) If for a physical system $\Sigma$ an hamiltonian operator exists, $H_0$, then the **natural algebraic framework** is $\mathcal{L}^+(\mathcal{D})[\tau_0]$ rather than $B(\mathcal{H})$ (indeed, if $\text{dim}(\mathcal{H}) = \infty$, $H_0 \notin B(\mathcal{H})$, $\delta$ does not map $B(\mathcal{H})$ into itself, and so on)
Let now $H = H_0 + B$, and suppose that the spectral decomposition of the *free hamiltonian* $H_0$ is explicitly known while the spectral decomposition of the *perturbed hamiltonian* $H$ cannot be exactly found.
Let now $H = H_0 + B$, and suppose that the spectral decomposition of the \textit{free hamiltonian} $H_0$ is explicitly known while the spectral decomposition of the \textit{perturbed hamiltonian} $H$ cannot be exactly found.

So the \textit{convenient algebraic structure} is again $\mathcal{L}^\dagger(\mathcal{D})$, with $\mathcal{D} = D^\infty(H_0)$, (since, if $H_0$ has discrete spectrum, we know an o.n. set in $\mathcal{D}$ and, as a consequence, we know $\mathcal{D}$) but the \textit{technically convenient topology}, $\tau$, is that given by the seminorms

$$\mathcal{L}^\dagger(\mathcal{D}) \ni A \mapsto \|A\|_{f,k}^f = \max\{\|H^k Af(H)\|, \|f(H)AH^k\|\}.$$ 

We proved that, if (1) $D(H_0) \subseteq D(B)$ and if $H = H_0 + B$ is self-adjoint on $D(H_0)$, and (2) if $D^\infty(H_0) = D^\infty(H)$ (for which we gave necessary and sufficient conditions), then $\tau_0 \equiv \tau$. 
Other approaches:


2. fixed point results [B., Publ.RIMS 2001]
X. Work in progress

Our point of view is now slightly different: the algebraic framework is fixed. Different hamiltonians give rise to different problems:

Let $S$ be a self-adjoint, unbounded, densely defined operator on $\mathcal{H}$. For simplicity we assume $S = \sum_{i=0}^{\infty} s_i P_i$

Let $D = D^\infty(S)$, $L^\dagger(D)$ and $\tau$ constructed as usual.

Let $H_L = \sum_{i=0}^{L} h_i P_i$ be our regular hamiltonian: $H_L \in B(\mathcal{H}), \forall L$.

In our previous attempt we always had $\tau - \lim H_L \in L^\dagger(D)$. We have just proven that this is not necessary, i.e., for each sequence $\{h_i\}$, if $\{s_i^{-1}\}$ is in $l^2(\mathbb{N}_0)$,
X. Work in progress

Our point of view is now slightly different: the algebraic framework is fixed. Different hamiltonians give rise to different problems:

Let $S$ be a self-adjoint, unbounded, densely defined operator on $\mathcal{H}$. For simplicity we assume $S = \sum_{l=0}^{\infty} s_l P_l$

Let $D = D^\infty(S)$, $\mathcal{L}^\dagger(D)$ and $\tau$ constructed as usual.

Let $H_L = \sum_{l=0}^{L} h_l P_l$ be our regular hamiltonian: $H_L \in B(\mathcal{H})$, $\forall L$.

In our previous attempt we always had $\tau - \lim H_L \in \mathcal{L}^\dagger(D)$. We have just proven that this is not necessary, i.e., for each sequence $\{h_l\}$, if $\{s_l^{-1}\}$ is in $l^2(\mathbb{N}_0)$,

1. $e^{iH_L t}$ $\tau$-converges to an element $T_t \in \mathcal{L}^\dagger(D)$;
X. Work in progress

Our point of view is now slightly different: the algebraic framework is fixed. Different hamiltonians give rise to different problems:

let $S$ be a self-adjoint, unbounded, densely defined operator on $\mathcal{H}$. For simplicity we assume $S = \sum_{l=0}^{\infty} s_l P_l$

Let $\mathcal{D} = D^\infty(S), \mathcal{L}^\dagger(\mathcal{D})$ and $\tau$ constructed as usual.

Let $H_L = \sum_{l=0}^{L} h_l P_l$ be our regular hamiltonian: $H_L \in B(\mathcal{H}), \forall L$.

In our previous attempt we always had $\tau - \lim H_L \in \mathcal{L}^\dagger(\mathcal{D})$. We have just proven that this is not necessary, i.e., for each sequence $\{h_l\}$, if $\{s_l^{-1}\}$ is in $l^2(\mathbb{N}_0)$,

1. $e^{iH_L t}$ $\tau$-converges to an element $T_t \in \mathcal{L}^\dagger(\mathcal{D})$;

2. $\forall X \in \mathcal{L}^\dagger(\mathcal{D})$ the sequence $e^{iH_L t} X e^{-iH_L t}$ $\tau$-converges to an element $\alpha^t(X) \in \mathcal{L}^\dagger(\mathcal{D})$;
Our point of view is now slightly different: the algebraic framework is fixed. Different hamiltonians give rise to different problems:

let $S$ be a self-adjoint, unbounded, densely defined operator on $\mathcal{H}$. For simplicity we assume $S = \sum_{l=0}^{\infty} s_l P_l$

Let $\mathcal{D} = D^\infty(S)$, $\mathcal{L}^\dagger(\mathcal{D})$ and $\tau$ constructed as usual. Let $H_L = \sum_{l=0}^{L} h_l P_l$ be our *regular hamiltonian*: $H_L \in B(\mathcal{H})$, $\forall L$.

In our previous attempt we always had $\tau - \text{lim} H_L \in \mathcal{L}^\dagger(\mathcal{D})$. We have just proven that this is not necessary, i.e., for each sequence $\{h_l\}$, if $\{s_l^{-1}\}$ is in $l^2(\mathbb{N}_0)$,

1. $e^{iH_L t} \tau$-converges to an element $T_t \in \mathcal{L}^\dagger(\mathcal{D})$;

2. $\forall X \in \mathcal{L}^\dagger(\mathcal{D})$ the sequence $e^{iH_L t} X e^{-iH_L t} \tau$-converges to an element $\alpha^t(X) \in \mathcal{L}^\dagger(\mathcal{D})$;

3. $\forall X \in \mathcal{L}^\dagger(\mathcal{D})$ we have $\alpha^t(X) = T_t X T_{-t}$;
4. if $Q_M = \sum_{i=0}^{M} X \in \mathcal{L}^\dagger(D)$, $X_M = Q_M X Q_M$ and
\[ \delta_L(X_M) = i[H_L, X_M] \] then

\[ \alpha^t(X) = \tau - \lim_{L,M,N} \sum_{j=0}^{N} \frac{t^j}{j!} \delta_L^j(X_M). \]
4. if $Q_M = \sum_{i=0}^{M} X \in \mathcal{L}^\dagger(\mathcal{D})$, $X_M = Q_M X Q_M$ and 
$\delta_L(X_M) = i[H_L, X_M]$ then 

$$\alpha^t(X) = \tau - \lim_{L,M,N} \sum_{j=0}^{N} \frac{t^j}{j!} \delta^j_L(X_M).$$

**Remarks:** (1) the time evolution of each element of $\mathcal{L}^\dagger(\mathcal{D})$ can be defined (in three different ways!) even if $H_L$ does not define an **hamiltonian** of the system $\Sigma$, i.e. even if $H_L$ does not converge in any **natural topology** (e.g. mean field spin models).
4. if $Q_M = \sum_{i=0}^{M} X \in \mathcal{L}^\dagger(\mathcal{D})$, $X_M = Q_M X Q_M$ and 
$\delta_L(X_M) = i[H_L, X_M]$ then

$$\alpha^t(X) = \tau - \lim_{L,M,N} \sum_{j=0}^{N} \frac{t^j}{j!} \delta^j_L(X_M).$$

**Remarks:**
(1) the time evolution of each element of
$\mathcal{L}^\dagger(\mathcal{D})$ can be defined (in three different ways!) even 
if $H_L$ does not define an *hamiltonian* of the system
$\Sigma$, i.e. even if $H_L$ does not converge in *any natural topology* (e.g. mean field spin models).

(2) However, here we are assuming that $S$ and $H_L$ admit the same spectral projections. **What if this is not the case?**
Suppose now that \( S = \sum_{i=0}^{\infty} s_i P_i \) and \( H_M = \sum_{i=0}^{M} h_i E_i \), with \( E_j \neq P_j \). Then

- if \( [E_i, P_j] = 0 \) for all \( i, j \), or if \( E_i \neq P_i \) only for a finite number of \( i \)'s, the above results still can be proved;
Suppose now that $S = \sum_{l=0}^{\infty} s_l P_l$ and $H_M = \sum_{l=0}^{M} h_l E_l$, with $E_j \neq P_j$. Then

- if $[E_l, P_j] = 0$ for all $l, j$, or if $E_l \neq P_l$ only for a finite number of $l$’s, the above results still can be proved;

- let’s call $\varphi_l$ and $\psi_l$ the eigensates of $S$ and $H_L$, then $P_l = |\varphi_l><\varphi_l|$ and $E_l = |\psi_l><\psi_l|$. $[E_l, P_j] \neq 0$ in general. Nevertheless, if $\psi_l$ is a finite linear combination of the $\varphi_j$’s, then again the above results still can be proved.
Suppose now that \( S = \sum_{i=0}^{\infty} s_i P_i \) and \( H_M = \sum_{i=0}^{M} h_i E_i \), with \( E_j \neq P_j \). Then

- if \([E_i, P_j] = 0\) for all \( i, j \), or if \( E_i \neq P_i \) only for a finite number of \( i \)'s, the above results still can be proved;

- let's call \( \varphi_i \) and \( \psi_i \) the eigensates of \( S \) and \( H_L \), then \( P_i = |\varphi_i><\varphi_i| \) and \( E_i = |\psi_i><\psi_i| \). \([E_i, P_j] \neq 0\) in general. Nevertheless, if \( \psi_i \) is a finite linear combination of the \( \varphi_j \)'s, then again the above results still can be proved.

**Further:** let \( \rho_L := \frac{e^{-\beta H_L}}{\text{tr}_L(e^{-\beta H_L})} \) be the density matrix of a Gibbs state. Then \( \tau - \lim_L \rho_L \) exists in \( \mathcal{L}^\dagger(\mathcal{D}) \). But: is this limit a KMS state? In which sense?
Suppose now that $S = \sum_{i=0}^{\infty} s_i P_i$ and $H_M = \sum_{i=0}^{M} h_i E_i$, with $E_j \neq P_j$. Then

- if $[E_i, P_j] = 0$ for all $i,j$, or if $E_i \neq P_i$ only for a finite number of $i$’s, the above results still can be proved;

- let’s call $\varphi_i$ and $\psi_i$ the eigensates of $S$ and $H_L$, then $P_i = |\varphi_i><\varphi_i|$ and $E_i = |\psi_i><\psi_i|$. $[E_i, P_j] \neq 0$ in general. Nevertheless, if $\psi_i$ is a finite linear combination of the $\varphi_j$’s, then again the above results still can be proved.

**Further:** let $\rho_L := \frac{e^{-\beta H_L}}{\text{tr}_L(e^{-\beta H_L})}$ be the density matrix of a Gibbs state. Then $\tau - \lim_L \rho_L$ exists in $L^\dagger(\mathcal{D})$. But: is this limit a KMS state? In which sense?

**Remark:**– not much is expected to change if $H_L$ and $S$ have *continuous spectra*. 
XI. Perspectives

The following lines are opened:

1. we need a deeper analysis of the previous results when $S$ and $H_L$ are essentially different;
XI. Perspectives

The following lines are opened:

1. we need a deeper analysis of the previous results when \( S \) and \( H_L \) are essentially different;

2. more on time evolution: we have seen that, given \( H \), then a *natural topological quasi-* algebra \((\mathcal{L}(\mathcal{D}, \mathcal{D}'), \mathcal{L}^\dagger(\mathcal{D}))\) exists in which \( e^{iHt}, \delta(), \alpha^t \) can all be defined. Few results exist if only \( \{H_V\} \) exists;
XI. Perspectives

The following lines are opened:

1. we need a deeper analysis of the previous results when $S$ and $H_L$ are essentially different;

2. more on time evolution: we have seen that, given $H$, then a natural topological quasi *-algebra $(\mathcal{L}(\mathcal{D},\mathcal{D}'), \mathcal{L}^\dagger(\mathcal{D}))$ exists in which $e^{iHt}$, $\delta()$, $\alpha^t$ can all be defined. Few results exist if only $\{H_V\}$ exists;

3. What about Goldstone’s theorem when $\alpha^t_V$ does not converge uniformly (or $\mathcal{F}$-strongly) to $\alpha^t$?
XI. Perspectives

The following lines are opened:

1. we need a deeper analysis of the previous results when $S$ and $H_L$ are essentially different;

2. more on time evolution: we have seen that, given $H$, then a *natural topological quasi *-algebra $(\mathcal{L}(\mathcal{D}, \mathcal{D}'), \mathcal{L}^\dagger(\mathcal{D}))$ exists in which $e^{iHt}$, $\delta()$, $\alpha^t$ can all be defined. Few results exist if only $\{H_V\}$ exists;

3. What about Goldstone’s theorem when $\alpha^t_V$ does not converge uniformly (or $\mathcal{F}$-strongly) to $\alpha^t$?

4. Can we define a KMS state when $\alpha^t_V$ does not converge uniformly (or $\mathcal{F}$-strongly) to $\alpha^t$?
XI. Perspectives

The following lines are opened:

1. we need a deeper analysis of the previous results when $S$ and $H_L$ are essentially different;

2. more on time evolution: we have seen that, given $H$, then a *natural topological quasi* ℱ-algebra $(\mathcal{L}(\mathcal{D},\mathcal{D}'), \mathcal{L}^\dagger(\mathcal{D}))$ exists in which $e^{iHt}$, $\delta()$, $\alpha^t$ can all be defined. Few results exist if only $\{H_V\}$ exists;

3. What about Goldstone's theorem when $\alpha^t_V$ does not converge uniformly (or $\mathcal{F}$-strongly) to $\alpha^t$?

4. Can we define a KMS state when $\alpha^t_V$ does not converge uniformly (or $\mathcal{F}$-strongly) to $\alpha^t$?

5. Is there any relation between these *Kms-like states* and the phase structure of the physical system?
6. Is there any relation between these *Kms-like states* and the Tomita-Takesaki theory? (something is discuss in [Antoine, Inoue, Trapani 2002])
6. Is there any relation between these Kms-like states and the Tomita-Takesaki theory? (something is discuss in [Antoine, Inoue, Trapani 2002])

7. What about local modifications? Does two states \( \rho \) and \( \chi \) which are only locally different generate unitarily equivalent representations? And what can be said about the related effective hamiltonians? (Some results are contained in [Trapani and B., JMP (1996)])
6. Is there any relation between these \textit{Kms-like states} and the Tomita-Takesaki theory? (something is discuss in [Antoine, Inoue, Trapani 2002])

7. What about local modifications? Does two states $\rho$ and $\chi$ which are only \textit{locally different} generate unitarily equivalent representations? And what can be said about the related effective hamiltonians? (Some results are contained in [Trapani and B., JMP (1996)])

\textit{Enough said!!}
C*-algebras:

An algebra $\mathcal{A}$ is a vector space over $\mathbb{C}$ with a multiplication law such that $\forall A, B \in \mathcal{A}, AB \in \mathcal{A}$. Also, two such elements can be summed up and the following properties hold:

\[
A(BC) = (AB)C, \quad A(B + C) = AB + AC, \quad (\alpha A)(\beta B) = \alpha\beta(AB)
\]

An involution is a map $* : \mathcal{A} \rightarrow \mathcal{A}$ such that

\[
A^{**} = A, \quad (AB)^* = B^*A^*, \quad (\alpha A + \beta B)^* = \overline{\alpha}A^* + \overline{\beta}B^*
\]

A *-algebra $\mathcal{A}$ is an algebra with an involution *. $\mathcal{A}$ is a normed algebra if there exists a map, the norm of the algebra, $\| \cdot \| : \mathcal{A} \rightarrow \mathbb{R}_+$, such that:

\[
\| A \| \geq 0, \quad \| A \| = 0 \iff A = 0, \quad \| \alpha A \| = |\alpha| \| A \|, \\
\| A + B \| \leq \| A \| + \| B \|, \quad \| AB \| \leq \| A \| \| B \|
\]
C*-algebras:

An algebra $\mathcal{A}$ is a vector space over $\mathbb{C}$ with a multiplication law such that $\forall A, B \in \mathcal{A}, AB \in \mathcal{A}$. Also, two such elements can be summed up and the following properties hold:

$$A(BC) = (AB)C, \quad A(B + C) = AB + AC, \quad (\alpha A)(\beta B) = \alpha\beta(AB)$$

An involution is a map $*: \mathcal{A} \rightarrow \mathcal{A}$ such that

$$A^{**} = A, \quad (AB)^* = B^*A^*, \quad (\alpha A + \beta B)^* = \bar{\alpha}A^* + \bar{\beta}B^*$$

A *-algebra $\mathcal{A}$ is an algebra with an involution *. $\mathcal{A}$ is a normed algebra if there exists a map, the norm of the algebra, $\|\cdot\|: \mathcal{A} \rightarrow \mathbb{R}_+$, such that:

$$\|A\| \geq 0, \quad \|A\| = 0 \iff A = 0, \quad \|\alpha A\| = |
\alpha| \|A\|, \quad \|A + B\| \leq \|A\| + \|B\|, \quad \|AB\| \leq \|A\| \|B\|$$

If $\mathcal{A}$ is complete wrt $\|\cdot\|$ then it is called a Banach algebra, or a Banach *-algebra if $\|A^*\| = \|A\|$.
**C*-algebras:**

An algebra \( \mathcal{A} \) is a vector space over \( \mathbb{C} \) with a multiplication law such that \( \forall A, B \in \mathcal{A}, AB \in \mathcal{A} \). Also, two such elements can be summed up and the following properties hold:

\[
A(BC) = (AB)C, \quad A(B + C) = AB + AC, \quad (\alpha A)(\beta B) = \alpha \beta (AB)
\]

An involution is a map \( * : \mathcal{A} \to \mathcal{A} \) such that

\[
A^{**} = A, \quad (AB)^* = B^* A^*, \quad (\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*
\]

A \( * \)-algebra \( \mathcal{A} \) is an algebra with an involution \( * \). \( \mathcal{A} \) is a normed algebra if there exists a map, the norm of the algebra, \( \| \cdot \| : \mathcal{A} \to \mathbb{R}_+ \), such that:

\[
\| A \| \geq 0, \quad \| A \| = 0 \iff A = 0, \quad \| \alpha A \| = |\alpha| \| A \|,
\]

\[
\| A + B \| \leq \| A \| + \| B \|, \quad \| AB \| \leq \| A \| \| B \|
\]

If \( \mathcal{A} \) is complete wrt \( \| \cdot \| \) then it is called a Banach algebra, or a Banach \( * \)-algebra if \( \| A^* \| = \| A \| \).

Finally, a \( C^* \)-algebra is a Banach \( * \)-algebra with the property \( \| A^* A \| = \| A \|^2 \).
Remarks:

(1) All the non abelian C*-algebras are isomorphic to a norm-closed, *-closed, algebra of bounded operators on a Hilbert space.
Remarks:

(1) All the non abelian C*-algebras are isomorphic to a norm-closed, *-closed, algebra of bounded operators on a Hilbert space.

(2) All the abelian C*-algebras are isomorphic to the *-algebra of continuous functions, over a locally compact Hausdorff space $X$, which vanishes at infinity, $C_o(X)$. □
Irreducible representations:

A representation $\pi$ in a Hilbert space $\mathcal{H}$ is *irreducible* if there exists no Hilbert subspace $\mathcal{K}$ of $\mathcal{H}$ which is stable under the action of $\pi$: $\pi(\mathcal{A})\mathcal{K} \subset \mathcal{K}$. $\square$
More on states:

Lanford and Ruelle introduced the notion of *states with short range correlations*: let $B$ be a local bounded observable, $\epsilon$ a positive number. Then $\rho$ is a src state if there exists a bounded region $\Lambda$ such that, for all $A$ bounded and localized in $\bar{\Lambda}$, then

$$|\rho(AB) - \rho(A)\rho(B)| \leq \epsilon\|A\|$$

Ruelle (1969) proved that each *pure state satisfies this requirement*.
More on states:

Lanford and Ruelle introduced the notion of states with short range correlations: let $B$ be a local bounded observable, $\epsilon$ a positive number. Then $\rho$ is a src state if there exists a bounded region $\Lambda$ such that, for all $A$ bounded and localized in $\bar{\Lambda}$, then

$$|\rho(AB) - \rho(A)\rho(B)| \leq \epsilon \|A\|$$

Ruelle (1969) proved that each pure state satisfies this requirement.

A weaker requirement is the asymptotic abelianess: for each local $A, B$

$$\|\rho(A\gamma_j(B)) - \rho(A)\rho(\gamma_j(B))\| \to 0,$$

when $|j| \to \infty$. □
GNS theorem:

Let $\rho$ be a state over a C*-algebra $\mathfrak{A}$ with unit. It follows that there exists a cyclic representation $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ of $\mathfrak{A}$ such that, $\forall A \in \mathfrak{A}$,

$$\rho(A) = \langle \Omega_\rho, \pi_\rho(A)\Omega_\rho \rangle,$$

and, consequently, $\|\Omega_\rho\| = \|\rho\| = 1$. Moreover the representation is unique up to unitary equivalence.
GNS theorem:

Let $\rho$ be a state over a C*-algebra $\mathcal{A}$ with unit. It follows that there exists a cyclic representation $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ of $\mathcal{A}$ such that, $\forall A \in \mathcal{A}$,

$$\rho(A) = \langle \Omega_\rho, \pi_\rho(A)\Omega_\rho \rangle,$$

and, consequently, $\|\Omega_\rho\| = \|\rho\| = 1$. Moreover the representation is unique up to unitary equivalence.

**Proof** $\mathcal{A}$ becomes a pre-Hilbert space wrt the following positive semidefinite scalar product: $(A, B) = \rho(A^*B)$. Let $\mathcal{I}_\rho = \{A \in \mathcal{A} : \rho(A^*A) = 0\}$. This is a left ideal of $\mathcal{A}$ ($A \in \mathcal{I}_\rho, X \in \mathcal{A}$ then $XA \in \mathcal{I}_\rho$). We introduce the equivalence classes: $[A] = A + I, I \in \text{call}_\rho$ which produce a complex vector space when equipped with the following operations: $[A] + [B] = [A + B], [\lambda A] = \lambda[A]$. In this way the set $\{[A], A \in \mathcal{A}\}$ is equipped with a positive definite scalar product: $\langle [A], [B] \rangle = (A, B) = \rho(A^*B)$. If we complete $\{[A], A \in \mathcal{A}\}$ wrt the norm inherited from $\langle, \rangle$ we get our Hilbert space $\mathcal{H}_\rho$.

The representation $\pi_\rho$ is defined by $\pi_\rho(A)[B] := [AB]$, while the cyclic vector $\Omega_\rho$ is simply $\Omega_\rho = [I]$. Notice that, incidentally, $\pi_\rho$ is a bounded representation (clearly!!): $\|\pi_\rho(A)[B]\| \leq \|A\|\|B\|$, for each $A, B \in \mathcal{A}$. □
KMS states, 1:

Let $\Sigma$ be a finite system. Then a state $\rho$ of $\Sigma$ is a KMS-state at an inverse temperature $\beta$ if, for all observables $A, B$ and for all $t \in \mathbb{R}$,

$$\rho(A_t B) = \rho(B A_{t+i\hbar\beta})$$

For infinite systems this definition does not work, in general, since, e.g., $A_{t+i\hbar\beta}$ may make no sense. Therefore, (Haag-Hugenholtz-Winnink) (1967) proposed a different definition:

for each $A, B \in \mathfrak{A}$ there exists a complex function $F_{AB}(z)$ which is analytical in the strip $\Im(z) \in [0, \hbar\beta]$, continuous on the boundaries, and such that

$$F_{AB}(t) = \rho(B A_t), \quad F_{AB}(t + i\hbar\beta) = \rho(A_t B)$$
KMS states, 2:

Physical interpretation: A KMS-state at an inverse temperature $\beta$, $\rho_\beta$, is a reservoir at that temperature. Given $\Sigma$ described by $\rho_\beta$ weakly coupled with a finite system $S$, in the limit $t \to \infty$, $S$ is described by a Gibbs state corresponding to the same inverse temperature $\beta$ of $\Sigma$. □
Entropy as mean surprise:

Let us consider a set of $M$ elementary events $\{\mathcal{E}_1, \ldots, \mathcal{E}_M\}$ and let $p_j$ be the probability that $\mathcal{E}_j$ occurs: $p_j \geq 0$, $\sum_{j=1}^{M} p_j = 1$. Let $u_j = -\log(p_j)$ be the surprise related to $\mathcal{E}_j$: if $p_j \simeq 0^+$ then $u_j \simeq +\infty$, while if $p_j \simeq 1^-$ then $u_j \simeq 0$.

**Interpretation**: if we are sure that $\mathcal{E}_j$ is going to happen ($p_j \simeq 1^-$), then there is no surprise at all ($u_j \simeq 0$). If $\mathcal{E}_j$ is extremely rare ($p_j \simeq 0^+$), then the surprise is very big ($u_j \simeq \infty$).

The *mean surprise* is defined as

$$MS = \frac{\sum_{i=1}^{M} N_i u_i}{\sum_{i=1}^{M} N_i} = \sum_{i=1}^{M} p_i u_i = - \sum_{i=1}^{M} p_i \log(p_i),$$

where $p_i = N_i / N$, which is exactly the Shannon expression of the entropy. □
Subadditivity of the entropy:

The entropy satisfy the following property (SA):

for each $V_1 \cap V_2 = \emptyset$ then

$$S_{V_1 \cup V_2}(\rho) \leq S_{V_1} + S_{V_2},$$

as well as the strong subadditivity property [Lieb, Ruskai]: for each $V_1, V_2$ then

$$S_{V_1 \cup V_2}(\rho) + S_{V_1 \cap V_2}(\rho) \leq S_{V_1} + S_{V_2}.$$
Some remarks about phase transitions:

Different states produce different GNS representations which describe different phases of the matter. A phase transition implies the transition from a state to another. For instance, for Ising model we are describing a *second order phase transition*.

The analysis of first order phase transition implies the analysis of the (first derivative) of some thermodynamical functional (e.g. the Gibbs functional). Sewell proved that *whenever the first derivatives of the Gibbs functional are continuous, there is no macroscopic degeneracy, while such a degeneracy appears whenever some derivative has some discontinuity*.

Therefore algebraic and thermodynamical languages matches!!
Goldstone’s theorem:

Let $\gamma_\lambda$ be a 1-parameter group of automorphisms of $\mathcal{A}_0$, commuting with the space translations, and generated by a local charge $Q_R$ on $\mathcal{A}_0$:

$$
\gamma_\lambda(A) = \lim_{R,\infty} e^{iQ_R\lambda} A e^{-iQ_R\lambda}, A \in \mathcal{A}_0,
$$

where this limit is uniform in $\lambda$, together with its derivative. Suppose further that $[\gamma_\lambda, \alpha^t_v] = 0$ and that $\alpha^t_v$ is norm-convergent to $\alpha^t$. Then $\gamma_\lambda$ has an unique extension to $\mathcal{A} = \overline{\mathcal{A}_0} \|\|$, commuting with $\alpha^t$. Moreover, if $\psi_0$ is a transactionally invariant state such that

$$
\lim_{R,\infty} \psi_0([Q_R, B]) \neq 0, B \in \mathcal{A},
$$

and if $Q_R = \int_{|\vec{x}| \leq R} j_o(\vec{x}, t) d^3x$ and $\psi_0([j_o(\vec{x}, t), B])$ is absolutely integrable in $\vec{x}$, then:

$\gamma_\lambda$ is spontaneously broken in $\pi\psi_0$ and there exist quasi particle excitations (Goldstone bosons) with infinite lifetime as $k \to 0$ and $\omega(k) \to 0$ when $k \to 0$. (No mass gap exists!)
Goldstone’s theorem 2:

More explicitly one can prove that a complete set of vectors $|\vec{k}, \omega_\gamma(k)\rangle$ does exist such that $\omega_\gamma(k) \to 0$ when $k \to 0$.

These states are labeled by $\gamma$, $\vec{k}$ and $\omega_\gamma(k)$ so that they are common eigenvectors of the symmetry, the hamiltonian and the momentum operators.

**In the elementary particle physicist language:** if a theory has a continuous symmetry of the lagrangian which is not a symmetry of the vacuum then we have a massless boson in the theory.

□
KMS and Gibbs states:

It is not evident that the infinite volume limit of a sequence of β-Gibbs states is a β-KMS state. Indeed, in [Haag, Local Quantum Physics, 1992], it is stated the following result (in a simplified version):

if \( \alpha^t_V \) is uniformly convergent to \( \alpha^t \) and if, calling \( \omega_{\beta}^{(V)}(A) = tr_V(\rho_{\beta,V}A) \), with \( \rho_{\beta,V} = \frac{e^{-\beta H_V}}{tr(idem)} \), the limit \( \lim_V \omega_{\beta}^{(V)}(A) = \omega_{\beta}(A) \) exists for each \( A \in \mathcal{A}_0 \), then:

a. \( \omega_{\beta} \) is a β-KMS state wrt \( \alpha^t \);
b. \( \omega_{\beta} \) is associated to a modular operator and a modular conjugation (in the sense of TT).

**Remark:** we are requiring uniform convergence of \( \alpha^t_V \) and the existence of the limit of \( \omega_{\beta}^{(V)}(A) \)!!

However the first assumption does not hold for long range interactions! □
Lindblad's theorem:

Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. A map $f : \mathcal{A} \rightarrow \mathcal{B}$ is **positive** if $f(A) > 0$ for each $A > 0$. It is **completely positive**, CP, if, for any finite matrix algebra $\mathcal{M}$, the mapping $f \otimes I : \mathcal{A} \otimes \mathcal{M} \rightarrow \mathcal{B} \otimes \mathcal{M}$ is positive.
Lindblad’s theorem:

Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. A map $f : \mathcal{A} \to \mathcal{B}$ is **positive** if $f(A) > 0$ for each $A > 0$. It is **completely positive**, CP, if, for any finite matrix algebra $\mathcal{M}$, the mapping $f \otimes I : \mathcal{A} \otimes \mathcal{M} \to \mathcal{B} \otimes \mathcal{M}$ is positive.

**Examples:** (1) The automorphisms of C*-algebras are CP. (2) Let $\mathcal{K} \subset \mathcal{H}$ be both Hilbert spaces and $P$ the projection operator from $\mathcal{H}$ into $\mathcal{K}$. Then $f(A) = PAP$ is CP.
Lindblad's theorem:

Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. A map $f : \mathcal{A} \rightarrow \mathcal{B}$ is **positive** if $f(A) > 0$ for each $A > 0$. It is **completely positive**, CP, if, for any finite matrix algebra $\mathcal{M}$, the mapping $f \otimes I : \mathcal{A} \otimes \mathcal{M} \rightarrow \mathcal{B} \otimes \mathcal{M}$ is positive.

**Examples:** (1) The automorphisms of C*-algebras are CP. (2) Let $\mathcal{K} \subset \mathcal{H}$ be both Hilbert spaces and $P$ the projection operator from $\mathcal{H}$ into $\mathcal{K}$. Then $f(A) = PAP$ is CP.

A **a quantum dynamical semi-group** is a set \( \{ T_t : t \geq 0 \} \) of completely positive, identity preserving elements maps of $\mathcal{A}$ such that $T_sT_t = T_{s+t}$ for all $s, t \geq 0$ and such that $T_0 = 1$. If $T_t$ is normwise continuous in $t$ for all $A \in \mathcal{A}$, then there exists an infinitesimal generator $L$ defined by the formula

$$\frac{d}{dt} T_tA = LT_tA = T_tL_A, \quad \forall A \in \mathcal{A}$$
Lindblad proved that, if $\mathcal{A} = B(\mathcal{H})$ (for some $\mathcal{H}$), then $L$ has necessarily the following expression:

$$LA = i[H, A] + \sum_j \left( V_j^*AV_j - \frac{1}{2}\{V_j^*V_j, A\} \right),$$

where $H$ is self-adjoint and $V_j, \sum_j V_j^*V_j \in \mathcal{A}$.

The technical difficulty in this case is that $T_t(AB) \neq T_t(A)T_t(B)$ if $T_t$ is only a semigroup. So we can cut-off the system (and the operators we get are bounded) but, because of this inequality, removing the cutoff is quite hard!
Tomita-Takesaki theory:

We start recalling that a *von Neumann algebra* is a set selfadjoint \( \mathcal{M} \subset B(\mathcal{H}) \) such that \( \mathcal{M} = \mathcal{M}'' \). Equivalently, \( \mathcal{M} \subset B(\mathcal{H}) \) is a VNA if it is weakly closed or strongly closed.

This implies that every VNA is a C*-algebra (\( \mathcal{M} \) weakly closed implies \( \mathcal{M} \) uniformly closed), while not any C*-algebra is a VNA (e.g. \( C_0(X) \)). Tomita-Takesaki theorem is given for \( \sigma \)-finite VNAs, for which a cyclic and separating vector surely exists. Let \( \Omega \) be this vector, and let \( S_0 \) and \( F_0 \) be the densely defined operators

\[
S_0 A \Omega = A^* \Omega, \quad F_0 A' \Omega = A'^* \Omega, \quad \forall A \in \mathcal{M}, \; \forall A' \in \mathcal{M}'.
\]

These operators are closable, \( S = \overline{S_0}, \; F = \overline{F_0} \), and the polar decomposition of \( S \), \( S = J \Delta^{1/2} \), produces the *modular conjugation* \( J \) and the *modular operator* \( \Delta \) associated to \( (\mathcal{M}, \Omega) \), since they depend only on the VNA \( \mathcal{M} \) and on the cyclic vector \( \Omega \).
Tomita-Takesaki theorem: With the above definitions we have

\[ JMJ = M', \quad \text{and} \quad \Delta^{it} M \Delta^{-it} = M, \]

for all \( t \in \mathbb{R} \).

\[ \square \]
Why $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$ instead of $\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$?

Even if it seems more natural to represent $(\mathcal{A}, \mathcal{A}_0)$ in a $(\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi), \mathcal{L}^\dagger(\mathcal{D}_\pi))$, it is more convenient to use $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$ for the following reasons:

1. notice that any partial *-algebra is a quasi *-algebra: therefore this choice is consistent;

2. if $a \in \mathcal{A}$ then $\pi(a) \in \mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$. Therefore $\forall \varphi \in \mathcal{D}_\pi \pi(a)\varphi \in \mathcal{H}_\pi$. Of course, if we decide to represent $a$ as an element of $\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$, we go out of the Hilbert space!

3. we also have a technical reason to use $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$: in the theorem on the derivations the $\tau_s$ topology plays a role, and this can be defined on $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$ but not on $\mathcal{L}(\mathcal{D}_\pi, \mathcal{D}'_\pi)$ □
Let $P(\mathcal{A})$ the set of all the sesquilinear forms satisfying the previous requirements. The following properties immediately follow:

(a) $\varphi(xa, b) = \varphi(a, x^* b)$, $\forall a, b \in \mathcal{A}$, $x \in \mathcal{A}_0$;
(b) $\varphi(a, b) = \varphi(b, a)$, $\forall a, b \in \mathcal{A}$;
(c) $|\varphi(a, b)|^2 \leq \varphi(a, a) \varphi(b, b)$, $\forall a, b \in \mathcal{A}$.

These imply that

$$N_\varphi = \{ a \in \mathcal{A} : \varphi(a, a) = 0 \} =$$

$$= \{ a \in \mathcal{A} : \varphi(a, b) = 0, \forall b \in \mathcal{A} \}.$$

Let $\lambda_\varphi(\mathcal{A}) = \mathcal{A}/N_\varphi$ and let us introduce a scalar product on this vector space as follows: $\langle \lambda_\varphi(a), \lambda_\varphi(b) \rangle = \varphi(a, b)$.

Let further $H_\varphi$ be the completion of $\lambda_\varphi(\mathcal{A})$ wrt the norm inherited by this scalar product.

One can check that $\lambda_\varphi(\mathcal{A}_0)$ is dense in $H_\varphi$.  

GNS-like representation
Let us finally define a map $\pi^0_\varphi$:

$$\pi^0_\varphi(a)\lambda_\varphi(x) = \lambda_\varphi(ax), \quad a \in \mathcal{A}, x \in \mathcal{A}_0 :$$

Then: $\pi^0_\varphi$ is a *-representation of $\mathcal{A}$ in $\mathcal{L}^\dagger(\lambda_\varphi(\mathcal{A}_0), \mathcal{H}_\varphi)$.

Moreover, if $(\mathcal{A}, \mathcal{A}_0)$ has a unit $e$ then
(i) $\lambda_\varphi(\mathcal{A}_0) = \pi^0_\varphi(\mathcal{A}_0)\lambda_\varphi(e)$ (i.e. $\lambda_\varphi(e)$ is ultra-cyclic);
(ii) $\varphi(a, b) = \langle \pi^0_\varphi(a)\lambda_\varphi(e), \pi^0_\varphi(b)\lambda_\varphi(e) \rangle, \quad \forall a, b \in \mathcal{A}.$ □
The algebra for the infinite system

Let $\Sigma$ be a continuous system. Then the construction of the C*-algebra for $\Sigma$ can be sketched as follows:

The construction in the topological quasi*-algebra situation goes as follows (for a spin system):
The algebra for the infinite system

Let $\Sigma$ be a continuous system. Then the construction of the C*-algebra for $\Sigma$ can be sketched as follows:

The construction in the topological quasi*-algebra situation goes as follows (for a spin system):

1. let $\mathcal{H} = \mathbb{C}^2$ and, $\vec{n} \in \mathbb{R}^3$, $|\vec{n}| = 1$, and $|\vec{n} \rangle \in \mathcal{H}$ fixed (but for a phase) by requiring that $(\vec{\sigma} \cdot \vec{n})|\vec{n} \rangle = |\vec{n} \rangle$ and $|\vec{n} \rangle$ is normalized. [This is just a way to extract a certain vector $|\vec{n} \rangle$ out of $\mathcal{H}$].
2. let \{\vec{n}_p\} be a sequence of normalized vectors in \(\mathbb{R}^3\) and \{|\vec{n}_p\rangle\} the related normalized vectors in \(\mathcal{H}_p\), copies of \(\mathcal{H}\). We put \(|\{n\}\rangle := \bigotimes_{p=1}^{\infty} |\vec{n}_p\rangle\).

Of course \(|\{n\}\rangle \in \mathcal{H}_\infty := \bigotimes_{p=1}^{\infty} \mathcal{H}_p\).
2. Let \( \{ \vec{n}_p \} \) be a sequence of normalized vectors in \( \mathbb{R}^3 \) and \( \{| \vec{n}_p >\} \) the related normalized vectors in \( \mathcal{H}_p \), copies of \( \mathcal{H} \). We put \(|\{n\} > = \bigotimes_{p=1}^\infty |\vec{n}_p >\). Of course \(|\{n\} > \in \mathcal{H}_\infty := \bigotimes_{p=1}^\infty \mathcal{H}_p \).

3. Let \( \pi \) be a natural realization of \( \mathfrak{A}_0 \): \( \pi (\sigma^\alpha_j) |\{n\} > = (\bigotimes_{p\neq j} |\vec{n}_p >) \otimes (\sigma^\alpha_j |\vec{n}_j >) \), and \( \mathcal{H}_{\{n\}} \) be the closure in \( \mathcal{H}_\infty \) of the space \( \pi (\mathfrak{A}_0) |\{n\} > \). This is a separable Hilbert space (\( \mathcal{H}_\infty \) is not!!)
2. Let \( \{ \vec{n}_p \} \) be a sequence of normalized vectors in \( \mathbb{R}^3 \) and \( \{ | \vec{n}_p > \} \) the related normalized vectors in \( \mathcal{H}_p \), copies of \( \mathcal{H} \). We put \( | \{ n \} > = \bigotimes_{p=1}^{\infty} | \vec{n}_p > \). Of course \( | \{ n \} > \in \mathcal{H}_\infty := \bigotimes_{p=1}^{\infty} \mathcal{H}_p \).

3. Let \( \pi \) be a natural realization of \( \mathcal{A}_0 : \pi(\sigma_j^\alpha) | \{ n \} > = (\bigotimes_{p \neq j} | \vec{n}_p >) \otimes (\sigma_j^\alpha | \vec{n}_j >) \), and \( \mathcal{H}_{\{n\}} \) be the closure in \( \mathcal{H}_\infty \) of the space \( \pi(\mathcal{A}_0) | \{ n \} > \). This is a separable Hilbert space (\( \mathcal{H}_\infty \) is not!!)

4. An o.n. basis of \( \mathcal{H}_{\{n\}} \) is given by the set \( | \{ m \}, \{ n \} > \) = \( \bigotimes | m_p, \vec{n}_p > \), where \( m_p = 0, 1 \) for each \( p \) and \( \sum_p m_p < \infty \). Here each vector \( | m, \vec{n} > := (\vec{\sigma} \cdot \vec{n}^-)^m | \vec{n} > \), \( m = 0, 1 \), where \( \vec{n}^- = \frac{1}{2}(\vec{n}^1 - i \vec{n}^2) \), \( \vec{n}^1, \vec{n}^2 \) and \( \vec{n} \) being an o.n. set in \( \mathbb{R}^3 \).
2. Let \( \{ \vec{n}_p \} \) be a sequence of normalized vectors in \( \mathbb{R}^3 \) and \( \{ | \vec{n}_p > \} \) the related normalized vectors in \( \mathcal{H}_p \), copies of \( \mathcal{H} \). We put \( \{ n \} := \bigotimes_{p=1}^{\infty} | \vec{n}_p > \). Of course \( \{ n \} \in \mathcal{H}_\infty := \bigotimes_{p=1}^{\infty} \mathcal{H}_p \).

3. Let \( \pi \) be a natural realization of \( \mathcal{A}_0 \): \( \pi(\sigma_\alpha^j)\{ n \} := (\bigotimes_{p \neq j} | \vec{n}_p >) \otimes (\sigma_\alpha^j | \vec{n}_j >) \), and \( \mathcal{H}_{\{ n \}} \) be the closure in \( \mathcal{H}_\infty \) of the space \( \pi(\mathcal{A}_0)\{ n \} \). This is a separable Hilbert space (\( \mathcal{H}_\infty \) is not!!)

4. An o.n. basis of \( \mathcal{H}_{\{ n \}} \) is given by the set \( \{ | \{ m \}, \{ n \} > \} = \bigotimes | m_p, \vec{n}_p > \), where \( m_p = 0, 1 \) for each \( p \) and \( \sum_p m_p < \infty \). Here each vector \( | m, \vec{n} > := (\vec{\sigma} \cdot \vec{n}^{-})^m | \vec{n} > \), \( m = 0, 1 \), where \( \vec{n}^- = \frac{1}{2}(\vec{n}^1 - i \vec{n}^2) \), \( \vec{n}^1, \vec{n}^2 \) and \( \vec{n} \) being an o.n. set in \( \mathbb{R}^3 \).

5. Let \( M_{\{ n \}}|\{ m \}, \{ n \} > = (1 + \sum_p m_p)|\{ m \}, \{ n \} > \) is unbounded and self-adjoint, greater than \( 1 \). We use this to define a dense subset of \( \mathcal{H}_{\{ n \}} \), \( \mathcal{D}_{\{ n \}} = \mathcal{D}_\infty(M_{\{ n \}}) \), and the \( \mathcal{O}^* \)-algebra \( \mathcal{L}^\dagger(\mathcal{D}_{\{ n \}}) \).
6. We find that $\pi(\mathcal{A}_0) \subset \mathcal{L}^\dagger(\mathcal{D}_{\{n\}})$. 
6. We find that \( \pi(\mathcal{A}_0) \subset \mathcal{L}^\dagger(\mathcal{D}_{\{n\}}) \).

7. We can introduce a topology \( \xi \) on \( \mathcal{A}_0 \) as follows: 
\( \forall X \in \mathcal{A}_0 \) we put
\[
\|X\|^{f,k}_{\{n\}} := \max \left\{ \|f(M_{\{n\}})\pi(X)M^k_{\{n\}}\|, \|M^k_{\{n\}}\pi(X)f(M_{\{n\}})\| \right\}
\]

As we see, these seminorms are labeled by \((f, k)\) and by \(\{n\}\).
6. We find that $\pi(\mathcal{A}_0) \subset \mathcal{L}^\dagger(\mathcal{D}_\{n\})$.

7. We can introduce a topology $\xi$ on $\mathcal{A}_0$ as follows: \( \forall X \in \mathcal{A}_0 \) we put

$$
\|X\|_{f,k}^{f,k} := \max \left\{ \|f(M_{\{n\}})\pi(X)M_{\{n\}}^k\|, \|M_{\{n\}}^k\pi(X)f(M_{\{n\}})\| \right\}
$$

As we see, these seminorms are labeled by \((f, k)\) and by \(\{n\}\).

8. Taking the completion $\mathcal{A}$ of $\mathcal{A}_0$ wrt the topology $\xi$ we get a topological *-algebra. The realization of $\mathcal{A}_0$ can be extended to $\mathcal{A}$ and we find that $\hat{\pi}(\mathcal{A}) \subset \mathcal{L}^\dagger(\mathcal{D}_\{n\})$. 
6. We find that $\pi(A_0) \subset \mathcal{L}^\dagger(D_{\{n\}})$. 

7. We can introduce a topology $\xi$ on $A_0$ as follows: $\forall X \in A_0$ we put 

$$\|X\|_{f,k}^{f,k} := \max \left\{ \|f(M_{\{n\}})\pi(X)M_{\{n\}}^k\|, \|M_{\{n\}}^k\pi(X)f(M_{\{n\}})\| \right\}$$

As we see, these seminorms are labeled by $(f, k)$ and by $\{n\}$. 

8. Taking the completion $\mathcal{A}$ of $A_0$ wrt the topology $\xi$ we get a topological $*$-algebra. The realization of $A_0$ can be extended to $\mathcal{A}$ and we find that $\hat{\pi}(A) \subset \mathcal{L}^\dagger(D_{\{n\}})$. 

9. In the analysis of concrete models (BCS) it is necessary to introduce a different topology, and some limits exist in some $\mathcal{L}(D_{\{n\}}, D'_{\{n\}})$. 

6. We find that $\pi(\mathcal{A}_0) \subset L^\dagger(\mathcal{D}_{\{n\}})$.

7. We can introduce a topology $\xi$ on $\mathcal{A}_0$ as follows: For all $X \in \mathcal{A}_0$ we put

$$\|X\|^{f,k}_{\{n\}} := \max \left\{ \|f(M_{\{n\}})\pi(X)M^k_{\{n\}}\|, \|M^k_{\{n\}}\pi(X)f(M_{\{n\}})\| \right\}$$

As we see, these seminorms are labeled by $(f,k)$ and by $\{n\}$.

8. Taking the completion $\mathcal{A}$ of $\mathcal{A}_0$ wrt the topology $\xi$ we get a topological $\ast$-algebra. The realization of $\mathcal{A}_0$ can be extended to $\mathcal{A}$ and we find that

$$\hat{\pi}(\mathcal{A}) \subset L^\dagger(\mathcal{D}_{\{n\}})$$

9. In the analysis of concrete models (BCS) it is necessary to introduce a different topology, and some limits exist in some $L(\mathcal{D}_{\{n\}}, \mathcal{D}'_{\{n\}})$.

**Remark:** while in the standard approach two cutoffs are needed (*one for the volume and one for the spectrum of the unbounded observables*), with this approach we only need a single cutoff. □
About the effective hamiltonian

Let us first remind that, if $A$ is a densely defined operator in $\mathcal{H}$, then $A$ is **symmetric** if $A \subseteq A^*$, that is, if $D(A) \subseteq D(A^*)$ and $A\varphi = A^*\varphi$ for each $\varphi \in D(A)$. $A$ is **self-adjoint** iff $A$ is symmetric and if $D(A) = D(A^*)$.

The effective hamiltonian $H$ found in [Bratteli and Robinson] is symmetric but not s.a., in general. In our theorem $H = H^\dagger = H^*_\pi \subseteq L^\dagger(C_{-}\pi, C'_{\pi})$. This means, because of the definition of $^\dagger$, that $H\varphi = H^*\varphi$ for each $\varphi \in \mathcal{D}_{\pi}$. It is furthermore clear that, since $H^\dagger$ is the **restriction** of $H^*$ to $\mathcal{D}_{\pi}$, $H^*$ is in general defined on a larger domain. Therefore $H$ is symmetric in the above sense. It could be self-adjoint or not!

Therefore, since the spectral theorem holds for s.a. operators, we cannot conclude that $e^{iHt}$ does exist! □
Some contributors to the algebraic approach

Some of the people that, during the years, gave contributions to this topics, and still keep giving, are:

G. Dell’Antonio, S. Doplicher, R. Longo, F. Strocchi, G. Morchio, C. Trapani,...


General facts (in functional analysis):

1. **Operators in a Hilbert space** $\mathcal{H}$: $A$ is defined on $D(A)$ which, if $A$ is bounded, can be taken to be all of $\mathcal{H}$. If $A$ is unbounded (i.e. if $\sup_{\phi \in \mathcal{H}} \|A\phi\| = \infty$), then $D(A)$ is a proper subspace of $\mathcal{H}$. (e.g. $D(\hat{x}), D(\hat{p}) \subset L^2(\mathbb{R})$, since $xf(x) \notin L^2(\mathbb{R})$ for each $f(x) \in L^2(\mathbb{R})$).

2. **Closed operator**: First the **graph** of an operator $A$ is the following set: $\Gamma(A) := \{(\phi, A\phi), \phi \in D(A)\} \subset \mathcal{H} \times \mathcal{H}$. This is a Hilbert space wrt the following scalar product: $\langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle := \langle \phi_1, \phi_2 \rangle + \langle \psi_1, \psi_2 \rangle$. $A$ is **closed** if $\Gamma(A)$ is closed (i.e. if all any sequence in $\Gamma(A)$ converging wrt the norm inherited by this scalar product converges in $\Gamma(A)$). More explicitly $A$ is closed iff, for each sequence $\{\phi_n\} \subset D(A)$ converging to $\phi$ and such that $A\phi_n$ converges to $\Psi$, then $\Psi = A\phi$.

3. **Extension of an operator**: Given two operators $A_1$ and $A_2$ on $\mathcal{H}$ such that $\Gamma(A_1) \supset \Gamma(A_2)$, then $A_1$ is an **extension** of $A_2$. We usually write $A_1 \supset A_2$. Equivalently $A_1 \supset A_2$ iff $D(A_1) \supset D(A_2)$ and if $A_1\phi = A_2\phi$ for each $\phi \in D(A_1)$. An operator $A$ is said **closable** if it has a closed extension. Every closable operator has a smallest closed extension, called its **closure**: $\overline{A}$. We have $\Gamma(\overline{A}) = \overline{\Gamma(A)}$. In other words, $A$ is closable iff for each sequence $\{\phi_n\} \subset D(A)$ converging to $0$ and such that $A\phi_n$ converges to $\Psi$, then $\Psi = 0$. $\square$
4. Example of a closed, unbounded and densely defined operator: Let $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ and $Q$ the operator of multiplication for $x$: $Qf(x) = xf(x), \forall f(x) \in \mathcal{D}$, where $\mathcal{D} = \{f(x) \in \mathcal{L}^2(\mathbb{R}) : xf(x) \in \mathcal{L}^2(\mathbb{R})\}$. $\mathcal{D}$ is dense in $\mathcal{L}^2(\mathbb{R})$. $Q$ is an example; we can prove that, if $\{f_n\} \subset \mathcal{D}$ is such that (i) $f_n \to f$ in the $\|\cdot\|_2$–norm, (ii) $xf_n \to g$ in $\|\cdot\|_2$, then $g(x) \in \mathcal{D}$ and $g(x) = xf(x)$.

5. Adjoint of an operator, bounded case: in this case $A^*$ is defined as $\langle f, A^*g \rangle = \langle Af, g \rangle$, for each $f, g \in \mathcal{H}$. If $A = A^*$ then $A$ is self-adjoint.

6. Adjoint of an operator, unbounded case: Again we put $\langle f, A^*g \rangle = \langle Af, g \rangle$, for each $f \in D(A)$ and $g \in D(A^*)$, where $D(A^*) = \{g \in \mathcal{H} : \exists g_A \in \mathcal{H}$ such that $\langle f, g_A \rangle = \langle Af, g \rangle\}$. Obviously we have $g_A = A^*g$. $A$ is s.a. iff $D(A) = D(A^*)$ and $A = A^*$.

7. Symmetric operator: let $A$ be densely defined in $\mathcal{H}$. $A$ is symmetric if $A \subset A^*$, that is, if $D(A) \subset D(A^*)$ and $A\varphi = A^*\varphi$ for each $\varphi \in D(A)$. Equivalently, $A$ is symmetric iff $\langle Af, g \rangle = \langle f, Ag \rangle$, for each $f, g \in D(A)$. $A$ is self-adjoint iff $A$ is symmetric and if $D(A) = D(A^*)$. A symmetric operator $A$ is called essentially self-adjoint if its closure $\overline{A}$ is self-adjoint. In this case there exists only one self-adjoint extension of $A$. 
8. **Density matrices and traces**: A density matrix, $\rho$, is an operator on $\mathcal{H}$ defined as $\rho = \sum_{n=1}^{\infty} w_n P_{\psi_n}$, where $P_{\psi_n}$ are orthogonal projectors on the o.n. set $\{\psi_n\}$ and $w_n \geq 0$ with $\sum_{n=1}^{\infty} w_n = 1$. Therefore $\rho$ is bounded and positive. Clearly $tr(\rho) = \sum_{n=1}^{\infty} \langle \psi_n, \rho \psi_n \rangle = \sum_{n=1}^{\infty} w_n = 1$. Remind that $tr$ does not depend on the choice of o.n. basis.

9. **One-parameter unitary groups**: Stone’s theorem: A 1pg of unitary transformations of $\mathcal{H}$ is a family $U_t$ of unitary operators in $\mathcal{H}$, $t \in \mathbb{R}$, such that $U_t U_s = U_{t+s}$ and $U_0 = I$. This is s-continuous, if $U_t \to I$ strongly when $t \to 0$. In this case Stone’s theorem states that there exists an unique self-adjoint operator $K$ in $\mathcal{H}$ such that, $\forall f \in D(K)$,

$$\frac{d}{dt} U_t f = iK U_t f = iU_t K f,$$

which formally can be written as $U_t = e^{iKt}$. $K$ is the infinitesimal generator of the group.

More in details we have the following: *let $U_t$ be a strongly continuous 1-parameter group of unitary operators. The vectors $\psi \in \mathcal{H}$ for which $\lim_{t,0} -i \frac{U_t - I}{t} \psi$ exists form a dense set $D$ in $\mathcal{H}$. This limit defines a self-adjoint operator $K$ which is the infinitesimal generator of the 1-parameter group.*

A related result is the following: *let $K$ be a self-adjoint operator with spectral resolution $E_{\alpha}$. Then the operators $U_t = e^{iKt}$...*
\[ \int_{\mathbb{R}} e^{i t \alpha} dE_{\alpha} \text{ form a 1-parameter group of unitary operators with } K \text{ as infinitesimal generator.} \]

10. **Spectral analysis**: if \( A = A^* \) has a discrete spectrum then it can be written as \( A = \sum_{n=1}^{\infty} \lambda_n P_{\psi_n} \), where \( \{\lambda_n\} \) and \( \{\psi_n\} \) are the eigenvalues and the eigenvectors of \( A \). If \( A \) has not discrete spectrum then we have \( A = \int \lambda dE(\lambda) \), in a weak sense, where \( \{E(\lambda)\} \) is a family of intercommuting operators, such that \( E(-\infty) = 0 \), \( E(\infty) = I \), \( E(\lambda) \leq E(\lambda') \) if \( \lambda \leq \lambda' \), and \( E(\lambda) \to E(\lambda') \) if \( \lambda \to \lambda' \) from above. (if \( A \) has discrete spectrum then \( E(\lambda) = \sum_{\lambda_n < \lambda} P_{\psi_n} \))

11. **Tensor products**: \( \mathcal{H}_1 \otimes \mathcal{H}_2 = \text{linear span}\{f_1 \otimes f_2, f_j \in \mathcal{H}_j, j = 1, 2\} \), with scalar product \( \langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle = \langle f_1, g_1 \rangle_1 + \langle f_2, g_2 \rangle_2 \). The operators \( A_j \in B(\mathcal{H}_j), j = 1, 2 \), define a bounded operator \( A_1 \otimes A_2 \) on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) as \( (A_1 \otimes A_2)(f_1 \otimes f_2) = A_1 f_1 \otimes A_2 f_2 \)

12. **Locally convex topology**: The topology \( \tau \) is *locally convex* if it is given by a family of seminorms separating points. □

13. **Strong dual topology**: Let \( E \) be a locally convex space and \( F \) its dual, i.e. the set of the bounded linear functionals on \( E \). The *strong dual topology* is a topology on \( F, \beta(F, E) \), defined by the following seminorms:

\[ F \ni f \to \rho_A(f) := \sup_{x \in A} |f(x)|, \]
which are labeled by the bounded subset of $E$, $A \subset E$.

For $\mathcal{D}$ and $\mathcal{D}'$ this becomes

$$\mathcal{D}' \ni z \to \rho_\mathcal{E}(z) := \sup_{x \in \mathcal{E}} |\langle x, z \rangle|,$$

where $\langle, \rangle$ is the form which puts in duality $\mathcal{D}$ and $\mathcal{D}'$ and $\mathcal{E}$ is a bounded set in $\mathcal{D}$. □

14. Why $X^\dagger = X^*_\mathcal{D}$? We need to introduce a map which, given an element $X \in L^\dagger(\mathcal{D})$, produces another element $X^\dagger \in L^\dagger(\mathcal{D})$. The most natural choice, $X^\dagger \equiv X^*$, is possible only if $L^\dagger(\mathcal{D}) = B(\mathcal{H})$.

Recalling that $D(X^*) \supseteq \mathcal{D}$, it is clear that $X^*_{\mid \mathcal{D}}$ is well defined. Further one can prove that $\dagger$ has the properties of an involution and maps $L^\dagger(\mathcal{D})$ into itself. □

15. Why $X \Box Y \equiv X^{\dagger*} Y$? The reason is essentially that, with this definition, the weak product is distributive while this is not the case if we adopt the strong multiplication $X \circ Y \equiv \overline{XY}$. Associativity, however, is not true in general for any of these multiplications. □

16. Example of $L(\mathcal{D}, \mathcal{D}')$: Suppose $\mathcal{D} \equiv S(\mathbb{R})$, the set of the test functions, and $\mathcal{D}' = S'(\mathbb{R})$. Since $S(\mathbb{R}) \subset S'(\mathbb{R})$, it is easy to check that $L^\dagger(S) \subset L(S, S')$. Let $\Psi(x) \in S'(\mathbb{R})$. We define $Z_\Psi$ as follows: $(Z_\Psi f)(x) = \Psi(x)f(x)$, $\forall f(x) \in S(\mathbb{R})$. Since
Ψ(x)f(x) ∈ S′(R) we conclude that Z_Ψ ∈ L(S, S'). It is clear that, for instance, Z_Ψ^2, does not exist. □

17. When L^†(D) ⊂ L(D, D')?: This inclusion is not always true. However it surely holds whenever the dual D' is constructed starting from D and using its graph topology. A typical example in which the inclusion holds is when D is the D^∞(H) for some unbounded, densely defined and self-adjoint operator H. Notice that this is, more or less, what we always do! □

18. We call them **metastable** because they have a long mean life and **good** thermodynamical properties. □

19. **GTS states**: the different GTS states may be constructed as limits of a Gibbs state (for those thermodynamical conditions) with different boundary conditions. □