Basic Core and Shapley

(Game Theory LM-77, S. Modica)

1. The Core may be empty (Rubi 259.2)

Players are a set N, with $\#N$ odd. Each one can do nothing but each pair can produce 1. The value function is then

$$v(S) = \begin{cases} \#S/2 & \#S \ \textit{even} \\ (\#S-1)/2 & \#S \ \textit{odd} \end{cases}$$

for $S \subseteq N$. In this game the Core is empty, because for any i we have $\sum_{j \neq i} x_j \geq v(N \setminus i) = v(N) = \sum x_j$ which implies $x_i = 0$, and then it cannot be $\sum x_i = v(N)$.

Exercise. Show that if $\#N$ is even the Core is the singleton where each gets 1/2.

2. Non-empty Core example: Outside option in exchange

There is one seller, player 1, of a good which is worth nothing to her and two potential buyers, players 2 and 3 who value the good at b and $100 > b$ respectively. Thus $v(12) = b, v(13) = 100 = v(N)$ and the other coalitions are worthless. Clearly to realize the value of 100 the horse should be sold at some price p to player 3, in other words the candidate imputations for the Core are of the form $(p, 0, 100 - p)$ (otherwise the sum would be at most b since 2 does not pay more than that). But player 2 is fundamental to determine the lower bound of p, intuitively because the seller can use her as a "threat" to the actual buyer. Indeed since $x_2 = 0$ a core imputation must be such that $x_1 = x_1 + x_2 \geq v(12) = b$ which means that $p \geq b$. The Core does not resolve the indeterminacy $b \leq p \leq 100$ which reflects the relative bargaining power of seller and buyer.

3. Core: a simple example from politics (this is Exercise 261.1)

Consider a situation where $v(S)$ can only be zero or one - just losing or winning - and $v(N) = 1$. Say i is a *veto player* if $S \not\ni i \Rightarrow v(S) = 0$ - no coalition can win without him. Then the situation is as follows:

(a) If there are no veto players the Core is empty. To show this suppose $x \in \text{Core}(v)$; then $\sum x_i = 1$, but if $S \not\ni i$ is winning (such an S exists since i is no veto player) then $\sum_{j \in S} x_j \geq v(S) = 1$ which implies $x_i = 0$; this should be true for all i, contradicting $\sum x_i = 1$.

(b) If there is a non-empty set V of veto players then for any x in the Core $x(V) = 1$ (the others get nothing). To prove this we show that these imputations are in the Core and there are no others. So suppose first $x(V) = 1$; is $x(S) \geq v(S)$ for all S? Certainly so if $v(S) = 0$; and if $v(S) = 1$ it must contain all the veto players that is $S \supseteq V$ which implies

$x(S) \geq x(V) = 1 = v(S)$. Suppose on the contrary that $x_i > 0$ for some $i \notin V$; then there is a winning coalition $S \not\ni i$ getting $x(S) < 1 = v(S)$ - so x is not in the Core.

Moral of the story here: if no player has any "power" the Core predicts chaos; if there are few veto players it predicts dire consequences for the others; if there are many it basically predicts nothing. A richer example we shall see is the *weighted majority game.*

4. Core and Shapley value in large gloves games

Consider the game on $N = 2m + 1$ with m players possessing a left glove and $m + 1$ a right one. All players in the "long" side of the market (the right gloves in this case) get zero in the Core, as we have seen in class.

The marginal contribution of a player in a coalition is either zero - if his glove type is redundant - or one - if there are fewer gloves of his type in the coalition. The Shapley value is expected marginal contribution hence it assigns a left-glover the probability that in a random coalition there are fewer left gloves than right ones; similarly for right glovers. These probabilities can be shown to converge to $1/2$ as $N \to \infty$.

5. The Shapley value in two-player games

This is a fundamental example. There are two partners $N = \{1, 2\}$, $v(\{1\}) = v_1, v(2) = v_2, v(N) = 1 > v_1 + v_2$. How to share the surplus $1 - (v_1 + v_2)$ arising from cooperation? The Core gives no hint: any sharing rule is in the Core. The reason is that the only Core restrictions on imputation (x_1, x_2) are $x_1 + x_2 = 1, x_1 \geq v_1, x_2 \geq v_2$ that is $v_1 \leq x_1 \leq 1 - v_2$.

On the other hand it is easy to show that Shapley prescribes sharing the surplus equally: $\psi_1 = v_1 + [1 - (v_1 + v_2)]/2, \psi_2 = v_2 + [1 - (v_1 + v_2)]/2$. In this case there is no doubt this is the "fair" solution. Notice that this is also implied directly by Proposition 291.3 because the balanced contributions property implies $\psi_1 - v_1 = \psi_2 - v_2$ which with $\psi_1 + \psi_2 = 1$ implies the result.[1]

6. A Shapley imputation in a game with empty Core.

This "power" game is called *the apex game.* There is one "big" player and four "small" ones. Precisely the value function is

$$v(S) = \begin{cases} 1 & \text{if } 1 \in S \text{ and } |S| \geq 2 \\ 1 & \text{if } |S| \geq 4 \\ 0 & \text{otherwise.} \end{cases}$$

In this game the Core is empty, with the usual argument: for any $i > 1$ it must be $x_1 + x_i \geq 1$ so $x_i = 0$ all $i > 1$; but $\sum_{i>1} x_i = 1$. For Shapley, by symmetry the four small

[1] Detail: $\psi_1 - v_1 = 1 - \psi_1 - v_2$ says $\psi_1 = (1 + v_1 - v_2)/2 = v_1 + [1 - (v_1 + v_2)]$.

ones must get the same; the big player's marginal contribution is 0 if he arrives first or last, which happens with probability 2/5, and 1 otherwise; so he gets 3/5. Then the others get 1/10. The Shapley imputation is thus $\psi(v) = (3/5, 1/10, 1/10, 1/10, 1/10)$.

7. Shapley is not equal to "voting" shares

Consider a game with 4 people holding 10, 30, 30, 40 votes (or shares in a company) respectively where the value $v(N) = 1$ can be obtained by any simple majority. The "voting shares" proportional imputation is $(1/11, 3/11, 3/11, 4/11)$. The Core is empty by Exercise 261.1 since this is a simple game with no veto player. To find the Shapley imputation we may observe that the marginal contribution of player 1 is always zero so $\psi_1 = 0$; and each of the others have the same marginal contribution in any coalition. Therefore $\psi(v) = (0, 1/3, 1/3, 1/3)$.

8. The Shapley imputation may not belong to a non-empty Core

Even in cases where the Core is not empty the Shapley imputation may not belong to it. Consider the "glove game" with $n + m$ players where n have a right glove and $m > n$ have a left glove. A pair of gloves is worth 1 (a single glove is worthless of course). A coalition S with n_1 right gloves and n_2 left ones has value $v(S) = \min\{n_1, n_2\}$. In particular $v(N) = n$ (the number of pairs of gloves).

Then the Core contains only the imputation where the owners of the right gloves get 1 and the others (in excess supply) get zero. The argument is very similar to the one in the previous example: any S including all players except some $m - n$ owners of a left glove must get n hence those owners of a left glove must get zero. This holds for any subset of owners of a left glove, so this is the only candidate core imputation. To show that it actually cannot be blocked observe that in any coalition worth something the right glovers cannot get more than in the given one. Now take the case of $n = 1, m = 2$ and show that the Shapley imputation is $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}) \neq (1, 0, 0)$.[2]

9. A little less basic: for convex games Shapley is in the Core

The aim here is to show that for convex games (a class which contains for example the bankruptcy game and the airport game) the Core is non-empty and contains the Shapley imputation. First, a set $A \subseteq \mathbb{R}^n$ is *convex* if $x, y \in A \Rightarrow \alpha x + (1 - \alpha)y \in A$ for $0 \leq \alpha \leq 1$. The point $\alpha x + (1 - \alpha)y$ is in the segment joining x and y. It is easy to show by induction that if A is convex and $x_i \in A$ for $i = 1, \ldots, n$ then the convex combination $\sum_i \alpha_i x_i$ where the α_i's are non-negative and sum up to 1 also belongs to A. First we observe

Lemma 1. *The Core imputations of a game v form a convex set.*

[2] *Solution:* this is a special case of the large glove game above, and the right glover's contribution is 0 if he is first and 1 otherwise - so he gets 2/3, and the two others share the rest (by symmetry).

Proof. If for all S we have $\sum_{i \in S} x_i \geq v(S)$ and $\sum_{i \in S} y_i \geq v(S)$ then $\alpha \sum_{i \in S} x_i + (1 - \alpha) \sum_{i \in S} y_i \geq v(S)$. □

A game v over N is *convex* if for all $S, T \subseteq N$ we have $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. We always use the convention $v(\emptyset) = 0$. To interpret this take disjoint sets: it says that cooperation is beneficial. The next lemma gives an equivalent definition of convexity - reminding of convexity in the sense of increasing derivative - which is usually easier to check.

Lemma 2. *v is convex if and only if for all $S \subset T$ not containing i*

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

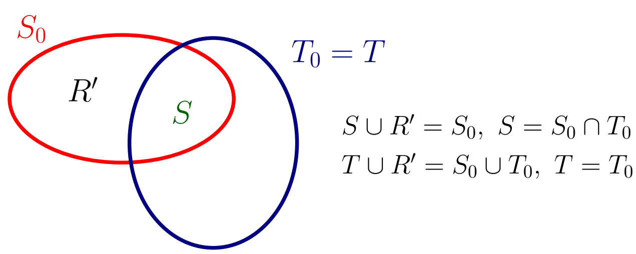
Proof. Suppose $v(S)+v(T) \leq v(S \cup T)+v(S \cap T)$ for all coalitions, equivalently $v(S_0) - v(S_0 \cap T_0) \leq v(S_0 \cup T_0) - v(T_0)$ for all $S_0, T_0 \subseteq N$. For $S \subset T$ not containing i, with $S_0 = S \cup \{i\}$ and $T_0 = T$ we get the inequality in the lemma. Conversely, assume the inequality. Take first $S \subset T$, let $R = N \setminus T \equiv \{i_1, \ldots, i_k\}$ and consider $R' = \{i_1, i_2\} \subset R$. By hypothesis

$$\begin{aligned} &v(S \cup \{i_1\}) - v(S) \leq v(T \cup \{i_1\}) - v(T) \\ &v(S \cup \{i_1, i_2\}) - v(S \cup \{i_1\}) \leq v(T \cup \{i_1, i_2\}) - v(T \cup \{i_1\}) \end{aligned}$$

so by summation we get

$$v(S \cup R') - v(S) \leq v(T \cup R') - v(T).$$

Since we can do the same with any subset of R the above inequality holds for any $R' \subseteq R$. Now take arbitrary sets $S_0, T_0 \subseteq N$, set $S = S_0 \cap T_0$ and $T = T_0$ and apply the inequality just proved. The figure below



makes it clear that we get $v(S_0) - v(S_0 \cap T_0) \leq v(S_0 \cup T_0) - v(T_0)$ as wanted. □

Now we can prove the result we were after:

Proposition. *If v is convex the Core is non-empty and it contains the Shapley imputation.*

Proof. We show that for any given order $\{i_1, \ldots, i_n\}$ of N the corresponding vector of marginal contributions

$$x_{i_k} = v(\{i_1, \ldots, i_k\}) - v(\{i_1, \ldots, i_{k-1}\}), \ k = 1, \ldots, n$$

is in the Core. This implies that the Shapley imputation is in the Core since it is a convex combination of marginal contribution vectors. We have to show that for any S we have $\sum_{i_k \in S} x_{i_k} \geq v(S)$. This is most easily seen in a special case; the general argument then only involves setting up the appropriate notation. Suppose $\#N = 7$ and that the order and S are as in the figure below:

$N:$	5	1	4	7	3	2	6
$S:$			4		3		6

If the game is convex then by the inequality in Lemma 2 we get

$$
\begin{aligned}
&v(\{4\}) - v(\emptyset) \leq v(\{5,1,4\}) - v(\{5,1\}) = x_4 \\
&v(\{4,3\}) - v(\{4\}) \leq v(\{5,1,4,7,3\}) - v(\{5,1,4,7\}) = x_3 \\
&v(\{4,3,6\}) - v(\{4,3\}) \leq v(\{5,1,4,7,3,2,6\}) - v(\{5,1,4,7,3,2\}) = x_6
\end{aligned}
$$

and by summing the three inequalities we obtain $v(S) \leq x_4 + x_3 + x_6$. The result follows because the argument is valid for any S. □

Example 1 (The airport/elevator game)**.** In this case the game is defined in terms of costs: the cost to the ith floor is c_i, with $c_1 < c_2 < \cdots < c_n$, and $c(S) = c_i$ for the highest i in S. To check convexity we have to define the value of a coalition, and this is just the opposite of cost: $v(S) := -c(S)$. Then convexity of v (in the formulation of Lemma 2) is equivalent to $c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$ for all $S \subset T$ not containing i. It is left as exercise to check convexity for the $n = 3$ case where we compute all quantities of interest, for $i = 1, 2, 3$. For example for $i = 2$: a pair $S \subset T$ with T not containing 2 must have $\#T \leq 2$ so $\#S \leq 1$; thus we have to check the cases listed in the table below:

	S	T
a	$\emptyset$	$\{1\}, \{3\}, \{1,3\}$
b	$\{1\}$	$\{1,3\}$
c	$\{3\}$	$\{1,3\}$

for example in case b we have to check that $c(\{1,2\}) - c(\{1\}) \geq c(N) - c(\{1,3\})$; this is $c_2 - c_1 \geq 0$ which we know to be true. The other cases are analogous.

Example 2 (The bankruptcy game)**.** This is defined by an estate E and n creditors with claims c_i such that $\sum c_i > E$. Letting $c(S) = \sum_{i \in S} c_i$ the value is defined as

$$
\begin{aligned}
v(S) &= \max\{0, E - c(N \setminus S)\} \\
&= \max\{0, \alpha + c(S)\}, \quad \alpha = E - c(N) < 0.
\end{aligned}
$$

Observe that $\max\{0,a\} + \max\{0,b\} = \max\{0, a, b, a+b\}$. Using Lemma 2, to show convexity of the game it suffices to show that for $S \subset T \not\ni i$ we have $v(S \cup \{i\}) + v(T) \leq v(T \cup \{i\}) + v(S)$.

But, using the above observation,

$$v(S \cup \{i\}) + v(T) = \max\{0, \alpha + c(S) + c_i, \alpha + c(T), 2\alpha + c(S) + c(T) + c_i\}$$
$$v(T \cup \{i\}) + v(S) = \max\{0, \alpha + c(T) + c_i, \alpha + c(S), 2\alpha + c(S) + c(T) + c_i\}$$

which clearly implies the wanted inequality since $\alpha + c(T) + c_i$ is larger than both $\alpha + c(S) + c_i$ and $\alpha + c(T)$.

10. Some Different Sets of Axioms for Shapley Value

The Shapley value has been the subject of a lot of research and alternative axioms sets exist; we mention a couple besides the set presented in Rubinstein-Osborne. We are not proving anything but it is still interesting to know about it.

Symmetry is always there - it is SYM in Rubinstein-Osborne: If $\Delta_i(S) = \Delta_j(S)$ for any S not containing i and j then $\psi_i(v) = \psi_j(v)$.

- A second single axiom suffices for Shapley: *Marginalism.* This says that payoff should depend only on marginal contributions: *If in games v, w on N it is $\Delta_i^v(S) = \Delta_i^w(S)$ for all S (where superscript indicates the relevant game) then $\psi_i(v) = \psi_i(w)$.* This result is due to Peyton Young (1985).

- The following sufficient set is presented in the textbook by Roger Myerson. Besides symmetry there is first a version of DUM saying that if in v a player never contributes anything he should get zero: *If $\Delta_i(S) = 0$ for any S then $\psi_i(v) = 0$.*

Lastly, additivity is modified to a *linearity* requirement. For $0 \leq p \leq 1$ define the game $z = pv + (1-p)w$ by $z(S) = pv(S) + (1-p)w(S)$. The axiom is that *for any i it should be $\psi_i(z) = p\psi_i(z) + (1-p)\psi_i(w)$.*

The justification for this is the following. Suppose players are not certain which game is played. Then they play z and i gets $\psi_i(z)$. If they wait until uncertainty is resolved they play v or w with probability p and $1-p$, and i's ex-ante expected payoff is then $p\psi_i(z) + (1-p)\psi_i(w)$. The axiom says there should be no advantage in waiting.