Bertrand competition with uncertainty

We start with the model without uncertainty. There are two price-setting firms with demand functions

$$d_1(p_1, p_2) = a - p_1 + \frac{1}{2}\frac{p_2}{p_1}$$
$$d_2(p_1, p_2) = b - p_2 + \frac{1}{2}p_1$$

where d_i, p_i are quantities and prices, and a, b are constants (parameters).[1] Production costs are zero so payoffs $u_i(p_1,p_2)$ are just gross revenues $p_i d_i$.

(i) Find the Nash equilibrium prices (best responses are found via the zeros of the derivatives). The answer is

$$p_1 = \frac{a}{2}, \quad p_2 = \frac{b}{2} + \frac{1}{4}p_1 = \frac{b}{2} + \frac{1}{8}a.$$

Now suppose a and b can take two values $\{a_H, a_L\}$ and $\{b_H, b_L\}$ respectively, where the value of a is known to firm 1 but not 2 and similarly b is known to firm 2 only. So each player has two types and we have to specify $p_1(a_H), p_1(a_L)$ for player 1 and $p_2(b_H), p_2(b_L)$ for player 2. The players' conditional probabilities π_i are as follows:

$$\pi_1(b_H \mid a_H) = \pi_2(a_H \mid b_H) = 4/5$$
$$\pi_1(b_L \mid a_L) = \pi_2(a_L \mid b_L) = 2/3$$

where of course $\pi_1(b_L \mid a_H) = 1/5$ etc. since probabilities must sum up to 1. Thus each firm believes that it is likely that its competitor faces the same demand condition as itself. In the obvious notation $(i, H), (i, L)$ for $i = 1, 2$ the payoff of $(1, H)$ is

$$\begin{aligned} Eu_{1,H} &= \pi_1(b_H \mid a_H)\left[a_H p_1 - p_1^2 + \frac{1}{2}p_2(b_H)\right] + \pi_1(b_L \mid a_H)\left[a_H p_1 - p_1^2 + \frac{1}{2}p_2(b_L)\right] \\ &= a_H p_1 - p_1^2 + \frac{1}{2}\left[\pi_1(b_H \mid a_H)p_2(b_H) + \pi_1(b_L \mid a_H)p_2(b_L)\right] \end{aligned}$$

where $p_1 = p_1(a_H)$; the others are defined similarly.

(ii) Find the Bayes-Nash equilibrium of the game. (*Hint:* you should find e.g. $p_2(b_L) = \frac{b_L}{2} + \left[\frac{1}{24}a_H + \frac{1}{12}a_L\right]$)

(iii) Suppose demand for our two firms has been constant for a while and then at some point firm 2 believes the demand conditions of its competitor may deteriorate. On the basis of the result above can we predict its price reaction?

[1] To be precise quantities demanded are $\max\{d_i, 0\}$ but we can neglect this formal point.