S Modica 8.IV.2002

Two Problems with Three boxes

Problem 1. You have preferences over money lotteries. A prize of M units of money is in one of three boxes b_1, b_2, b_3, and the probability of each box containing the prize is 1/3. In chronological order, the following happens:

- you indicate one box among the three, a first time
- a referee opens an <u>empty</u> box among the two not indicated; if he has to choose between two empty boxes, he picks the one with lower index with probability ρ
- you indicate one box among the three, a second time
- you get what is inside the box chosen the second time.

(i) Assume your preferences obey elementary stochastic dominance. What should you do in this situation?

(ii) Assume that you are a risk neutral expected utility maximizer, with $u(x) = x$, $x \in \mathbb{R}$. How much is this 'game' worth to you (that is, how much would you be willing to pay to enter this situation)?

Problem 2. As before there are three boxes and there is a prize in one of them, with probability 1/3 each. Here we assume $M = 1$, and also, as in part (ii) above, that your vNM utility is $u(x) = x$.

You may enter a fair bet where you put down 1/3 and get 1 if the prize is in box 1. A person informed of the location of the prize proposes to show you, before you decide whether to enter the bet or not, one empty box among b_2, b_3, but wants some money in exchange. If both b_2 and b_3 are empty, the person will open up b_2 with probability ρ. Let $f(\rho)$ be the maximum amount you will offer him. What is this $f\colon [0,1] \to [0,1]$?

For this problem I suggest you work with state space $S = \{b_1, b_2, b_3\} \times \{o_2, o_3\}$, where b_i means the prize is in box i and o_j indicates that the person opens box j. First step is to observe that the data determine a probability P on S (it may be helpful to draw S with e.g. the b's on the orizontal axis and o's on the vertical one). You can deduce this from the fact that, writing $b_i o_j$ for the singleton $\{(b_i, o_j)\} \subseteq S$ and letting $\beta_i = \{b_i\} \times \{o_2, o_3\}$ and $\eta_j = \{b_1, b_2, b_3\} \times \{o_j\}$, the hypotheses say:

$$\begin{aligned} &\mathsf{P}(\beta_i) = 1/3, \quad \mathsf{P}(b_2 o_2) = \mathsf{P}(b_3 o_3) = 0, \\ &\mathsf{P}(\eta_2 \mid \beta_1) = \rho, \quad \mathsf{P}(\eta_3 \mid \beta_1) = 1 - \rho. \end{aligned} \tag{1}$$

Note. Problem 2 sometimes appears in the following formulation: Two of three prisoners b_1, b_2, b_3 are to be released, and each has the same probability 1/3 of not being selected. You may enter a fair bet where you put down 1/3 and get 1 if b_2 and b_3 are set free. A guard knowing what has to happen proposes to let you know, before you decide whether to enter the bet or not, one of b_2, b_3 who is going to be released, but wants some money in exchange. If you know that if both b_2 and b_3 are to be freed the guard says b_2 with probability ρ, you will offer him at most $f(\rho)$. How much is $f(\rho)$?

Solution to Problem 1. This is easy: let $S = \{b_1, b_2, b_3\}$, where b_i occurs if the prize is in box i; on S we have the probability $\mathsf{P} = (1/3, 1/3, 1/3)$. The choice of sticking to the indicated box gives probability of winning equal to 1/3. The choice of switching gives zero if the prize is in the indicated box, and it gives M otherwise —think about it—; thus it has winning probability equal to 2/3. Conclusion, you ought to switch if you obey elementary stochastic dominance. For part (ii), since the expected value of the optimal choice is $2M/3$, this is the value of the game under expected value maximization.

Solution to Problem 2. Since $\mathsf{P}(b_i o_2) + \mathsf{P}(b_i o_3) = \mathsf{P}(\beta_i)$, relations (1) determine the other values of P uniquely:

$$\mathsf{P}(b_1 o_2) = \rho/3, \quad \mathsf{P}(b_1 o_3) = (1-\rho)/3, \quad \mathsf{P}(b_2 o_3) = \mathsf{P}(b_3 o_2) = 1/3.$$

You win if β_1 occurs, and elementary calculations give

$$\mathsf{P}(\beta_1 \mid \eta_2) = \frac{\rho}{1+\rho}, \quad \mathsf{P}(\beta_1 \mid \eta_3) = \frac{1-\rho}{2-\rho}.$$

If $\rho = 1/2$ both of these are 1/3, so $f(1/2) = 0$. Now suppose $\rho > 1/2$. Then $\mathsf{P}(\beta_1 \mid \eta_3) < 1/3 < \mathsf{P}(\beta_1 \mid \eta_2)$, and you enter the bet only under η_2. In the $(x_1, p_1; \dots; x_n, p_n)$–notation, you then get the lottery

$$\left(0, \mathsf{P}(\eta_3); \tfrac{2}{3}, \mathsf{P}(b_1 o_2); -\tfrac{1}{3}, \mathsf{P}(b_3 o_2)\right) = \left(0, \tfrac{2-\rho}{3}; \tfrac{2}{3}, \tfrac{\rho}{3}; -\tfrac{1}{3}, \tfrac{1}{3}\right).$$

You will offer (up to) the expected value of this lottery, so $f(\rho) = \frac{2}{9}\left(\rho - \frac{1}{2}\right)$ for $\rho > \frac{1}{2}$. For $\rho < \frac{1}{2}$ work with $\tau = 1 - \rho$ to conclude that the problem is symmetric around 1/2. Conclusion:

$$f(\rho) = \tfrac{2}{9}\left|\rho - \tfrac{1}{2}\right|, \quad 0 \le \rho \le 1.$$