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THE ALLOCATION OF DEBTS AND TAXES

H. Peyton Young

1. THE "RIGHTS" PROBLEM

In a bankruptcy proceeding, creditors advance claims that exceed the total value of a failed firm. Or an individual dies leaving debts in excess of the value of his estate. How should the total be divided among the creditors or heirs: Proportionally to the claims? Equally? Or by some other rule?

The following fascinating example of the settlement of such cases is mentioned in a Mishna from the Babylonian Talmud [Aumann and Maschler, 1985].[1] Three widows make claims of 100, 200, and 300 on their deceased husband's estate. The Mishna gives divisions for three possible sizes of the estate as shown in Table 1. These examples suggest that if an estate is "small" relative to the claims, then it should be divided equally; whereas if it is medium-sized it should be allocated proportionally. The case between small and medium is harder to fathom.

TABLE 1

		Claim		
		100	200	300
	100	33-1/3	33-1/3	33-1/3
Estate	200	50	75	75
	300	50	100	150

In modern U.S. bankruptcy law, the standard formula is proportional division among the creditors--i.e., so many cents on the dollar--subject to meeting various "priorities" on the claims. Yet, as the Talmudic example shows, there are situations in which equal treatment may seem more equitable.

But what is meant by "equal treatment"? Is it equal payment? Or should the focus be on the amount of the deficits? These two points of view play an important role in the theory. In some sense they are polar opposites, and proportional allocation the meridian, in a larger world of "reasonable" methods.

Let $N = \{1,2,\ldots,n\}$ represent a set of individuals, and let $u_i > 0$ be i's <u>claim</u> or <u>right</u> against a total amount T, where $0 \leq T \leq \sum_N u_i$. $(\mathbf{u},T)$ is a <u>rights problem</u>. A <u>solution</u> to $(\mathbf{u},T)$ is a vector $\mathbf{t} = (t_1,\ldots,t_n) \in R^n$ such that $\sum_N t_i = T$ and $0 \leq t_i \leq u_i$ for all $i \in N$. An allocation <u>method</u> or <u>rule</u> is a function F that associates to every problem $(\mathbf{u},T)$ a single solution $\mathbf{t} = F(\mathbf{u},T)$.

The <u>proportional</u> allocation method is to let $t_i = \lambda u_i$ where $\lambda = T/\sum u_i$. The (constrained) <u>egalitarian</u> allocation method is to let $t_i = \min\{\lambda,u_i\}$ where λ is a uniquely determined parameter such that
[1] $\sum t_i = T$. This method was proposed as early as the twelfth century by the great Jewish scholar Maimonides [Aumann and Maschler, 1985]. The <u>levelling</u> method is the rule that equalizes the claimants' deficits <u>ex post</u>, subject to no debt being exceeded; that is, $t_i = \max\{u_i - \lambda, 0\}$ where λ is a uniquely
[2] determined parameter such that $\sum t_i = T$. Solutions by these three methods for $T = 200$ are compared with the Mishna solution in Table 2.

TABLE 2

Four methods of allocating an estate of 200 among three creditors.

Claim	100	200	300
Proportional	33-1/3	66-2/3	100
Equal	66-2/3	66-2/3	66-2/3
Levelling	0	50	150
Mishna	50	75	75
Shapley	33 1/3	83 1/3	83 1/3

What is the logic of the Mishna solution? Aumann and Maschler provide a beautiful answer: it is the nucleolus of the following cooperative game: For each class $S \subseteq N$ of creditors let $v(S) = \max\{0, T - \sum_{N-S} u_i\}$. Thus $v(S)$ is the amount left over to S after the claims of $N - S$ have been met (or the total T exhausted). It is the least that the members of S can assure themselves collectively. The nucleolus of this game is easy to compute. Fix $\mathbf{u}$, and imagine the size of the total T increasing from zero. When $T = 0$, $\mathbf{t} = 0$. As T grows increase $\mathbf{t}$ incrementally as follows

[1] ⇒ $\lambda = 66\,2/3$ [2] ⇒ $\lambda = 150$

(1)
(a) for $0 < T < \sum u_i/2$, give equal incremental increases to all widows i such that $t_i < u_i/2$;

(b) for $T = \sum u_i/2$, let all $t_i = u_i/2$;

(c) for $\sum u_i/2 < T \leqq \sum u_i$, give equal incremental increases to all widows i whose shortfall $u_i - t_i$ is a maximum, until at $T = \sum u_i$ all $t_i = u_i$.

The reader may verify that this procedure yields the Mishna solutions of Table 1. For estates of $T = 400$ and $T = 500$ the solutions are shown below (using (1c). Notice that the solution for $T = 500$ agrees with the levelling method.

(2)

		Claim		
		100	200	300
Estate	400	50	125	225
	500	66-2/3	166-2/3	266-2/3

A possible rationale of the nucleolus or "Mishna" method is that the attention of the creditors tends to focus on the amount of the awards if the estate is small relative to the claims, but on the size of the deficits if the estate is large relative to the claims. This point of view seems intuitively appealing. Yet in other contexts, such as taxation, it does not produce satisfactory results.

2. TAXATION: THE OBLIGATIONS PROBLEM

The obverse side of the "rights" problem is the "obligations" problem. A person's taxable income u_i can be viewed as a potential liability or obligation to society. Let T be the total amount of tax to be levied. A taxation method is a function $F(u,T) = t$ where $0 \leqq t \leqq u$ and $\sum t_i = T$. F is defined for all pairs (u,T) such that $u > 0$ and $0 \leqq T \leqq u$. We shall not concern ourselves here with the issue of how taxable incomes are defined or of how large T should be.

The foregoing methods can be interpreted in the context of taxation. Proportional allocation (the flat tax) is frequently advocated in theory (but less often practiced). The egalitarian method is well-known under the name head or poll tax. The head tax is a typical method for assessing fees to use a public facility (e.g. a subway token, a ticket to the zoo). The levelling tax