Airport Game: Shapley, Nucleolus ($n = 3$) and Core ($n = 3$)

(Game Theory LM-77, S. Modica)

The game is usually described in terms of the cost of runaways of various lengths in an airport. We phrase it in terms of the more familiar elevator cost. The cost to first floor is c_1, cost to i-th floor is c_i and so on up to the cost c_n to last n-th floor; so $c_1 < c_2 \cdots < c_n$ and the elevator costs c_n. The problem is how this cost is to be shared among the families living in the building (one for each floor), that is we look for cost imputations $x = (x_1, \dots, x_n)$ with $\sum_i x_i = c_n$.

Shapley

A "reasonable" solution is that since every family uses the stretch from ground to first floor then everybody should share that; families from 2 to n use the second stretch so they should share the increment $c_2 - c_1$, and so on up to the last stretch from $n-1$ to n which only the uppermost family should pay for.

That is: letting x_i the share payed by i, where $\sum_i x_i = c_n$, and letting also $\delta_i = c_i - c_{i-1}$ (with $c_0 \equiv 0$), the proposal is $x_1 = \frac{\delta_1}{n}, x_2 = \frac{\delta_1}{n} + \frac{\delta_2}{n-1}$ and so on, that is

$$x_i = \sum_{j=1}^{i} \frac{\delta_j}{n-(j-1)} = \frac{\delta_1}{n} + \frac{\delta_2}{n-1} + \cdots + \frac{\delta_i}{n-(i-1)} \quad i = 1, \dots, n$$

where notice that $n-(j-1) = \#\{j, j+1, \dots, n\}$ (the number of families who share δ_j). Since all terms contain δ_1/n, all except the first (that is a total of $n-1$ terms) contain $\delta_2/(n-1)$ and so on we get $\sum_i x_i = \sum_i \delta_i = c_n$ as it should be.

We next show that this is actually the Shapley imputation of the corresponding cooperative game. This is defined in terms of costs, where for $S \subseteq N$ we have $c(S) = c_i$ with i the highest index in S. The argument is the following (adapted from a book by H. Moulin).[1] Consider a randomly ordered N, say $(2, 1, 3, 5, 7, \dots)$; here 2 pays $\delta_1 + \delta_2$, 1 pays nothing, 3 pays δ_3, 5 pays $\delta_4 + \delta_5$ and 7 pays $\delta_6 + \delta_7$; looking at 5, if 3 was not before it as in $(2, 1, 5, 7, \dots)$ then 5 should pay δ_3 as well (with a total of $\delta_3 + \delta_4 + \delta_5$); in general, for $j \leq 5$ family 5 must pay δ_j if in the given order there is nobody in $\{j, j+1, \dots, n\}$ appearing before it. So in $(2, 1, 3, 5, 7, \dots)$ it pays δ_4 since it is the first family in $\{4, \dots, n\}$ appearing in the order and δ_5 since it is the first family in $\{5, \dots, n\}$; and in $(2, 1, 5, 7, \dots)$ it also pays δ_3 because it is also the first in $\{3, \dots, n\}$ appearing in the order. Since in a random order family i is the first in $\{3, \dots, n\}$ with probability $1/(n-2)$, and in general it is the first in $\{j, \dots, n\}$ with probability $1/[n-(j-1)]$ we get that i pays δ_j with probability $1/[n-(j-1)]$, for $j \leq i$. This is the formula displayed above.

[1] *Axioms for Cooperative Decision Making*, Cambridge University Press 1988

Nucleolus ($n = 3$)

For $n = 3$ the Shapley imputation is

$$x_1 = \frac{\delta_1}{3},\ x_2 = \frac{\delta_1}{3} + \frac{\delta_2}{2},\ x_3 = \frac{\delta_1}{3} + \frac{\delta_2}{2} + \delta_3.$$

The "problem" with this is that coalition 23 has a total net gain of $c(23) - (x_2 + x_3) = \delta_1/3$ while family 1 gets more: $c(1) - x_1 = 2\delta_1/3$. As we shall see by avoiding this potential "complaint" we arrive at the Nucleolus.

To work with positive numbers we let $\eta(x, S) = c(S) - \sum_{i \in S} x_i$ (the gain of S) and successively try to maximize the lowest value. We start from the Shapley imputation $x_{Sh} = (\frac{\delta_1}{3}, \frac{\delta_1}{3} + \frac{\delta_2}{2}, \frac{\delta_1}{3} + \frac{\delta_2}{2} + \delta_3)$, as in the table below.

S	$c(S)$	$\eta(x,S)$	$\eta(x_{Sh},S)$	$(\frac{1}{2}\delta_1, \frac{1}{4}\delta_1 + \frac{1}{2}\delta_2, \frac{1}{4}\delta_1 + \frac{1}{2}\delta_2 + \delta_3)$
1	c_1	$\delta_1 - x_1$	$\frac{2}{3}\delta_1$	$x_1 = \frac{1}{2}\delta_1$
2	c_2	$\delta_1 + \delta_2 - x_2$	$\frac{2}{3}\delta_1 + \frac{1}{2}\delta_2$	$\frac{3}{4}\delta_1 + \frac{1}{2}\delta_2$
3	c_3	$\delta_1 + \delta_2 + \delta_3 - x_3$	$\frac{2}{3}\delta_1 + \frac{1}{2}\delta_2$	$\frac{3}{4}\delta_1 + \frac{1}{2}\delta_2$
12	c_2	$\delta_1 + \delta_2 - x_1 - x_2$	$\frac{1}{3}\delta_1 + \frac{1}{2}\delta_2$	$\frac{1}{4}\delta_1 + \frac{1}{2}\delta_2$
13	c_3	$\delta_1 + \delta_2 + \delta_3 - x_1 - x_3$	$\frac{1}{3}\delta_1 + \frac{1}{2}\delta_2$	$\frac{1}{4}\delta_1 + \frac{1}{2}\delta_2$
23	c_3	$\delta_1 + \delta_2 + \delta_3 - x_2 - x_3$	$\frac{1}{3}\delta_1$	$\frac{1}{2}\delta_1$

Coalition 23 has the lowest gain so we raise it by lowering $x_2 + x_3$ - which means raising x_1 - until the gain of 23 becomes equal to that of 1; this gives $x_1 = \delta_1/2$. We are left with an the extra $\delta_1/6$ to divide between 2 and 3; and we divide it evenly between between them (by lowering their x_i) since both have the same excesses; this results in the imputation in the $5th$ column. And that is the nucleolus, since there 2 and 3 have the same excesses so we cannot touch either. Therefore the difference compared to the Shapley imputation is in this case that the cost to the first floor is shared equally between 1 and 23. Thus the Nucleolus is given by

$$x_1 = \frac{\delta_1}{2},\ x_2 = \frac{\delta_1}{4} + \frac{\delta_2}{2},\ x_3 = \frac{\delta_1}{4} + \frac{\delta_2}{2} + \delta_3.$$

Core ($n = 3$)

Observe that $c_i = \sum_{j=1}^{i} \delta_j$ for all i. The inequalities defining the Core are the following:

$$\sum_i x_i = \sum_i \delta_i \quad x_i \le \sum_{j=1}^{i} \delta_j$$
$$x_1 + x_2 \le \delta_1 + \delta_2 \quad x_1 + x_3,\ x_2 + x_3 \le \sum_i \delta_i$$

Now given $\sum_i x_i = \sum_i \delta_i$: $x_2 + x_3 \le \sum_i \delta_i$ implies $x_1 \ge 0$, and $x_1 + x_3 \le \sum_i \delta_i$ implies $x_2 \ge 0$. Since x_3 may be computed as the difference $c_3 - (x_1 + x_2)$ we can draw the Core in (x_1, x_2)

space, where it is characterized by the inequalities $0 \leq x_1 \leq \delta_1$, $0 \leq x_1 + x_2 \leq \delta_1 + \delta_2$. Note that the Core is large, in particular it contains the imputation where $x_1 = x_2 = 0$ and 3 pays the whole cost and at the other extreme also the point where 3 pays only δ_3; and also the distribution of costs between 1 and 2 has virtually no restrictions.

Comparison in a diagram

We visualize the three solutions in the figure below (for x_1 and x_2). The Core is the yellow region; Shapley and Nucleolus imputations are the marked points.