Mixed extension and Rubi Lemma 33.2

We argued in the matching pennies example that the only equilibrium is that the players randomize 50-50. We called that an equilibrium in mixed strategies, and we now proceed more formally. The idea is to extend the strategy spaces to lotteries on the pure strategies, and assume that players randomize independently and are expected utility maximizers with vNM utility u_i. We thus arrive at the "mixed extension" of the original game, and a mixed equilibrium of the original game will then be a Nash equilibrium of the extended game.

We start from the game $G = \langle N, (A_i), (u_i) \rangle$. For each i first we extend the strategy space to the set $\Delta(A_i)$ of all lotteries on A_i; we call α_i an element of this set, so $\alpha_i(a_i)$ is the probability that α_i puts on action a_i and since α_i is a probability distribution $\sum_{a_i \in A_i} \alpha_i(a_i) = 1$. We call α_i a *mixed strategy.* Then consider the profile $\alpha = (\alpha_1, \ldots, \alpha_n)$; the assumption that players mix independently means that α will be a distribution on the set of profiles $A = A_1 \times \cdots \times A_n$ assigning to the profile $a = (a_1, \ldots, a_n)$ the probability

$$\alpha(a) = \alpha_1(a_1) \cdot \alpha_2(a_2) \cdot \cdots \cdot \alpha_n(a_n).$$

You can check that this is indeed a probability distribution, that is $\sum_{a \in A} \alpha(a) = 1$. Lastly we use the expected utility assumption, which is that player i will assign to the mixed profile α the utility $U_i(\alpha)$ given by

$$U_i(\alpha) = \sum \alpha(a) u_i(a) = \sum_a \prod_j \alpha_j(a_j) u_i(a).$$

We have thus arrived at the *mixed extension* of G which we will denote by G^Δ:

$$G^\Delta = \langle N, (\Delta(A_i)), (U_i) \rangle .$$

The extension from A_i to $\Delta(A_i)$ greatly enlarges the strategies available to player i. In matching pennies for example there are two pure strategies, and the set of mixed strategies is the whole interval $[0, 1]$: indeed a lottery on $\{H, T\}$ is specified by the probability $p \in [0, 1]$ assigned to H. Notice that playing $p = 1$ effectively means playing the pure strategy H. You can now read Definition 32.3, which says that a mixed equilibrium of G is just a Nash equilibrium of G^Δ.

For the results we are going to study it is useful to express $U_i(\alpha)$ in a convenient way - equation (32.2) in the book. To this end observe that a mixed strategy α_i may put all probability on a particular a_i - that is you may play a degenerate lottery. In the book this is denoted by $e(a_i)$ but we can simply call it a_i, as we have done in the decision theory part of the course. If i plays a_i she plays the α_i defined by $\alpha_i(a_i) = 1$ and $\alpha_i(a_i') = 0$ for $a_i' \neq a_i$.[1] So if i plays a_i and the others play α_{-i} the resulting mixed profile $\alpha = (a_i, \alpha_{-i})$ gives to the pure profile (a_i, a_{-i}) probability $\prod_j \alpha_j(a_j) = 1 \cdot \prod_{j \neq i} \alpha_j(a_j)$ and to profiles (a_i', a_{-i}) with $a_i' \neq a_i$ probability zero. Therefore player i gets $U_i(a_i, \alpha_{-i}) = \sum_{a_{-i}} \prod_{j \neq i} \alpha_j(a_j) u_i(a_i, a_{-i})$. From this we can deduce that if i plays the general mixed strategy α_i we have

$$\begin{aligned} U_i(\alpha) &= \sum_{a_i} \sum_{a_{-i}} \prod_j \alpha_j(a_j) u_i(a) = \sum_{a_i} \alpha_i(a_i) \sum_{a_{-i}} \prod_{j \neq i} \alpha_j(a_j) u_i(a_i, a_{-i}) \\ &= \sum_{a_i \in A_i} \alpha_i(a_i)\, U_i(a_i, \alpha_{-i}). \end{aligned}$$

This is equation (32.2), which shows that if i plays α_i she gets a lottery with values $U_i(a_i, \alpha_{-i})$ and probabilities $\alpha_i(a_i)$. To proceed we need some notation:

1. For $S \subseteq A_i$ let - as usual after all - $\alpha_i(S) = \sum_{a_i \in S} \alpha_i(a_i)$.
2. $\text{supp}(\alpha_i) = \{a_i \colon \alpha_i(a_i) > 0\}$ - the set S such that $\alpha_i(S) = 1$.
3. $\bar{U}_i(\alpha_{-i}) = \max_{a_i} U_i(a_i, \alpha_{-i})$ - highest utility you can get against α_{-i}
4. $B_i(\alpha_{-i}) = \{a_i \colon U_i(a_i, \alpha_{-i}) = \bar{U}_i(\alpha_{-i})\}$ - actions which yield highest U_i against α_{-i}

Recall the defining property of positive numbers: if $z > 0$ then $a < b \iff za < zb$. Now the Lemma in question may be stated as follows:

Proposition (Lemma 33.2). *α is Nash if and only if*

$$\text{supp}(\alpha_i) \subseteq B_i(\alpha_{-i}) \quad \forall i$$

Proof. By definition $U_i(a_i, \alpha_{-i}) \leq \bar{U}_i(\alpha_{-i})$, with equality if and only if $a_i \in B_i(\alpha_{-i})$. So

$$U_i(\alpha_i, \alpha_{-i}) = \sum_{a_i \in B_i(\alpha_{-i})} \alpha_i(a_i)\, U_i(a_i, \alpha_{-i}) + \sum_{a_i \notin B_i(\alpha_{-i})} \alpha_i(a_i)\, U_i(a_i, \alpha_{-i}) \ \leq \bar{U}_i(\alpha_{-i})$$

[1] Incidentally, by identifying a degenerate lottery with the corresponding pure strategy we can say that A_i is contained in $\Delta(A_i)$.

with equality if and only if $\text{supp}(\alpha_i) \subseteq B_i(\alpha_{-i})$. This says that you cannot profitably deviate if and only if $\text{supp}(\alpha_i) \subseteq B_i(\alpha_{-i})$, which proves the Lemma. $\square$

Note that the result implies that any action which i plays with positive probability in equilibrium gives her the same payoff - namely $\bar{U}_i(\alpha_{-i})$.