Mixed Extension of Zero-sum Games

Given a zero-sum game

$$G = \langle \{1, 2\}, (A_1, A_2), (u, -u)\rangle$$

where $u_1 = u$ and $u_2 = -u$, consider its mixed extension, which we denote by G^Δ. Since

$$U_2(\alpha) = \sum_a \alpha(a)u_2(a) = -\sum_a \alpha(a)u(a) = -U(\alpha)$$

the mixed extension is also zero-sum: $G^\Delta = \langle \{1, 2\}, (\Delta(A_1), \Delta(A_2)), (U, -U)\rangle$.

Now Proposition 33.1 asserts that every game G with finite strategy sets has a mixed Nash equilibrium, therefore if A_1, A_2 are finite G^Δ has an equilibrium. As we have just seen if G is zero-sum so is G^Δ, so we can apply Proposition 22.2 (assuming finite A_i, which we do) which implies that: G^Δ has a value $v(G^\Delta)$, all equilibria (α_1^*, α_2^*) are conservative and $U(\alpha_1^*, \alpha_2^*) = v(G^\Delta)$. Here of course α_1^* is conservative if it solves $\max_{\alpha_1} \min_{\alpha_2} U(\alpha_1, \alpha_2)$ and α_2^* is conservative if it solves $\min_{\alpha_2} \max_{\alpha_1} U(\alpha_1, \alpha_2)$.

We have also seen that (repetita iuvant) considering pure strategies as degenerate mixed strategies the following holds:

Proposition 1. *a^* is an equilibrium of G if and only if it is an equilibrium of G^Δ.*

Proof. If a^* is an equilibrium of G^Δ there are no profitable deviations in the sets $\Delta(A_i)$ so a fortiori there aren't in the smaller sets A_i. If on the other hand a^* is an equilibrium of G, that is for each i one has $u_i(a_i, a_{-i}^*) \leq u_i(a_i^*, a_{-i}^*)$ for all a_i, then (equation 32.2) $U_i(\alpha_i, a_{-i}^*) = \sum_{a_i} \alpha_i(a_i)u_i(a_i, a_{-i}^*) \leq u_i(a^*)$ so a^* is also an equilibrium of G^Δ. □

The relation between values of G and G^Δ is the following, where we let $\underline{v}(G), v(\bar{G})$ the lower and upper values of G.

Proposition 2. $\underline{v}(G) \leq v(G^\Delta) \leq v(\bar{G})$.

Proof. Let (a_1^*, a_2^*) be conservative. Then

$$\begin{aligned}\underline{v}(G) &= \max_{a_1\in A_1} \min_{a_2\in A_2} u(a_1, a_2) = \min_{a_2\in A_2} u(a_1^*, a_2) = \min_{a_2\in A_2} U(a_1^*, a_2)\\ &= \min_{\alpha_2\in\Delta(A_2)} U(a_1^*, \alpha_2) \leq \max_{\alpha_1\in\Delta(A_1)} \min_{\alpha_2\in\Delta(A_2)} U(\alpha_1, \alpha_2) = v(G^\Delta).\end{aligned}$$

The other half is analogous. □

For example for matching pennies as we have seen $-1 = \underline{v}(G) < v(\bar{G}) = 1$ (and there is no equilibrium) while in the mixed extension it is $v(G^\Delta) = 0$ which is what players get in equilibrium. Since G having a value means $\underline{v}(G) = \bar{v}(G)$ we get the following

Corollary. *If G has a value $v(G)$ then $v(G) = v(G^\Delta)$.*