

Li Calzi 3-person game exercise n.48 p.50

System for fully mixed equilibrium is

$$\begin{aligned}2x + 3y - 5xy &= 1 \\2y + 3z - 5yz &= 1 \\2z + 3x - 5zx &= 1\end{aligned}$$

1. $x, y, z \neq 0$

$x = 0 \Rightarrow y = 1/3, z = 1/2 \Rightarrow 2y + 3z - 5yz \neq 1$. Analogously $y, z \neq 0$

2. $x = y = z$

1st equation is $x(2 - 5y) = 1 - 3y$. Now $2 - 5y = 0$ & $x \neq 0 \Rightarrow x(2 - 5y) \neq 1 - 3y$, so $2 - 5y \neq 0$. Analogously $2 - 5x \neq 0, 2 - 5z \neq 0$. We then derive

$$x = \frac{1 - 3y}{2 - 5y}, y = \frac{1 - 3z}{2 - 5z}, z = \frac{1 - 3x}{2 - 5x}$$

so with a little algebra we get

$$\begin{aligned}x - y &= \frac{1 - 3y}{2 - 5y} - \frac{1 - 3z}{2 - 5z} = \dots = \frac{z - y}{(2 - 5y)(2 - 5z)} \\z - y &= \dots = \frac{z - x}{(2 - 5x)(2 - 5z)} \\z - x &= \dots = \frac{y - x}{(2 - 5x)(2 - 5y)}\end{aligned}$$

whence

$$x - y = \frac{y - x}{(2 - 5x)^2(2 - 5y)^2(2 - 5z)^2}$$

Thus if $x \neq y$ we get a contradiction (-1 equal positive number). Analogously $y = z$.

3. Then the three equations are all $5x - 5x^2 = 1$ which is solved by $x = (5 \pm \sqrt{5})/10$. So the solution of the system is $x = y = z = (5 \pm \sqrt{5})/10$.