Pareto Optima of the classical Tragedy of the Commons

There are n players with strategy sets $x_i \in [0,1]$ and utility

$$u_i(x) = r \sum_j x_j - x_i$$

where $x = (x_1, \ldots, x_n)$ and $r < 1$ but $nr > 1$. Let $m < n$ be such that $mr < 1 \leq (m+1)r$.

Proposition. *x is Pareto optimal if and only if $x_i < 1$ for at most m players.*

Proof. Suppose the first $m+1$ players have $x_i < 1$ (the argument is the same for any subset of that size or larger). Then take y with $y_i = x_i$ for $i > m+1$ and $y_i = x_i + \epsilon \leq 1$ for $i \leq m+1$. Then clearly $u_j(y) > u_j(x)$ for $j > m+1$, and for $i \leq m+1$ we have $u_i(y) - u_i(x) = (m+1)r\epsilon - \epsilon \geq 0$. Hence x is not Pareto optimal.

Suppose conversely that at x the first $n-m$ players have $x_i = 1$ and consider an alternative allocation y. Let $y_i = x_i + \alpha_i$ and $\alpha_j = \max \alpha_i$. We show that $u_j(y) < u_j(x)$. If $\alpha_j < 0$ then

$$u_j(y) - u_j(x) = r \sum_i \alpha_i - \alpha_j \leq (nr-1)\alpha_j < 0.$$

If on the other hand $\alpha_j \geq 0$ then, since $\alpha_i \leq 0$ for $i \leq n-m$

$$u_j(y) - u_j(x) = r \sum_i \alpha_i - \alpha_j \leq (mr-1)\alpha_j \leq 0$$

and one of the two inequalities must be strict, because: if $\alpha_j = 0$ then for at least one i it must be $\alpha_i < 0$ (otherwise $y = x$) hence the first inequality is strict; if $\alpha_j > 0$ the second one is strict since $mr - 1 < 0$. $\square$