On Rubinstein–Osborne page 22

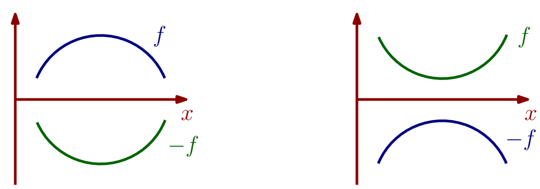
Let us first restate Lemma 22.1.

Lemma (Rubinstein-Osborne 22.1)**.** *We have*

$$\max_{y\in A_2}\min_{x\in A_1} u_2(x,y) = -\min_{y\in A_2}\max_{x\in A_1} u_1(x,y)$$

and the y-solution is the same for both problems.

Proof. We know that for any function f we have $-\max f = \min(-f)$ and $\max(-f) = -\min f$, and that the solutions are the same - see the picture below:



Thus, recalling that $u_2 = -u_1$, we have

$$\max_{y\in A_2}\min_{x\in A_1} u_2(x,y) = \max_{y\in A_2}[-\max_{x\in A_1} u_1(x,y)] = -\min_{y\in A_2}\max_{x\in A_1} u_1(x,y)$$

as asserted. □

Now we use notation $u_1 = u, u_2 = -u$. We are talking about a two–person zero–sum game with action sets A_1, A_2 and $u_1 = u,\ u_2 = -u$. In this context (x^*, y^*) is Nash iff

$$\forall x, y \quad u(x, y^*) \le u(x^*, y^*) \le u(x^*, y). \tag{1}$$

The definitions of "conservative" strategy ("maxminimizer" in the text), and of $\underline{v}$ and $\overline{v}$ are the following:

- $x^* \in A_1$ is conservative if $\max_{x\in A_1}\min_{y\in A_2} u(x,y) = \min_{y\in A_2} u(x^*, y)$;
- $y^* \in A_2$ is conservative if $\min_{y\in A_2}\max_{x\in A_1} u(x,y) = \max_{x\in A_1} u(x, y^*)$;
- $\underline{v} \equiv \max_{x\in A_1}\min_{y\in A_2} u(x,y)$ - called the lower value of th game
- $\overline{v} \equiv \min_{y\in A_2}\max_{x\in A_1} u(x,y)$ - the upper value.

Remark. $\underline{v} \leq \overline{v}$. *Proof:* for any x, y it is $\max_x u(x,y) \geq u(x,y)$, whence for any x (taking min with respect to y on both sides) it is $\overline{v} \geq \min_y u(x,y)$, and from this (take max with respect to x) $\overline{v} \geq \underline{v}$. In case $\underline{v} = \overline{v}$ this common number is denoted by v, or $v(G)$ if needed, and is called the *value* of the game.

Observe that in any equilibrium player 1 should get $u_1 = u \geq \max_{x\in A_1} \min_{y\in A_2} u(x,y) = \underline{v}$ - otherwise she would deviate to conservative play. By the same token player 2 must get $u_2 = -u \geq \max_{y\in A_2} \min_{x\in A_1} u_2(x,y) = -\bar{v}$. Hence in any Nash equilibrium (x^*, y^*) we must have $\underline{v} \leq u \leq \bar{v}$. Proposition 22.2 considerably strengthens these, and also establishes that equilibria coincide with the profiles of conservative strategies in two-person zero-sum games - unlike in general two-person games, think of the crossroads game for instance. We may re–phrase the proposition as follows.

Proposition (Rubinstein-Osborne 22.2)**.** *In a two–person zero–sum game, (x^*, y^*) is a Nash equilibrium if and only if x^* and y^* are conservative and $\underline{v} = \overline{v}$. Moreover, if (x^*, y^*) is Nash then $u(x^*, y^*) = v$, the value of the game.*

Proof. Assume (x^*, y^*) is Nash. Then

$$\overline{v} \leq \max_x u(x, y^*) = u(x^*, y^*) = \min_y u(x^*, y) \leq \underline{v}$$

(equalities by definition of equilibrium, inequalities by definition of $\underline{v}, \overline{v}$); hence $\overline{v} = \underline{v}$. The two inequalities displayed above are then in fact equalities, and this directly implies that x^* and y^* are conservative. This also shows that $u(x^*, y^*) = v$.

Conversely, assume that $\underline{v} = \overline{v}$ and x^* and y^* are conservative. Then

$$\min_y u(x^*, y) = \underline{v} = \overline{v} = \max_x u(x, y^*)$$

so that the inequalities $\min_y u(x^*, y) \leq u(x^*, y^*) \leq \max_x u(x, y^*)$ are in fact equalities, which means that (x^*, y^*) is Nash. □

Thus in a zero-sum game the equilibria coincide with the profiles of conservative strategies, and in all of them player 1 gets $v(G)$ and 2 the opposite. The other obvious but important consequence of the proposition is that if $\underline{v}(G) < \overline{v}(G)$ then G has no Nash equilibria; for a simple example think of 'matching pennies' (in cases like this —A_i finite— we know that there exists a Nash equilibrium in mixed strategies, action sets becoming $\Delta(A_i)$).