Preliminaries on discounting flows

We collect here a couple of basic facts about discounting streams. We have infinite streams $(v_t)_{t\geq 1}$, assumed bounded in the sense that for some $\bar{v}$ we have $|v_t| < \bar{v}$ for all t. Preferences are given by

$$(w)_{t\geq 1} \succ (v_t)_{t\geq 1} \iff (1-\delta)\textstyle\sum_{t=1}^{\infty}\delta^{t-1}w_t > (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}v_t \quad 0 < \delta < 1$$

The closer δ is to 1 the more "patient" is the decision maker. Recall geometric sums:

$$\textstyle\sum_{t=1}^{\infty}\delta^{t-1} = \dfrac{1}{1-\delta}$$

so multiplication by $1-\delta$ means dividing by the sum of weights. This is convenient because it implies that if $v_t = c\ \forall t$ then $(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}v_t = c$. Obviously preferences are unaffected by this factor.

1. The basic observation is that when flows are discounted as above, for δ close enough to 1 no one-period gain γ (however large) is enough to offset a loss of ϵ (however small) suffered indefinitely in the future. Essentially you gain γ but lose $\epsilon\delta/(1-\delta)$, and the net gain $\gamma - \epsilon\delta/(1-\delta) < 0$ for

$$\delta > \underline{\delta} \equiv \frac{\gamma/\epsilon}{1+\gamma/\epsilon}$$

as you can (and should) easily check (note that $\underline{\delta} < 1$ for any γ, ϵ by the way). This is at the heart of the proofs that grim trigger strategies are Nash in infinitely repeated games. Formally, if

$$w_t = \begin{cases} v_t & t < t_0 \\ v_t + \gamma & t = t_0 \\ v_t - \epsilon & t > t_0 \end{cases}$$

then $(w_t) \prec (v_t)$ for $\underline{\delta} < \delta < 1$. Proof (the one we have just seen): neglecting $1-\delta$ for ease of reading,

$$\begin{aligned} \textstyle\sum_{t=1}^{\infty}\delta^{t-1}w_t &= \textstyle\sum_{t=1}^{t_0-1}\delta^{t-1}v_t + \delta^{t_0-1}(v_{t_0}+\gamma) + \sum_{t=t_0+1}^{\infty}\delta^{t-1}(v_t-\epsilon) \\ &= \textstyle\sum_{t=1}^{\infty}\delta^{t-1}v_t + \delta^{t_0-1}\gamma - \epsilon\sum_{t=t_0+1}^{\infty}\delta^{t-1} \\ &= \textstyle\sum_{t=1}^{\infty}\delta^{t-1}v_t + \delta^{t_0-1}\gamma - \epsilon\delta^{t_0}\sum_{s=1}^{\infty}\delta^{s-1} \\ &= \textstyle\sum_{t=1}^{\infty}\delta^{t-1}v_t + \delta^{t_0-1}[\gamma - \epsilon\delta/(1-\delta)] < \sum_{t=1}^{\infty}\delta^{t-1}v_t \end{aligned}$$

for $\underline{\delta} < \delta < 1$, as we know.

2. The second point we make is that if $(w_t) \succ (v_t)$ then for T large enough if we replace w_t by v_t for $t > T$ the resulting stream is still better than (v_t), the reason being that what happens in the far future is almost irrelevant. This is what is needed to extend the one-deviation property of subgame perfect equilibria to infinitely repeated games with the discounting criterion.

To prove it observe that for any (v_t) we have

$$\textstyle\sum_{t=1}^{\infty}\delta^{t-1}v_t = \sum_{t=1}^{T}\delta^{t-1}v_t + \sum_{t=T+1}^{\infty}\delta^{t-1}v_t = \sum_{t=1}^{T}\delta^{t-1}v_t + \delta^T\sum_{s=1}^{\infty}\delta^{s-1}v_{T+s}$$

and the last term is δ^T times a number which in absolute value is less than $\bar{v}/(1-\delta)$.[1] Therefore

$$\textstyle\delta^T\sum_{s=1}^{\infty}\delta^{s-1}v_{T+s} \to 0 \quad \text{as } T \to \infty$$

so that given ϵ, for T large enough (precisely: for $T > T_\epsilon$ for some T_ϵ) we have

$$\textstyle\left|\sum_{t=T+1}^{\infty}\delta^{t-1}v_t\right|, \left|\sum_{t=T+1}^{\infty}\delta^{t-1}w_t\right| < \epsilon$$

In this sense far off future payoffs count little. Now suppose $(w_t) \succ (v_t)$, say $\sum_{t=1}^{\infty}\delta^{t-1}w_t - \sum_{t=1}^{\infty}\delta^{t-1}v_t = K > 0$. Take $\epsilon = K/2$ and $T > T_\epsilon$. Then the value of the modified stream is[2]

$$\begin{aligned}\textstyle\sum_{t=1}^{T}\delta^{t-1}w_t + \sum_{t=T+1}^{\infty}\delta^{t-1}v_t &= \textstyle\sum_{t=1}^{\infty}\delta^{t-1}w_t + \sum_{t=T+1}^{\infty}\delta^{t-1}v_t - \sum_{t=T+1}^{\infty}\delta^{t-1}w_t \\ &> \textstyle\sum_{t=1}^{\infty}\delta^{t-1}w_t - 2\epsilon = \sum_{t=1}^{\infty}\delta^{t-1}w_t - K = \sum_{t=1}^{\infty}\delta^{t-1}v_t\end{aligned}$$

as was to be shown.

3. This last point we haven't actually used it, it serves to extend arguments from single outcomes of the constituent games to rational convex combinations of them. The fact is that if an infinite streams is composed of repetitions of a given finite sequence, then for δ close to 1 the discounted sum is approximately equal to the average of the sequence.

Consider for example the stream $(v_t) = (2, 1, 2, 1, \ldots, 2, 1 \ldots)$ given by repetitions

[1] Because

$$\textstyle\sum_1^\infty \delta^{s-1}v_{T+s} \le \left|\sum_1^\infty \delta^{s-1}v_{T+s}\right| \le \sum_1^\infty \delta^{s-1}|v_{T+s}| \le \bar{v}\sum_1^\infty\delta^{s-1} = \bar{v}/(1-\delta)$$

[2] Just in case: if two numbers $a, b \in (-\epsilon, \epsilon)$ then $a - b < 2\epsilon$.

of the sequence $(2,1)$. Its average is 1.5. Now look at the discounted value. It is

$$\begin{aligned}&(1-\delta)[2+\delta\cdot 1+\delta^2\cdot 2+\delta^3\cdot 1+\cdots+\delta^{2n}\cdot 2+\delta^{2n+1}\cdot 1+\dots]\\=&(1-\delta)[2\textstyle\sum_{n=0}^{\infty}(\delta^2)^n+\delta\textstyle\sum_{n=0}^{\infty}(\delta^2)^n]=\frac{1-\delta}{1-\delta^2}(2+\delta)=\frac{2+\delta}{1+\delta}\approx\frac{3}{2}\end{aligned}$$

when $\delta \approx 1$. This is the essence of the point.

For a general statement, consider a stream given by infinite repetitions of the sequence $(v_1, v_2, \dots v_C)$. Its discounted value is given by

$$\begin{aligned}&(1-\delta)[v_1+\delta v_2+\cdots+\delta^{C-1}v_C+\delta^C v_1+\delta^{C+1}v_2\cdots+\delta^{2C-1}v_C+\cdots+\delta^{2C}v_1+\dots]\\=&(1-\delta)[v_1\textstyle\sum_0^\infty(\delta^C)^n+\delta v_2\textstyle\sum_0^\infty(\delta^C)^n+\cdots+\delta^{C-1}v_C\textstyle\sum_0^\infty(\delta^C)^n]\\=&\frac{1-\delta}{1-\delta^C}[v_1+\delta v_2+\cdots+\delta^{C-1}v_C]=\frac{v_1+\delta v_2+\cdots+\delta^{C-1}v_C}{1+\delta+\delta^2+\cdots+\delta^{C-1}}\approx\frac{\sum_{n=1}^C v_n}{C}\end{aligned}$$

when $\delta \approx 1$, where we have use the fact that $1-\delta^C = (1-\delta)(1+\delta+\cdots+\delta^{C-1})$.