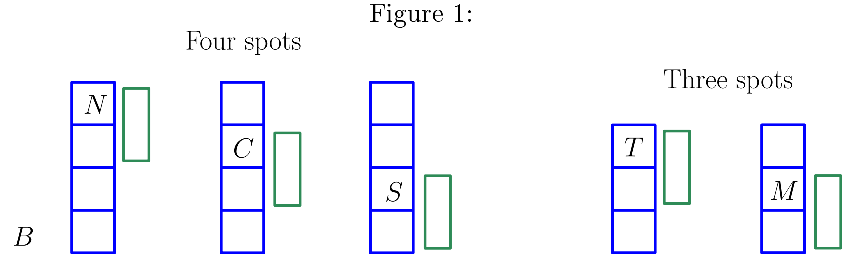
Trivial Battleships (Based on Binmore's *Fun and Games*)

As we all know there are two players who both bomb and hide, and the one who sinks all the other's ships first wins. We are going to consider the case where there is only one ship, and will start by looking at the further simplified case where one player bombs and the other one hides. In fact, since for each player the two roles of bomber and hider are totally separate, the analysis of this asymmetric case effectively solves the bombing-and-hiding case. We describe the details next.

One bombs, the other hides

John has a ship of length 2 that he may place in two of four aligned squares of length one, as in the left panel of the figure. Alberta cannot see it but she wants to sink it (two good hits for that), and she can shoot as many times as she wants, John truthfully reporting the outcome after each shot. John wants to live as long as possible, while Alberta wants to win as quickly as she can. So it is a zero-sum game, where we may take the number of shots up to sinking as John's payoff. Obviously John has 3 strategies, which we call N, C, S (the position of the prow); the tricky part is to show that Alberta is basically in the same position, in the sense that she has only three undominated strategies (she has four, but one is redundant because it replicates one of the other three). At each shot she can target N, C, S and the bottom spot B. Start thinking of Alberta's strategies and you will realize that it takes a little effort to describe them conveniently. Once we get to them the game is trivial.

Figure 1:



To start organizing thoughts it is useful to look at an even simpler version of this game, with three spots instead of four, as in the right panel. John 's strategies are T and M; Alberta's targets are here T, M, B. But M is a sure and necessary hit, so she really has two strategies, represented by first-shot choices T or B.

(a) Write down the two-by-two matrix of the 3-spot game with John's payoff as entries, and find equilibrium and value (just to check, the value is $2\frac{1}{2}$, understandably). (b) Guided: turn to the 4-spot game and do the same, that is write the game matrix and find equilibrium and value (value is $2\frac{2}{3}$, John is better off because he has more room to hide). Start by arguing that in any equilibrium John has to fully mix; to prove this a full specification of Alberta's strategies is not needed. This implies that Alberta will not play weakly dominated strategies in equilibrium so we can ignore them. To find a dominated strategy consider starting with target N. Show that however you continue you can do weakly better by starting with C instead. To do this use what you know about the 3-spot game to show that there are two possibilities starting with N, one for example is $N_m C$ meaning "start with N, if you miss target C" (it is obvious what to do if you hit); there are obviously two starting with C, one being $C_h N$ which means "start with C, if you hit target N" (it is obvious what to do if you miss). Write down the corresponding payoffs under John's N, C, S and compare.

Analogously, starting with B must be weakly dominated too. At this point you can get down to 4 strategies for Alberta (two $C_h X$ and two $S_h X$). Write down the 3×4 matrix and observe that two strategies are equivalent. Reduce the matrix to a 3×3, and you are done.

Bombing and hiding

(c) Who should be expected to win? Assume that both play the Nash equilibrium as bombers and hiders (it can be argued that this will actually be the case in the long run).