BoS with uncertainty, or: Yes, girls are smarter than boys

Recall that the game (from Osborne section 9.1) is an incomplete information version of the battle of sexes. For concreteness we think of player 1 as a boy and player 2 as a girl. The situation is that she may or may not like to be with him and he does not know it. So the two possibilities are those in the picture below

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

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where the left game - where she is happy to meet him - has probability $0 < \pi < 1$ (in the book $\pi = 1/2$). Then the payoff matrix for the players (three of them, player 1 and the two types of player 2) is the following, where a strategy of player 2 is a pair of actions (one for each type), the payoff of player 1 is his expected payoff and the two numbers for player 2 denote the payoff for her left and right type respectively:

	BB	***BS***	***SB***	***SS***
B	2; $1, 0$ •	2π; $1, 2$ • •	$2 - 2\pi$; $0, 0$	0; $0, 2$ •
S	0; $0, 1$ •	$1 - \pi$; $0, 0$	π; $2, 1$ • •	1; $2, 0$ •

Pure Equilibria

As we know both types of player 2 must respond optimally so $BR_2(B) = BS$ and $BR_2(S) = SB$. They are strictly better than the others hence for pure strategy equilibria we can restrict attention to these two. Best response of 1 to each of those depend on π. We have

$$2\pi > 1 - \pi \iff \pi > 1/3$$
$$2 - 2\pi > \pi \iff \pi < 2/3.$$

Therefore there are three cases:

a) $\pi < 1/3$: there is no equilibrium in pure strategies. In this case we are essentially in the right game, where there is no pure equilibrium.

b) $1/3 < \pi < 2/3$: equilibrium is (B, BS). Here π is large enough so that 1 grabs the occasion to meet 2 on the left if 2 plays BS, and $1-\pi$ is large enough that if 2 plays SB then enjoying B in good company in the right game has higher expected utility than enjoying S in the left game.

c) $\pi > 2/3$: there are two equilibria, (B, BS) and (S, SB) - those of the left game.

Mixed Equilibrium when $\pi < 1/3$

Now let us look at the case $\pi < 1/3$: there is no pure equilibrium, so we look for mixed ones. If both types of player 2 use a pure strategy then 1 cannot mix since his two strategies give different payoffs. So at least one type of player 2 must mix. In the left game we know that for 2 to mix 1 has to play B with probability $p = 2/3$. Similarly, the indifference condition of player 2 in the right game gives $1 - p = 2p$ that is $p = 1/3 \neq 2/3$. Therefore there is no equilibrium in which both types of player 2 mix. Suppose the left type of player 2 (only) mixes; then 1 must mix with $p = 2/3$. What is the best reply to this by the right type of player 2? If she plays B she gets $1-p = 1/3$; if she plays S she gets $2p = 4/3$ so her best reply is S. Let q be the probability that left type of 2 plays B. We derive the indifference condition for player 1: if he plays B he gets $\pi \cdot 2q + (1-\pi) \cdot 0$ while playing S yields $\pi \cdot (1-q) + (1-\pi) \cdot 1$; thus the condition is $2\pi q = \pi \cdot (1-q) + (1-\pi) \cdot 1$ or $q = 1/[3\pi]$. Since $\pi < 1/3$ this is larger than 1 so there is no equilibrium where only the left type of 2 mixes.

Try with only the right type of 2 mixing. Then $p = 1/3$; we know that then 2-left will play S because she is indifferent if $p = 2/3$.[1] Letting q the probability of B for the right type of 2: if player 1 plays B he gets $\pi \cdot 0 + (1-\pi) \cdot 2q$; if he plays S he gets $\pi \cdot 1 + (1-\pi)(1-q)$ so the indifference condition gives $q = 1/[3(1-\pi)]$; this is less than $1/2$ since $\pi < 1/3$.

In conclusion the unique mixed equilibrium of the game is this: player 1 mixes with $p = 1/3$, the left type of 2 plays S and her right type mixes with $q = 1/3(1-\pi) < 1/2$.

Lets us look at the equilibrium payoff of the left-type girl (the one who likes him). She plays S and he plays B with probability $1/3$ so she gets $(2/3) \cdot 2 \approx 1.3$. In the original BoS without uncertainty the mixed equilibrium is $p = 2/3, q = 1/3$ so that her payoff is $pq + 2(1-p)(1-q) = 2/9 + 2 \cdot 2/9 = 2/3 \approx 0.67$. In other words: if the girl who likes him makes him believe she quite possibly does not then she ends up much better off. Since - as we all know too well - this behavior is the rule, **the moral of this story is: sorry guys, but girls are so much smarter than us!**

[1] Double check: if the left type of 2 plays B she gets $p = 1/3$; if she plays S she gets $2(1-p) = 4/3$.