Clarifications on correlated equilibrium, OR pp.45-47

1. This is just to make sure you understand what the book says after formula (45.2). To sum over all $\omega \in \Omega$ we can sum the ω's in each $P_i \in \mathcal{P}_i$ and then sum over the P_i's. That is we can always write $\sum_{\omega\in\Omega} \cdots = \sum_{P_i\in\mathcal{P}_i} \sum_{\omega\in P_i} \ldots$ Of course if $\pi(\omega) = 0$ for all ω in a given P_i - that is $\pi(P_i) = 0$ - those terms do not count. Hence in (45.2) the sums on both sides are really $\sum_{\{P_i\in\mathcal{P}_i|\pi(P_i)>0\}} \sum_{\omega\in P_i} \ldots$ Now for $\pi(P_i) > 0$ and $\omega \in P_i$ we can write $\pi(\omega) = \pi(P_i) \cdot \pi(\omega|P_i)$. Since $\sigma_i(\omega)$ is constant on each P_i we can conveniently write $\sigma_i(P_i)$. Therefore the left member of (45.2) is

$$\sum_{\{P_i\in\mathcal{P}_i|\pi(P_i)>0\}} \sum_{\omega\in P_i} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \\ = \sum_{P_i\in\mathcal{P}_i|\pi(P_i)>0} \pi(P_i) \sum_{\omega\in P_i} \pi(\omega|P_i) u_i(\sigma_{-i}(\omega), \sigma_i(P_i))$$

and similarly for the right member. Thus (45.2) requires that for any τ_i it must be

$$\sum_{P_i\in\mathcal{P}_i|\pi(P_i)>0} \pi(P_i) \sum_{\omega\in P_i} \pi(\omega|P_i) u_i(\sigma_{-i}(\omega), \sigma_i(P_i)) \\ \geq \sum_{P_i\in\mathcal{P}_i|\pi(P_i)>0} \pi(P_i) \sum_{\omega\in P_i} \pi(\omega|P_i) u_i(\sigma_{-i}(\omega), \tau_i(P_i))$$

The last thing to realize is that if you can profitably deviate from σ_i on at least *one* P_i you can deviate: just act as in σ_i everywhere except on that P_i. Therefore the above inequality holds if for each P_i, for any $a_i \neq \sigma_i(P_i)$ it is

$$\sum_{\omega\in P_i} \pi(\omega|P_i) u_i(\sigma_{-i}(\omega), \sigma_i(P_i)) \geq \sum_{\omega\in P_i} \pi(\omega|P_i) u_i(\sigma_{-i}(\omega), a_i).$$

In other words we can check no deviation separately in each P_i taking conditional probabilities, as we have done in the examples.

2. When $\Omega = A$, the information structure is canonical - $P_i(a) = \{b \in A \mid b_i = a_i\}$ - and $\sigma_i(a) = a_i$ the no-deviation property is

$$\sum_{a\in A} \pi(a) u_i(a_i, a_{-i}) \geq \sum_{a\in A} \pi(a) u_i(a_i', a_{-i}) \ \forall a_i'.$$

In conditional form, letting $\pi(a_i) \equiv \pi(\{b \in A \mid b_i = a_i\})$ and $\pi(a \mid a_i) = \pi(a)/\pi(a_i)$ for

a_i with $\pi(a_i) > 0$, the requirement is that for all a_i with $\pi(a_i) > 0$ it must be

$$\sum\nolimits_{a\in A} \pi(a \mid a_i)u_i(a) \geq \sum\nolimits_{a\in A} \pi(a \mid a_i)u_i(a_i', a_{-i}) \ \forall a_i'.$$

This is the form we use all the time.

3. Proposition 47.1 says that there is no loss of generality in using the canonical construction above. We can expand a little on the construction used to prove it. If players play $\sigma'(a) = a$ then the probability on profiles is clearly the same as the original one. We should check that no player has a profitable deviation from this profile.

This time we break the sum $\sum_{\omega\in\Omega} \pi(\omega)u_i(\sigma(\omega))$ by partitioning Ω into the sets where σ is constant, that is write $\sum_{\omega\in\Omega} \cdots = \sum_{a\in A} \sum_{\{\omega|\sigma(\omega)=a\}}$. Thus

$$\sum\nolimits_{\omega\in\Omega} \pi(\omega)u_i(\sigma(\omega)) = \sum\nolimits_{a\in A} \sum\nolimits_{\{\omega|\sigma(\omega)=a\}} \pi(\omega)u_i(\sigma(\omega)) = \sum\nolimits_{a\in A} \pi'(a)u_i(a).$$

Does i want to deviate from $\sigma_i'(a) = a_i$? Deviating means choosing a $a_i' \in A_i$ possibly different than a_i for each a_i; equivalently, choosing $\tau_i(\omega) = a_i'$ for ω such that $\sigma_i(\omega) = a_i$. By breaking the sum as before we see that i's payoff becomes

$$\begin{aligned}\sum\nolimits_{a\in A} \pi'(a)u_i(a) &= \sum\nolimits_{\omega\in\Omega} \pi(\omega)u_i(\sigma(\omega)) \geq \sum\nolimits_{\omega\in\Omega} \pi(\omega)u_i(\tau_i(\omega), \sigma_{-i}(\omega)) \\ &= \sum\nolimits_{a\in A} \sum\nolimits_{\{\omega|\sigma(\omega)=a\}} \pi(\omega)u_i(\tau_i(\omega), \sigma_{-i}(\omega)) = \sum\nolimits_{a\in A} \pi'(a)u_i(a_i', a_{-i})\end{aligned}$$

so the answer is no, i does not want to deviate.