Game Theory S Modica 20 January 2017

1. Consider a preference relation $\succcurlyeq$ on lotteries. When is such a preference said to be *risk averse*? Give a formal definition. You may denote a lottery by x and its expected value by Ex.

Solution. (From textbook) $Ex \succcurlyeq x$ for all x

2. A two-person zero-sum game is defined by the condition $u_2(a) = -u_1(a)$ for any strategy profile a. When is the *value* of such a game well defined?

Solution. When a Nash equilibrium exists.

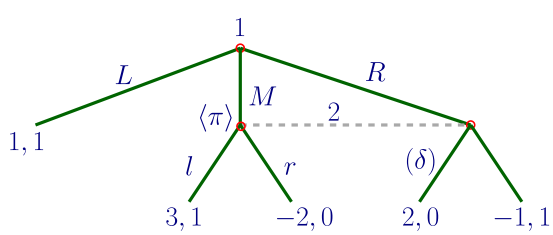
3. Consider a duopoly where the two firms have linear production cost $c_i(q_i) = c_i q_i, i = 1, 2$ with $c_1 = 3, c_2 = 2$ and demand price is given by $p(q_1, q_2) = 16 - q_1 - q_2$. Find the Nash equilibrium of the game between the two firms. You may recall that firm i maximizes profit $\pi_i(q_1, q_2) = p(q_1, q_2)q_i - c_i q_i$ with respect to q_i *given* $q_j, j \neq i$; this gives its best response function $q_i = b_i(q_j)$, and Nash is the pair q_1^*, q_2^* such that $q_i^* = b_i(q_j^*), i = 1, 2$. (*Answer:* $q_1^* = 4, q_2^* = 5$)

Solution. $\partial\pi_1/\partial q_1 = 13 - q_2 - 2q_1$; setting this equal to zero gives $b_1(q_2) = 7.5 - q_2/2$; similarly, maximization of π_2 gives $b_2(q_1) = 7 - q_1/2$. Solving the system $q_1 = b_1(q_2), q_2 = b_2(q_1)$ gives the result.

4. Suppose infinite payoff streams $w = (w_1, w_2, \dots)$ are evaluated according to the discounted sum $U(w) = \sum_{t=1}^{\infty} \delta^{t-1} w_t$ where $0 < \delta < 1$. Show that there exists a $\underline{\delta} < 1$ such that for $\delta > \underline{\delta}$ any gain $\gamma > 0$ in period 1 is more than offset by a loss $\epsilon > 0$ from 2 onwards, in the sense that if $w = (\gamma, -\epsilon, -\epsilon, \dots)$ - that is $w_1 = \gamma, w_t = -\epsilon, t > 1$ - we have $U(w) < 0$ for $\delta > \underline{\delta}$.

Solution. For the given w we have $U(w) = \gamma - \delta\epsilon/(1-\delta) < 0$ for $\delta > \frac{\gamma/\epsilon}{\gamma/\epsilon+1} \equiv \underline{\delta}$

5. Consider the two-player game represented below. δ denotes the behavioral probability of l and π is the probability attached to the left node of 2's information set. As usual we also write $\succcurlyeq_i$ for i's preferences.



(a) Draw the normal form of the game and find all Nash equilibria. (*Hint:* show first that 1 cannot mix; second hint: there are three Nash equilibria) (b) Check that there are consistent π's which make all of them sequential.

Solution. (a) The normal form is the following:

	l	r
L	1,1	1,1
M	3,1	-2,0
R	2,0	-1,1

We look at the possible support of 1's mixed strategies in equilibrium. $M \sim_1 R$ easily implies $\delta = 1/2$; then both M and R give 1 payoff 1/2 so $M \prec_1 L$. Thus full support and $\{M, R\}$ are excluded. Also, if 1 mixes on $\{L, M\}$ then $l \succ_2 r$ and if he mixes on $\{L, R\}$ then $r \succ_2 l$ so also all the two-point supports are excluded. Conclusion is that 1 cannot mix.

The pure equilibria are (M, l) and (L, r), and there are no equilibria where 1 plays R; now for the mixed ones. These can only be with 1 playing L and 2 mixing. In these equilibria it must be $L \succcurlyeq_1 M, R$, and it is easily checked that $L \succcurlyeq_1 M \iff \delta \leq 3/5$ and $L \succcurlyeq_1 R \iff \delta \leq 2/3$; so the mixed equilibria are with 1 playing L and 2 mixing with $\delta \leq 3/5$ (they include the pure (L, r) for $\delta = 0$.

(b) The (M, l) equilibrium is sequential with $\pi = 1$. In the equilibria where 1 plays L player 2's information set is not reached so we need consistent beliefs supporting her choice. In the pure (L, r) equilibrium we need $\pi \leq 1/2$, in the ones where 2 mixes we need $\pi = 1/2$. But in this game any $0 < \pi < 1$ is easily made consistent: indeed, the equilibrium strategy of player 1 is the distribution $(1, 0, 0)$ which is approximated by any $((1 - \epsilon_n), \pi\epsilon_n, (1 - \pi)\epsilon_n)$ for any sequence $\epsilon_n \to 0$ and $0 < \pi < 1$; and along the sequence $\pi_n = \pi$.

Remark. For the sake of curiosity, also $\pi = 0, 1$ are consistent: for $\pi = 0$ let the approximating sequence be $(1 - \epsilon_n - \epsilon_n^2, \epsilon_n^2, \epsilon_n)$, which implies that

$$\pi_n = \frac{\epsilon_n^2}{\epsilon_n^2 + \epsilon_n} = \frac{1}{1 + 1/\epsilon_n} \to 0$$

For $\pi = 1$ take $(1 - \epsilon_n - \epsilon_n^2, \epsilon_n, \epsilon_n^2)$.