The heritage game[1]

(Game Theory LM-77, S. Modica)

There are "sisters" $i = 1, \dots n$ who inherit "houses" $h_j, j = 1, \dots m$. We assume for simplicity that $m \leq n$. Sister i has utility $u_i(h_j) \geq 0$ for house h_j. If $m < n$ some sisters are assigned no house; in this case we say they are assigned house h_0 and take $u_i(h_0) = 0$ for all i. Money transfers $t_1, \dots, t_n$ with $t_1 + \cdots + t_n = 0$ are allowed. Suppose sister i is assigned house $\varsigma(i)$; then overall utility of i is $u_i(h_{\varsigma(i)}) + t_i$.

1. One house, n sisters

Here we can simplify notation and let u_i sisters i's utility for the house. Assume $u_1 < u_2 < \cdots < u_n$. The cooperative game is defined by $v(S) = u_i$ for the highest $i \in S$. The value $v(N) = u_n$ is obtained by assigning the house to the sister who values it most. Of course she will have to compensate her sisters as we shall see. Observe that the game has the same structure as the airport game, the only difference being that it is formulated in terms of utilities instead of costs. Therefore letting $u_0 = 0$ and $\delta_i = u_i - u_{i-1}$ we know that the Shapley imputation is given by

$$x_i = \sum_{j=1}^{i} \frac{\delta_j}{n-(j-1)} = \frac{\delta_1}{n} + \frac{\delta_2}{n-1} + \cdots + \frac{\delta_i}{n-(i-1)} \quad i = 1, \dots, n$$

that is $x_1 = \frac{\delta_1}{n}, x_2 = \frac{\delta_1}{n} + \frac{\delta_2}{n-1}$ and so on up to the highest utility sister who gets $\frac{\delta_1}{n} + \cdots + \frac{\delta_{n-1}}{2} + \delta_n$. Since the house goes to the n-th sister the x_i's for $i < n$ are to be interpreted as transfers t_i from her, while since $\sum_i x_i = v(N) = u_n$ sister n has $x_n = u_n - \sum_1^{n-1} x_i$ that is her utility from getting the house minus the compensations paid to her sisters. Using the more natural notation $x_i = t_i$ for $i < n$ we may and will write any imputation in this game as

$$(t_1, \dots, t_{n-1}, u_n - \sum_{i=1}^{n-1} t_i).$$

The case $n = 3$: Shapley versus Nucleolus

In the $n = 3$ case Shapley prescribes that sister 3 pays

$$t_1 = \frac{u_1}{3},\ t_2 = \frac{u_1}{3} + \frac{u_2 - u_1}{2}$$

while we know from the airport game that the Nucleolus prescribes

$$t_1 = \frac{u_1}{2},\ t_2 = \frac{u_1}{4} + \frac{u_2 - u_1}{2}$$

that is more to the sister who value the house the least and less to the other. The total transfer from sister 3 is higher in the Nucleolus.

[1]Adapted from H. Moulin *Cooperative Microeconomics.*

2. Envy

As it happens, a common problem in heritage situation is envy. Are the imputations we have considered envy-free? The answer for $n > 2$ is clearly no because in both cases sisters 1 and 2 get no house and in general $t_1 \neq t_2$. In particular Shapley has $t_2 > t_1$ so sister 1 surely envies at least sister 2. In the Nucleolus envy is less pronounced. In the case $n = 2$, in an imputation $(t_1, u_2 - t_1)$ sister 1 does not envy 2 if she prefers t_1 to having the house and paying t_1 that is $t_1 \geq u_1 - t_1$ or $t_1 \geq u_1/2$; and similarly sister 2 does not envy 1 if $u_2 - t_1 \geq t_1$ that is $t_1 \leq u_2/2$. So the no-envy conditions in this case are

$$\frac{u_1}{2} \leq t_1 \leq \frac{u_2}{2}.$$

In the $n = 2$ case Shapley has $t_1 = u_1/2 < u_2/2$ so no-envy obtains; and since excesses in this imputation, $v_1 - x_1 = u_1/2$ and $v_2 - x_2 = u_2 - (u_2 - u_1/2) = u_1/2$, are equal the Nucleolus coincides with Shapley.

Which imputations $(t_1, \ldots, t_{n-1}, u_n - \sum_1^{n-1} t_i)$ are envy-free in the general case? Sisters 1 to $n-1$ get (no house and) t_i so to avoid envy among them we must impose $t_1 = \cdots = t_{n-1} \equiv t$. Avoiding envy between sister n and the others then gives $u_n - (n-1)t \geq t$ and $t \geq u_{n-1} - (n-1)t$ (if $n-1$ does not envy n nor will the others since u_{n-1} is highest among them). Thus we get the no-envy conditions

$$t_1 = \cdots = t_{n-1} = t \quad \textit{and} \quad \frac{u_{n-1}}{n} \leq t \leq \frac{u_n}{n}.$$

Consider for example the case $n = 3$ with $u_1 = 0, u_2 = \alpha, u_3 = 100$ with $0 < \alpha < 100$. In this simple example Shapley and the Nucleolus coincide because $t_1 = 0$. In both cases 1 envies both 2 and 3. No envy of 3 on the part of 2 gives $t_2 \geq u_2 - t_2 - t_1$ that is $t_2 \geq \alpha/2$ which is satisfied with equality. The condition that 3 should not envy 2 is $u_3 - t_2 - t_1 \geq t_2$ that is $t_2 \leq u_3/2$ which is satisfied strictly. Notice that in this case full no envy gives $t_1 = t_2$ and $\frac{\alpha}{3} \leq t \leq 33\frac{1}{3}$ which is not very plausible. We should mention that in practice the utilities u_i are not easy to pin down, so that the model may be difficult to apply.

3. General case

Generalizing the case of one house we define $v(N)$ to be the highest sum of utilities over all possible assignments ς:

$$v(N) = \max_{\varsigma} \sum_i u_i(h_{\varsigma(i)}).$$

To take a concrete case consider the following case where $n = m = 3$ and utilities are in the following table and the optimal assignment is in boldface and gives total utility of $v(N) = 27$ (check that it is indeed the highest possible sum):

	u_1	u_2	u_3
h_1	3	9	**9**
h_2	**12**	6	6
h_3	9	**6**	3

An imputation x is determined by an optimal assignment ς and a vector of zero-sum transfers $t = (t_1, \ldots, t_n)$ giving $x_i = u_i(h_{\varsigma(i)}) + t_i$. No envy can be defined for any (not necessarily optimal) assignment and transfer set: *a pair* (ς, t) *is envy-free if no* i *envies any* i', that is if $u_i(h_{\varsigma(i)}) + t_i \geq u_i(h_{\varsigma(i')}) + t_{i'}$ for any i, i'. It is remarkable that in any envy-free pair (ς, t) the assignment ς is automatically optimal. The proof is in footnote.[2]

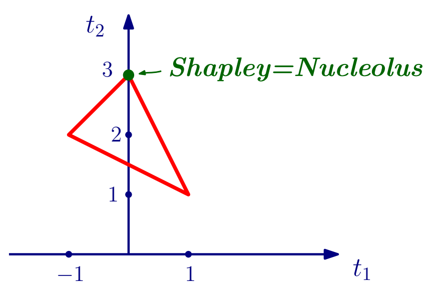
Given an optimal assignment ς we can find the set of transfers which make the pair (ς, t) envy free. In the above example we get the following inequalities:

$$\begin{array}{lll} \textit{no envy by } 1: & 12 + t_1 \geq 9 + t_2 & 12 + t_1 \geq 3 + t_3 \\ \textit{no envy by } 2: & 6 + t_2 \geq 9 + t_3 & 6 + t_2 \geq 6 + t_1 \\ \textit{no envy by } 3: & 9 + t_3 \geq 6 + t_1 & 9 + t_3 \geq 3 + t_2 \end{array}$$

You can check that the three inequalities in the last column are implied by the other three, and since $t_3 = -(t_1 + t_2)$ we end up with the following conditions on (t_1, t_2):

$$t_2 \leq 3 + t_1 \quad 3 \leq t_1 + 2t_2 \quad 2t_1 + t_2 \leq 3$$

which we can draw in the (t_1, t_2) plane. It is the triangular region in the figure below:



We have also drawn the transfers implied by Shapley imputation, which we can easily compute. In the same spirit as in the bankruptcy game the v function is obtained here by taking for any $S \subseteq N$ the sum of the utilities resulting from an optimal assignment within the coalition *after* the others are given their optimal assignment; for example sister 1 may leave houses 1 and 3 to the others (who could do no better on their own) so that $v(1) = u_1(2) = 12$; the other values are obtained similarly and the result is the following (besides $v(N) = 27$):

$$\begin{array}{ccc} v(1) = 12 & v(2) = 6 & v(3) = 3 \\ v(12) = 18 & v(13) = 15 & v(23) = 15 \end{array}$$

The Shapley imputation of this game is $x = (12, 9, 6)$. These are the overall utilities, which means the implied transfers are $t_1 = 0, t_2 = 3, t_3 = -3$ (sister 3 pays 3 to sister 2). In this case it happens that Shapley prescribes an envy-free allocation of houses and transfers. By writing down the excesses it is easy to see that the Shapley imputation is in this case also equal to the Nucleolus.

[2] Consider $n = m$ first, assume (ς, t) is envy free and consider any other assignment $\tilde{\varsigma}$; sister i in ς does not envy the i' who gets her house in $\tilde{\varsigma}$ - that is the i' such that $\varsigma(i') = \tilde{\varsigma}(i)$. So $u_i(h_{\varsigma(i)}) + t_i \geq u_i(h_{\tilde{\varsigma}(i)}) + t_{i'}$ - and by summing up these inequalities we get $\sum u_i(h_{\varsigma(i)}) \geq \sum u_i(h_{\tilde{\varsigma}(i)})$ for the transfers sum to zero; this shows that ς is optimal. If $m < n$ this argument breaks down because if i gets the empty house in $\tilde{\varsigma}$ there are more than one who get this in ς; but the fix is just to consider the lowest i' who gets the empty house in ς, and the rest of the argument is the same.

The Core of this game also is easily computed: it is the set of x such that

$$x_1 = 12 \quad 6 \leq x_2 \leq 12 \quad 3 \leq x_3 \leq 9 \quad x_2 + x_3 = 15$$

In terms of transfers this is $t_1 = 0,\ 0 \leq t_2 \leq 6,\ t_3 = -t_2$. Shapley and Nucleolus select the midpoint of the admissible (x_2, x_3) pairs in the Core.