Game Theory S Modica 26 January 2018

1. Consider a lottery by x with expected value Ex. Rosetta owns the lottery and is risk averse, while Joe owns money (more than Ex) and is risk neutral. Is there a trade which makes both better off? At what prices?

Solution. Rosetta can sell x to Joe at any price p between $Ex - \pi$ and Ex where π is the risk premium of Rosetta for x, equivalently between Rosetta's certainty equivalent of x (less than Ex) and Ex.

2. Consider an n-person game where an action for player i is a contribution $0 \leq x_i \leq 1$ to a public good whose marginal benefit is r; that is, i's payoff is given by $u_i(x_1, \ldots, x_n) = r \sum_{j=1}^n x_j - x_i$. Show that for $1/n < r < 1$ the group's total utility $\sum_i u_i(x_1, \ldots, x_n)$ is maximum at $x_i = 1$ for all i but the only Nash equilibrium has $x_i = 0$ for all i. This is the "tragedy of the commons".

Solution. $\sum_i u_i(x_1, \ldots, x_n) = (rn-1) \sum_{i=1}^n x_i$ which if $rn - 1 > 0$ is increasing in $\sum_{i=1}^n x_i$ - whence its maximum as in the statement. On the other hand $u_i(x_1, \ldots, x_n) = (r-1)x_i + r \sum_{j \neq i} x_j$ which if $r < 1$ is decreasing in x_i hence has maximum at $x_i = 0$ for any x_{-i}, and this is then the only Nash equilibrium of the game.

3. Recall the game of matching pennies with spying, whose strategic form is

	HH	HT	TH	TT
H	1	$2\pi - 1$	$1 - 2\pi$	-1
T	-1	$2\pi - 1$	$1 - 2\pi$	1

where numbers are player 1's payoffs (recall that $\pi > 1/2$). We concluded in class that in any equilibrium player 1 must mix - say play H with probability p - and player 2 must play TH for sure. We did not solve for the equilibrium probabilities p. Please fill this last missing step (of course we can ignore strategy HT which is worse than TH for player 2 for any p). *Hint:* the starting point is that $-1 < 1 - 2\pi < 1$.

Solution. Strategy TH must be better than HH, so $2\pi - 1 \geq 1 - 2p$ that is $p \geq 1 - \pi$; and it must be better than TT, whence $2\pi - 1 \geq 2p - 1$ that is $p \leq \pi$. We therefore conclude that there is a continuum of equilibria, with player 2 playing TH for sure and 1 mixing with $1 - \pi \leq p \leq \pi$.

4. Consider the infinitely repeated prisoner's dilemma with stage game

	C	D
C	2,2	0,3
D	3,0	1,1

with the discounting criterion for the payoff in the repeated game. Consider the symmetric profile where each player plays the "grim trigger strategy" which consists of starting with C and reverting to D forever after a stage in which (C, C) is not played. (a) Show that this is a Nash equilibrium for discount $\delta > 1/2$. (b) Show that for the same δ's the profile is subgame perfect. *Hint for (b):* using the one-deviation property consider deviations lasting one period only; you must examine, at any given period: histories where (C, C) has always been played in the past; and any other history.

Solution. In Osborne's textbook.

5. Consider the cooperative game among players $1, \ldots, 5$ with value function

$$v(S) = \begin{cases} 1 & \textit{if } 1 \in S \textit{ and } \#S \geq 2 \textit{ or if } \#S \geq 4 \\ 0 & \textit{otherwise} \end{cases}$$

that is $v = 1$ if player 1 associates with at least one other player, or if all of the other players are in. (a) Show that the Core of the game is empty; (b) Show that the Shapley imputation is $(0.6, 0.1, 0.1, 0.1, 0.1)$.

Solution. (a) $x_1 + x_i \geq 1$ for all $i > 1$ implies $x_j = 0$ for all $j > 1$; but $\sum_{j>1} x_i \geq 1$, contradiction. (b) if 1 does not arrive first or last - probability 3/5 her marginal contribution is 1, otherwise it is zero. So player 1 gets 3/5; each of the others has the same marginal contribution to any preceding coalition so they get the same payoff, which is then $(1/4)(2/5) = 1/10$.

6. The most basic sharing context is the cooperative game with two players $i = 1, 2$ where $v(12) > v(1) + v(2)$. (a) Specify the imputation (φ_1, φ_2) which corresponds to the Shapley imputation and to the Nucleolus. (b) Show that it corresponds to the Shapley imputation; (c) Show that it is also the Nucleolus of the game

Solution. (a) The rule is given by

$$\varphi_i = v(i) + \frac{1}{2}[v(12) - (v(1) + v(2))]$$

(b),(c) done in class.