Shapley and Nucleolus in the bankruptcy game

(Game Theory LM-77, S. Modica)

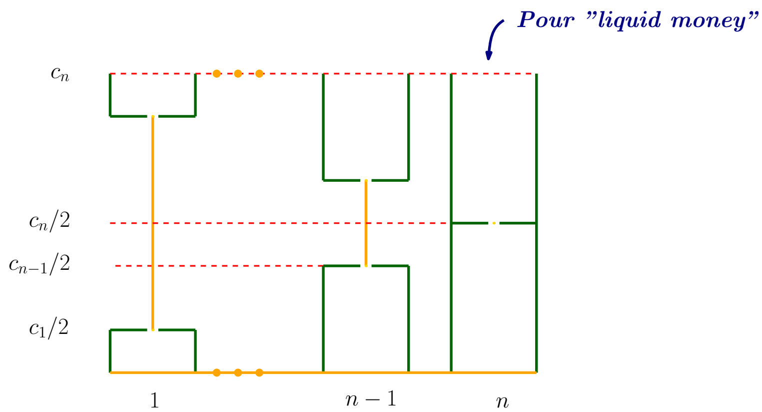
The estate is E and claims are $c_i,\ i = 1, \dots, n$ with $\sum c_i > E$. The game is defined by

$$v(S) = \max\{0, E - \sum_{i \notin S} c_i\}.$$

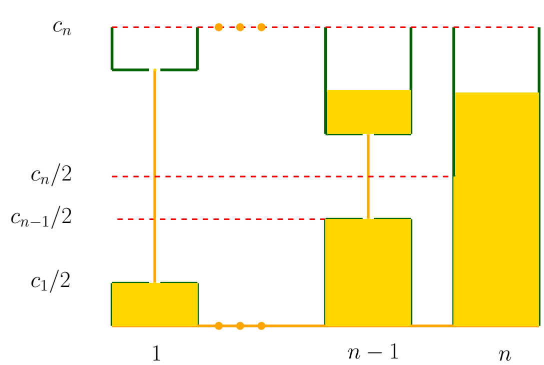
Notice that $v(N) = E$. We assume to fix ideas that $c_1 < c_2 < \cdots < c_n$. In this game the Nucleolus can be computed as follows:[1]

- If $E \leq \sum c_i/2$ then give equal incremental increases to all i such that $x_i < c_i/2$
- If $E > \sum c_i/2$ then above $c_i/2$ distribute equally to all i such that $c_i - x_i$ is highest.

This is most easily done with the help of the following picture:



these are supposed to be containers of width 1; the rightmost has height c_n so to fill it up it takes a quantity c_n of "water", and it is divided in two equal parts; the next one on its left has total height c_{n-1} and it is also divided in two equal parts. Water poured from above freely passes through the different halves of the containers, as indicated by the orange lines and the vertical pipes are supposed to have width zero. So if you pour water from above into any container the desired allocation is realized. An example is visualized in the figure below:



[1] Aumann-Maschler JET 1985

The Shapley value is computed as usual. So we can compare imputations for different values of E (for the Nucleolus you should draw the appropriate pictures). We do so for a few cases in the table below; claims are fixed at $c_1 = 100, c_2 = 200, c_3 = 300$.[2]

		x_1	x_2	x_3
$E = 200$	Shapley	$33\frac{1}{3}$	$83\frac{1}{3}$	$83\frac{1}{3}$
	Nucleolus	50	75	75
$E = 300$	Shapley	50	100	150
	Nucleolus	50	100	150
$E = 350$	Shapley	$58\frac{1}{3}$	$108\frac{1}{3}$	$183\frac{1}{3}$
	Nucleolus	50	100	200
$E = 400$	Shapley	$66\frac{2}{3}$	$116\frac{2}{3}$	$216\frac{2}{3}$
	Nucleolus	50	125	225
$E = 500$	Shapley	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$
	Nucleolus	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$

There is not much we can learn from these numbers really. That is why axioms are needed.

One more point about bankruptcy and Nucleolus

There is a a fairly strong argument in favor of the Nucleolus in the bankruptcy game, which we now present. Letting $v_i = v(\{i\})$ we first show the following

Lemma 1. $\sum v_i \leq E$.

Proof. First observe that $v_i < c_i$. For if $\sum_{j \neq i} c_j \geq E$ then $v_i = 0 < c_i$; otherwise $v_i = E - \sum_{j \neq i} c_j < \sum c_i - \sum_{j \neq i} c_j = c_i$. Next proceed by induction. Take $n = 2$. If $c_i \geq E$ then $v_1 + v_2 = v_i \leq E$; if on the other hand $c_1, c_2 \leq E$ then $v_1 + v_2 = 2E - (c_1 + c_2) < 2E - E = E$. Now assume the inequality is true for $n - 1$; then for n: if $c_1 \geq E$ then $\sum v_i = v_1 \leq E$; if $c_1 < E$ then

$$\sum_2^n v_i = \sum_2^n \max\{0, (E - c_1) - \sum_{j \neq i, \geq 2}^n c_j\} \leq E - c_1$$

by the induction hypothesis (on the game with the $n-1$ players from 2 to n and estate $E - c_1$); therefore $\sum v_i < c_1 + \sum_2^n v_i \leq E$ (using $v_1 < c_1$). □

[2] For exemplification we find the Shapley value for $E = 200$; in this case $v(N) = 200, v(2,3) = 100$ and otherwise $v(S) = 0$. Then the usual procedure gives the following table:

orderings			imputations		
1	2	3	0	0	200
1	3	2	0	200	0
2	1	3	0	0	200
2	3	1	100	0	100
3	1	2	0	200	0
3	2	1	100	100	0
Shapley			$33\frac{1}{3}$	$83\frac{1}{3}$	$83\frac{1}{3}$

Now consider the special case of $n = 2$ and the following so-called "Contested Garment Rule" on estate E:

$$x_i = v_i + \frac{1}{2}\left[E - (v_1 + v_2)\right] \tag{CGR}$$

This is pretty compelling: each player is granted her value and the non-negative excess $E - (v_1 + v_2)$ is split equally between the two.[3] It is easy to check directly from the definitions that both Shapley and the Nucleolus agree with the rule in the two-player case. For Shapley the usual table is

orderings	imputations	
12	v_1	$E - v_1$
21	$E - v_2$	v_2
Shapley	$\frac{v_1+E-v_2}{2}$	$\frac{v_2+E-v_1}{2}$

and the solution is the one we want; for the Nucleolus, the excesses of the two players at the given imputation are equal: $e(i, x) = v_i - x_i = -\left[E - (v_1 + v_2)\right]/2$.

In the general n-player game the two solutions may differ. We say that the imputation x is *consistent with the Contested Garment Rule* if for any pair of players i, j the rule over estate $x_i + x_j$ (and credits c_i and c_j) gives (x_i, x_j). And the result is the following:

Theorem (Aumann and Maschler (JET 1985))**.** *In the bankruptcy game the only imputation consistent with the Contested Garment Rule is the nucleolus.*

This is a fairly strong point in favor of the Nucleolus in this game. In fact, with Shapley there is also another problem: it is not *"population monotonic"*. The idea is that if each player is replicated and the estate is doubled then in the resulting game the imputation should give each the same payoff as in the original game; and Shapley (but not the Nucleolus as is easy to check) fails this test. Consider this example[4]: $c_1 = 200, c_2 = 300, E = 300$. Then Shapley is $x = (100, 200)$; but if there are two players with credit 100 and two with credit 200 and the estate is 600 the Shapley imputation is not $(100, 100, 200, 200)$ - in fact you can check that it is $(116\frac{2}{3}, 116\frac{2}{3}, 183\frac{1}{3}, 183\frac{1}{3})$. The Nucleolus give in both cases 100 to the first type and 200 to the second type.

Remark. Going back to the two-player case: the model also applies to the division of surplus $v(N) = v(12)$ between two partners, each of which can make $v_i \geq 0$ on her own and $v_1 + v_2 \leq v(12)$. In this case the (CGR) formula - here $x_i = v_i + [v(12) - (v_1 + v_2)]/2$ - can be useful in practice if v_i can be reliably assessed.

[3] Note that $x_i \geq v_i$ by the previous lemma.
[4] Taken from Young, *Equity,* p.71