Rubi Ex 289.2 Nucleolus in the 4-player weighted majority game
(S. Modica)

We consider the case of Example 294.1: $w = (1,1,1,2)$ and $q = 3$. We will start from the Shapley imputation $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2})$. The first three players play identical roles in any coalition so they must have the same payoff. Any one or two of them will be denoted by i or i,j. In obvious notation we then have

$$v(i) = v(ij) = 0, \quad v(i4) = v(ij4) = v(123) = v(N) = 1$$

therefore their excesses at the Shapley imputation are

$$e(i) = -\frac{1}{6},\ e(4) = -\frac{1}{2},\ e(i4) = \frac{1}{3} = \frac{10}{30},\ e(ij4) = \frac{1}{6},\ e(123) = \frac{1}{2} = \frac{15}{30}.$$

We must lower the last one by raising the three x_i's by the same amount - thereby lowering x_4 by three times as much as x_i since $\sum \Delta x_i = 0$ (the sum $\sum x_i$ is fixed at 1) - so that $\Delta e(4) = -\Delta x_4 = 3\Delta x_i = -\Delta e(123)$. $\Delta e(4)$ goes up from $-1/2$, but the excess which halts the fall in $e(123)$ is $e(i4)$. We have $\Delta e(i4) = -\Delta(x_i + x_4) = 2\Delta x_i$ and that goes up from $1/3$. For $\Delta x_i = 1/30$ we have $\Delta e(123) = -3/30$ and $\Delta e(i4) = 2/30$ so that $e(123) + \Delta e(123) = e(i4) + \Delta e(i4) = 12/30$. And we have no more freedom since Δx_i determines the whole Δx and we get

$$x + \Delta x = (\frac{1}{6} + \frac{1}{30}, \frac{1}{6} + \frac{1}{30}, \frac{1}{6} + \frac{1}{30}, \frac{1}{2} - \frac{3}{30}) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}).$$

This is the Nucleolus of the game. Of course it is a special case of the general formula $x_i = w_i / \sum w_j$. Compared to the Shapley imputation the Nucleolus takes something away from the strong player 4 to distribute it to the other, weaker players in the game. It seems that Shapley is more in the spirit of a measure of "power" and the Nucleolus a "fair" allocation of $v(N)$.

For the sake of curiosity, observing that $\Delta e(ij4) = \Delta x_i$, the resulting excess vector in the Nucleolus is

$$e(i) = -\frac{1}{5},\ e(4) = -\frac{2}{5},\ e(i4) = \frac{2}{5},\ e(ij4) = \frac{1}{5},\ e(123) = \frac{2}{5}.$$