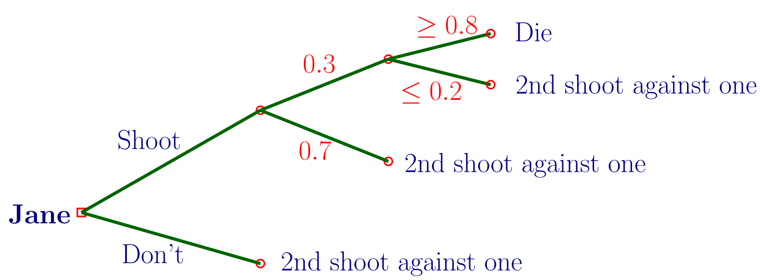
**Three-way duels (Or: the games of your life)**

There are three players in a duel: *Jane*, a young girl, who kills with probability 0.3 if she shoots (that's you); *Cowboy* who kills with probability 0.8; and *Killer*, who kills with probability 1. There are two rounds, in each of which each alive player has a shot; first Jane, then Cowboy then Killer. At any round the player who plays may choose to shoot at any alive player or not to shoot. At the end of the second round the players who are alive share a prize $M$. Players have equal preferences, represented by a $u$ such that $u(0) < u(M/3) < u(M/2) < u(M)$.

(a) Find the subgame perfect equilibrium of the game. In particular, what is Jane going to do in the first round? (No need to draw the whole game!). (b) Compute, for Jane and Killer, the probability of remaining alive at the end of the game (*Answer:* 41.2% and 14% respectively!) and the probability of getting $M$ (*Answer:* 30% and 14% respectively).

**Solution.** (a) In the first stage, if after Jane's turn both Cowboy and Killer are alive Cowboy will shoot at Killer (proved later), and if he fails Killer will kill him (proved later). In any case only one of them remains alive at the end of the round, and Jane has her second shot against him in the second round. If instead Jane kills one of them then the one alive will shoot at her: she will then die with probability $\geq .8$, or have a second shot at him (with probability $\leq .2$). Jane prefers the second shot against a single man to dying, since in that case she has a positive chance to remain alive at the end of the game. Therefore in the first stage it is in Jane's interest not to kill, and she won't shoot. Jane's decision problem is depicted in the figure below:




The rest of the equilibrium: in stage 2 Jane shoots at the single other player alive, and if she fails he shoots back.

(b) Probability of remaining alive for Killer:

$$0.2[\text{prob. Cowboy fails}] * 0.7[\text{prob. Jane fails}] = 0.14$$

this is also equal to the probability that he wins $M$. For Jane we compute the probability of dying (starting at her second turn):

$$0.2[\text{Cowboy misses}] * 0.7[\text{she misses}] * 1[\text{Killer kills her}] \\ + 0.8[\text{Cowboy kills Killer}] * 0.7[\text{she misses}] * 0.8[\text{Cowboy kills her}]$$

that is $0.14 + 0.56 \cdot 0.8 = 0.588$ so that Jane has $1 - 0.588 = 0.412 = 41.2\%$ chance of remaining alive. The probability of wining $M$ is

$$0.2[\text{Cowboy misses}] * 0.3[\text{she kills Killer}] \\ + 0.8[\text{Cowboy kills Killer}] * 0.3[\text{she kills Cowboy}] = 0.30$$

that is 30% chance of winning - more than twice as much as the Killer.

To finish we have to prove the following

*Claim.* In first round Cowboy shoots at Killer if both are alive, and if Killer remains alive he will kill Cowboy.

*Proof.* Observe that the first part follows easily from the second: if Cowboy does not shoot at Killer he is dead for sure, while if he does he will remain alive with positive probability. To prove the second part we preliminarily show that: *in the second round, if Killer and Cowboy are alive and it is Cowboy's turn he will shoot at Killer.* Proof of this. There are two cases: Jane is alive or dead. (a) Jane alive. Then if Killer dies Cowboy gets $M/2$ for sure; if Killer remains alive he will kill one so Cowboy gets 0 if killed and $M/2$ if alive.[1] Hence in this case Cowboy prefers Killer dead. (b) Jane is dead: if Killer dies Cowboy gets $M$, if alive Cowboy will die for sure. So again Cowboy prefers Killer dead than alive. He also prefers Killer dead than Jane dead, for if he kills Jane he will die for sure (by Killer's hand).

Now the second part of the claim: in the first round, if Killer and Cowboy are alive and it is Killer's turn he will kill Cowboy. To prove it observe that if Cowboy dies then Killer gets $M$ with probability at least 0.7 (for sure if Jane is dead); if on the other hand Cowboy remains alive then (as we have shown above) he will shoot at Killer, who will then confront the following prospect: die with probability 0.8, and get at most $M$ with probability 0.2. Hence Killer prefers Cowboy dead than alive. He also prefers Cowboy dead to Jane dead if she is alive, for killing Jane would leave Cowboy alive. This ends the proof. $\square$

The other scenario we alluded to above is the following. Consider the second stage and suppose Killer is alive and it is Cowboy's turn. If Jane is dead nothing changes from above. Suppose all are alive. Of course Cowboy will not shoot at Jane for if he kills her he's dead (by Killer a minute after), and if not there is a chance Killer will kill him anyway - better to shoot at killer, in which case at least if he kills him he split $M$ with Jane. So his alternatives are either to shoot at killer or not shoot. And Killer may say to Cowboy: "If you don't shoot I will kill Jane and we split $M$; if you shoot at me and you miss I will kill you and split the prize with Jane". This is convenient for Cowboy since then he gets $M/2$ for sure, otherwise he won't (you may easily spell out why). So if Jane is alive, at the second stage she will shoot at Killer - for if she kills neither she is dead; and whoever is alive will shoot at her, so it better be Cowboy. Now consider the first round and suppose it is Killer's turn. If there is only one other left he will kill him or her and win $M$. If both Cowboy and Jane

[1] There is another possibility which we discuss at end of proof.

are alive: if Killer kills Jane then Cowboy will shoot at him and he gets $0.2M$; if he kills Cowboy Jane will shoot and he'll get $0.7M$; if he does not shoot Jane will shoot at him and if she misses then the agreement between he and Cowboy will guarantee him $M/2$; so he gets $0.7 \cdot M/2$; therefore Killer kills Cowboy. The conclusion is that also in this alternative - and after all more plausible - equilibrium at the first stage if Killer has the turn he will kill Cowboy if the latter is alive.

**Moral of the Story and a Modification of the game**

When you, young and small, start moving in a working environment with lots of bigger guys around, don't rush to raise your voice too much. It may be better to stay covered for a while. Unless... well, unless if you skip your first chance you have no other, as in the following modification if the above game.

The modification is simply that if you choose not to shoot in the first round you will not be given the move in the second. Show that in this case Jane has to shoot in the first round. *Hint.* It suffices to show that shooting for example at Killer is better than no shooting. So compute expected payoff from not shooting (*Sol.* $0.84u(0) + 0.16u(M/2)$) and the probability of dying if you shoot at Killer (*Sol.* 0.6852). Conclude from this that shooting is better.

**Solution.** If you don't shoot in the first round, since the rest of the game is the same for the others, you get

$$0.8[\text{Cowboy kills Killer}] * \Big(0.8u(0)[\text{Cowboy kills you}] + 0.2u(M/2)[\text{Cowboy misses you}]\Big) \\ + 0.2u(0)[\text{Cowboy misses Killer and Killer kills you}]$$

that is $0.84u(0) + 0.16u(M/2)$. If you shoot at Killer the probability of dying is

$$\begin{aligned} &0.3[\text{u kill K}] * \Big[0.8[\text{Cb kills u}] + 0.2[\text{Cb misses u}] * 0.7[\text{u miss Cb}] * 0.8[\text{Cb kills u}]\Big] \\ &\quad + 0.7[\text{u miss K}]\Big[0.8[\text{Cb kills K}]\big(0.7[\text{u miss Cb}] * 0.8[\text{Cb kills u}]\big) \\ &\quad\quad + 0.2[\text{Cb misses K}] * 0.7[\text{u miss K \& K kills u}]\Big] \\ &= 0.3\Big[0.8 + 0.2 * 0.7 * 0.8\Big] + 0.7\Big[0.8 * 0.7 * 0.8 + 0.2 * 0.7\Big] \\ &= 0.24 + 0.14 * 0.24 + 0.64 * 0.49 + 0.14 * 0.7 = 0.6852 \end{aligned}$$

Now for any distribution $(\alpha, 1-\alpha)$ of $x \equiv 1 - 0.6852$ between $u(M/2)$ and $u(M)$ from $u(0) < u(M/2) < u(M)$ we derive

$$\begin{aligned} &0.6852u(0) + \alpha x u(M/2) + (1-\alpha)xu(M) > 0.6852u(0) + xu(M/2) \\ &= 0.6852u(0) + (1 - 0.6852)u(M/2) > 0.84u(0) + 0.16u(M/2) \end{aligned}$$