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CONSENSUS FOR NETWORKS WITH UNKNOWN BUT BOUNDED DISTURBANCES*

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Abstract. We consider stationary consensus protocols for networks of dynamic agents. The measure of the neighbors' states is affected by unknown but bounded disturbances. Here the main contribution is the formulation and solution of what we call the ϵ-consensus problem, where the states are required to converge in a target set of radius ϵ asymptotically or in finite time. We introduce as a solution a dead-zone policy that we denote as the *lazy rule*.

Key words. networks, unknown but bounded, consensus, dynamic agents

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1. Introduction.

Consensus protocols are distributed control policies based on neighbors' state feedback that allow the coordination of multiagent systems. According to the usual meaning of consensus, the system state components must converge, in finite time or asymptotically, to an equilibrium point where they all have the same value lying somewhere between the minimum and maximum of their initial values [1, 3, 6, 7, 8, 10, 9, 11, 12].

The novelty of our approach is in the presence of unknown but bounded (UBB) disturbances [2] in the neighbors' state feedback. Actually, despite the extensive literature on consensus, only a few approaches have considered a disturbance affecting the measurements. In our approach we have assumed a UBB noise, because it requires the least amount of a priori knowledge of the disturbance. Only the knowledge of a bound on the realization is assumed, and no statistical properties need to be satisfied. Moreover, we recall that, starting from [2], the UBB framework has been used in many different fields and applications, such as mobile robotics, vision, multi-inventory, data fusion, and unmanned air vehicles, and in estimation, filtering, identification, and robust control theory.

Because of the presence of UBB disturbances, convergence to equilibria with all equal components is, in general, not possible. The main contribution is then the introduction and solution of the ϵ-consensus problem, where the states converge in a target set of radius ϵ asymptotically or in finite time. In solving the ϵ-consensus problem we focus on linear protocols and present a rule, called the *lazy rule*, for estimating the average from a compact set of candidate points such that the optimal estimate for the ith agent is the one which minimizes the distance from x_i.

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The system under consideration consists of n dynamic agents that reach consensus on a group decision value by implementing *distributed* and *stationary control policies* based on disturbed neighbors' state feedback. Neighborhood relations are defined by the existence of bidirectional communication links between nearby agents. Here, we assume that the set of communication links are bidirectional and define a time-invariant connected communication network.

We solve the ϵ-consensus asymptotically and in finite time. To be more precise, for a given protocol, we find a target set of minimum radius that the agents reach asymptotically. Trivially, any target set of radius strictly greater than the minimum radius can be reached in finite time. We do this by introducing polyhedra of equilibrium points and studying their stability. The above result means that, in general, the value of ϵ cannot be chosen arbitrarily small. We point out its relation with the amplitude ξ of the disturbances. We also consider additional assumptions on the disturbance realization, beyond its inclusion in D, and show that different types of disturbances lead to different values for the minimum radius. For certain disturbance realizations the agents are shown to asymptotically reach 0-consensus.

The paper is organized as follows. In section 2, we set up the new framework of networks under UBB disturbances and formulate the ϵ-consensus problem. In section 3 we introduce the linear protocol and the lazy rule. In section 4, we study the equilibrium points. In section 5, we analyze the asymptotic stability of the equilibrium points. In section 6, we discuss the feasibility conditions for the ϵ-consensus problem. Finally, in section 7, we draw some conclusions.

2. System model. Consider a system of n dynamic agents $\Gamma = \{1, \dots, n\}$ with bidirectional communication links. Assume that we can model the interaction topology among agents through an *undirected* network $G = (\Gamma, E)$, where each edge (i, j) in the edgeset $E \subseteq \Gamma \times \Gamma$ means that there is bidirectional communication from j to i, namely, that agent i can receive information from agent j and vice versa. Assume also that network G is *connected*; that is, there is a path connecting any two arbitrary vertices i and j of the network. Hereafter, we define a path from i_0 to i_r as a sequence $i_0, i_1, i_2, \dots, i_{r-1}, i_r$ of vertices in Γ, such that for each pair of consecutive vertices i_s, i_{s+1}, $s = 0, \dots, r-1$, there exists an edge in E joining i_s with i_{s+1}. Connectivity implies that each agent can spread information, at least indirectly, to all other agents.

Let $A = [a_{ij}]_{i,j\in\Gamma} \in \{0, 1\}^{n\times n}$ be the adjacency matrix of network G; i.e., $a_{ij} = 1$ if there exists an edge connecting i with j in E and $a_{ij} = 0$ otherwise. Without loss of generality we assume that no vertices have loops, which means that all diagonal entries a_{ii}, $i \in \Gamma$, are equal to zero. For each agent i we define *neighborhood of* i as the subset of other agents $N_i = \{j \in \Gamma : a_{ij} \neq 0\}$ to which agent i is connected (with one edge). We also denote the Laplacian matrix as $L = \mathrm{diag}(|N_1|, \dots, |N_n|) - A$, where $|N_i|$ is the degree of vertex i, that is, the cardinality of set N_i.

2.1. Unknown but bounded disturbances. For all $i \in \Gamma$, we consider the family of *first-order* dynamical systems controlled by a *distributed* and *stationary* control policy,

$$\dot{x}_i = u_i(x_i, y^{(i)}) \quad \forall t \geq 0, \tag{1}$$

where $y^{(i)}$ is the information vector from the agents in N_i with generic component j defined as follows:

$$y_j^{(i)} = \begin{cases} y_{ij} & \text{if } j \in N_i, \\ 0 & \text{otherwise.} \end{cases}$$

In the above equation, y_{ij} is a disturbed measure of x_j obtained by agent i as

$$y_{ij} = x_j + d_{ij},$$

and d_{ij} is a UBB disturbance; i.e., $-\xi \le d_{ij} \le \xi$ with a priori known $\xi > 0$. All agents have a perfect measure of their own state; that is, $d_{ii} = 0$ for all $i \in \Gamma$.

The control policy in (1), also called *protocol*, has the structure

$$u_i(x_i, y^{(i)}) = \sum_{j \in N_i} a_{ij} \phi(y_{ij} - x_i), \tag{2}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a generic nonlinear scalar valued function. Then, it is worth noting that the system (1) is a dynamical system with an additive disturbance $d_{ij}(t)$ and a multiplicative disturbance which is the adjacency matrix of the network.

Hereafter, we denote by $x(t) = [x_i(t)]_{i \in \Gamma} \in \mathbb{R}^n$ the state vector, by $d = [d_{ij}]_{i,j \in \Gamma} \in \mathbb{R}^{n \times n}$ the matrix of disturbances, and by D the hypercube $D = \{d : -\xi \le d_{ij} \le \xi$ for all $(i,j) \in \Gamma \times \Gamma$ such that $i \ne j$, $d_{ii} = 0$ for all $i \in \Gamma\}$ of the possible disturbance matrices. We assume that any disturbance realization $\{d(t) \in D,\ t \ge 0\}$ is continuous over time. Note that both d and D are independent of the topology of network G, and, in particular, for each $(i,j) \in \Gamma \times \Gamma$, d_{ij} may differ from d_{ji}. The continuity hypothesis on the disturbance realizations can be weakened, and most of our results keep holding true. However, we hold the continuity assumption to make the proofs of our results simpler and more readable.

2.2. Problem formulation. Before stating the problem, we need to introduce the notions of equilibrium point for given realizations of disturbance $d(t)$, and of ϵ-consensus.

DEFINITION 1. *A point x^* is an* equilibrium point *for given realizations of disturbance $\{d(t) \in D,\ t \ge 0\}$ if there exists $\bar{t} \ge 0$ such that $u_i(x_i^*, y^{(i)}) = 0$ for all $i \in \Gamma$, for all $t \ge \bar{t}$.*

Note that the equilibrium point depends on both the additive disturbance $d_{ij}(t)$ and the multiplicative disturbance a_{ij}.

Consensus means all agents agreeing on a value that lies somewhere between the minimum and maximum of their initial values. In other words, letting the initial state be $x(0)$, the system state must converge to an equilibrium point $x^* \in \{x \in \mathbb{R}^n : x \in span\{\mathbf{1}\}, \min_{j \in \Gamma} x_j(0) \le x_i \le \max_{j \in \Gamma} x_j(0)$ for all $i \in \Gamma\}$ in finite time or asymptotically. Here and in the following, $\mathbf{1}$ stands for the vector $(1, 1, \dots, 1)^T$, and the set $span\{\mathbf{1}\} = \{x :$ there exists $\pi \in \mathbb{R}$ such that $x = \pi \mathbf{1}\}$.

Because of the presence of UBB disturbances, solving the consensus problem is, in general, not possible. This motivates the following definition of ϵ-consensus, describing the cases where the system state is driven within a bounded target set of radius ϵ:

$$T = \left\{ x \in \mathbb{R}^n : |x_i - x_j| \le 2\epsilon \ \forall\, i,j \in \Gamma,\ \min_{j \in \Gamma} x_j(0) \le x_i \le \max_{j \in \Gamma} x_j(0) \ \forall i \in \Gamma \right\}. \tag{3}$$

Note that T is confined between $\max_{j \in \Gamma} x_j(0)$ and $\min_{j \in \Gamma} x_j(0)$. Also, it will be clearer later that the value of the parameter ϵ depends on the maximal amplitude ξ of the disturbances. More precisely, ϵ is a nondecreasing function of ξ and is zero for $\xi = 0$.

To visualize the geometric structure of T, we can see it as the intersection of an unbounded set $T_1 = \{x \in \mathbb{R}^n : |x_i - x_j| \le 2\epsilon$ for all $i, j \in \Gamma\}$ and a bounded set

$T_2 = \{x \in \mathbb{R}^n : \min_{j\in\Gamma} x_j(0) \leq x_i \leq \max_{j\in\Gamma} x_j(0) \text{ for all } i \in \Gamma\}$. The latter is a hypercube, and the former is the sumset of a zonotope $Z = \{x \in \mathbb{R}^n : x = Au,\ 0 \leq u \leq 2\epsilon,\ u \in \mathbb{R}^n\}$ with $span\{\mathbf{1}\}$, where

$$A = \begin{bmatrix} I & -\mathbf{1} \\ \mathbf{0}^T & 0 \end{bmatrix},$$

where $I \in \mathbb{R}^{(n-1)\times(n-1)}$ is the identity matrix, $\mathbf{0} \in \mathbb{R}^{n-1}$ is the null vector, and $-\mathbf{1} \in \mathbb{R}^{n-1}$ is the vector with all components equal to -1. To see that $T_1 = Z + span\{\mathbf{1}\}$, note that $T_1 \supseteq Z + span\{\mathbf{1}\}$ holds trivially, whereas $T_1 \subseteq Z + span\{\mathbf{1}\}$ can be derived as follows. Write $x \in T_1$ as $x = z + x_n\mathbf{1}$, where $z \in Z$, with ith component $z_i = u_i - u_n$ and with $0 \leq u_i = x_i - \min_{j\in\Gamma} x_j \leq 1$, and $x_n\mathbf{1} \in span\{\mathbf{1}\}$.

The above consideration implies that T_1 is an unbounded polyhedron which includes $span\{\mathbf{1}\}$ and whose rays are parallel to $span\{\mathbf{1}\}$ (see, e.g., Figure 1). We remind the reader that a *ray* of a closed set S is a vector v such that if $x \in S$, then $x + \pi v \in S$ for any $\pi \in \mathbb{R}^+$.

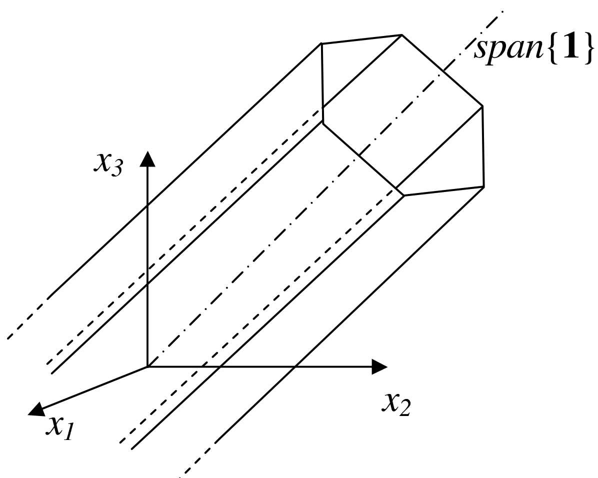

FIG. 1. *Structure of* T_1.

DEFINITION 2. *We say that a protocol* $u_i(x_i, y^{(i)})$, *for all* $i \in \Gamma$, *makes the agents reach* ϵ-consensus in finite time *if there exists a finite time* $\bar{t} > 0$ *such that the system state* $x(t) = x^* \in T$ *for all* $t \geq \bar{t}$, *with* x^* *an equilibrium point in* T. *Furthermore, we say that a protocol* $u_i(x_i, y^{(i)})$, *for all* $i \in \Gamma$, *makes the agents reach* ϵ-consensus asymptotically *if the system state* $x(t) \to x^* \in T$ *for* $t \to \infty$.

The above definition for $\epsilon = 0$ (call it 0-consensus) coincides with the usual definition of consensus [1, 8, 10, 12].

PROBLEM 1 (ϵ-consensus problem). *Given the switched system* (1), *determine a (distributed stationary) protocol* $u_i(x_i, y^{(i)})$, *for all* $i \in \Gamma$, *that makes the agents reach* ϵ-*consensus in finite time or asymptotically for any initial state* $x(0)$. *Furthermore, study the dependence of the target set of radius* ϵ *on the sets* E *and* D.

In the rest of this paper we focus on linear protocols and present a rule for estimating the average from a compact set of candidate points, called the lazy rule, such that the optimal estimate for the ith agent is the one which minimizes the distance from x_i.

3. Linear protocols and lazy rule. A typical consensus problem is the average consensus problem, i.e., the system state converges to the average of the initial state. It is well known that the average consensus can be solved by linear protocols, in the absence of disturbances and under certain conditions on the network. With this in mind, let the following linear protocol be given as

$$u_i(x_i, y^{(i)}) = \sum_{j \in N_i} (\tilde{y}_{ij} - x_i) \quad \forall i \in \Gamma, \tag{4}$$

where $\tilde{y}_{ij}$ is the estimate of state x_j on the part of agent i. For a given disturbed measure y_{ij}, the state x_j and consequently its estimate $\tilde{y}_{ij}$ must belong to the interval

$$y_{ij} - \xi \leq \tilde{y}_{ij} \leq y_{ij} + \xi. \tag{5}$$

The crucial point is how to select $\tilde{y}_{ij}$ from the above interval. In general there may not exist equilibria, and this is easily seen if we take $\tilde{y}_{ij} = y_{ij} = x_j + d_{ij}$ and then rewrite the dynamics (1) under protocol (4) in a compact form as

$$\dot{x}(t) = -Lx(t) + \zeta,$$

where ζ is a vector given by the diagonal of the matrix $Ld(t)$.

Let $\tilde{y}^{(i)} = [\tilde{y}_{ij}]_{j \in N_i}$ be defined according to the following lazy rule:

$$\tilde{y}^{(i)} = \arg \min_{y_{ij} - \xi \leq \tilde{y}_{ij} \leq y_{ij} + \xi} \left| \sum_{j \in N_i} (\tilde{y}_{ij} - x_i) \right|. \tag{6}$$

We call (6) lazy since, according to this rule, each agent i estimates its neighbors' states as equal to the values $\tilde{y}_{ij}$ that induce the minimal value of $|\dot{x}_i|$.

Note that as $u(.,.)$ in protocol (4) depends on $\sum_{j \in N_i} \tilde{y}_{ij}$, the existence of multiple solutions $\tilde{y}^{(i)}$ for (6) is not an issue. This is clearer if one observes that multiple solutions induce the same value $\sum_{j \in N_i} \tilde{y}_{ij}$. Given the lazy rule (6), protocol (4) turns out to have a feedback structure

$$\begin{aligned} \dot{x}_i(t) = u_i(x_i, y^{(i)}) &= \mathcal{D}_{|N_i|\xi} \left(\sum_{j \in N_i} (y_{ij} - x_i) \right) \\ &= \mathcal{D}_{|N_i|\xi} \left(\sum_{j \in N_i} (x_j + d_{ij} - x_i) \right), \end{aligned} \tag{7}$$

where $\mathcal{D}_{|N_i|\xi}$ is a dead-zone function of type

$$\mathcal{D}_{|N_i|\xi}(x) = \begin{cases} 0 & \text{if } |x| \leq |N_i|\xi, \\ x - |N_i|\xi & \text{if } x \geq |N_i|\xi, \\ x + |N_i|\xi & \text{if } x \leq -|N_i|\xi. \end{cases}$$

Hereafter, we simply refer to system (7) when considering the linear protocol (4) with $\tilde{y}^{(i)}$ as in (6).

We conclude this section by noticing that the problem at hand is analogous to a problem in adaptive control in which disturbances cause drift in the parameter estimates. Solutions to this problem in the adaptive control literature include leakage, projection, and dead-zone (see [5]). While the lazy rule solution proposed here parallels the dead-zone solution in adaptive control, it is not straightforward to specialize the leakage or projection method to the consensus problem. Projection does not work because of the distributed setting of the consensus problem. Leakage provides a worse performance with respect to the dead-zone solution in the absence of disturbances. To see this, note that for $\xi = 0$ (no disturbances), system (7) reaches consensus with $\epsilon = 0$ because of the connectivity of the network. With a leakage term (take, for instance, the simplest form of leakage, called "fixed sigma-modification"), the eigenvalue at zero is shifted to minus sigma, so a bounded disturbance will now produce a bounded error (instead of an error that drifts), but in the absence of disturbances the system no longer reaches consensus with $\epsilon = 0$.

The problem at hand can be linked, for comparison, to a different consensus problem which consists of reaching consensus on the average of an exogenous and time varying input $\tilde{d}(t) = [\tilde{d}_i(t)]_{i\in\Gamma}$ on the assumption that the measurement errors of state $x_j(t)$ are all equal to the jth input, i.e., $d_{1j}(t) = \cdots = d_{nj}(t) = \tilde{d}_j(t)$. Such a problem is known as a dynamic average consensus estimators problem [4] and is solved using a proportional integral (PI) protocol. This PI protocol is subject to drift in the presence of the interagent disturbances, but with the key difference that this drift occurs only in states that are unobservable from the estimation errors. Thus, there is a finite l_1-norm from the interagent disturbances to the estimation errors. Not only does this approach not require knowledge of the bound on these disturbances, but it also has the property that if the input becomes small (or zero) at some point, then the errors also become small (or vanish) after that point. Unfortunately, the PI protocol does not exhibit the same nice property if consensus has to be reached on the average of the initial states rather than on the average of an exogenous input. The two problems have different natures though they are both called consensus problems. To make the two problems more similar, one might think of imposing the input equal to the average on the initial states, but this would be in contrast to the exogenous nature of the input.

4. Equilibrium points. We prove that the equilibrium points exist and belong to polyhedra depending on the type of disturbance realization. In particular, we state a first result in the case of constant disturbance d and extend such a result to the case where the disturbance d takes on values in specific subsets of D.

LEMMA 1. *Given system* (7)*, if the disturbance d is constant over time, then*

(i) *a point x is an equilibrium point for $u(.)$ if and only if it belongs to the polyhedron*

$$P(d,E) = \left\{x : -\frac{\sum_{j\in N_i} d_{ij}}{|N_i|} - \xi \le \frac{\sum_{j\in N_i} x_j}{|N_i|} - x_i \le -\frac{\sum_{j\in N_i} d_{ij}}{|N_i|} + \xi \quad \forall i \in \Gamma\right\}; \tag{8}$$

(ii) *$P(d,E)$ includes all the points in span$\{\mathbf{1}\}$; in addition, span$\{\mathbf{1}\} = \bigcap_{d\in D} P(d,E)$;*

(iii) *$P(d,E)$ has $\mathbf{1}$ as the only ray up to multiplication by a nonzero scalar.*

Proof. (i) When the disturbance is constant over time, a point x is an equilibrium point if and only if $u_i(x_i, y^{(i)}) = 0$ for all $i \in \Gamma$. From (7), this condition is equivalent to

$$\left|\sum_{j\in N_i} (x_j + d_{ij} - x_i)\right| \le |N_i|\,\xi \quad \forall i \in \Gamma \tag{9}$$

from which we derive the polyhedron $P(d, E)$ as in (8).

(ii) For any $x \in span\{\mathbf{1}\}$ it holds that $\frac{\sum_{j\in N_i} x_j}{|N_i|} - x_i = 0$. Also, $-\frac{\sum_{j\in N_i} d_{ij}}{|N_i|} - \xi \leq 0 \leq -\frac{\sum_{j\in N_i} d_{ij}}{|N_i|} + \xi$ because $-\xi \leq d_{ij} \leq \xi$ for any $i \in \Gamma$, $j \in N_i$. Then, $span\{\mathbf{1}\} \subseteq P(d, E)$. To prove that $span\{\mathbf{1}\} = \bigcap_{d\in D} P(d, E)$ we show that $P(d^\xi, E) = span\{\mathbf{1}\}$, where $d^\xi = [d^\xi_{ij}] \in \mathbb{R}^{n\times n}$ denotes the matrix of disturbances when these disturbances are all equal to ξ; i.e., $d^\xi_{ij} = \xi$ for all $i \neq j$ and $d_{ii} = 0$ for all $i \in \Gamma$. To see this last argument, from (8) with $d_{ij} = \xi$ for all i and j we have that $\frac{\sum_{j\in N_i} x_j}{|N_i|} - x_i \leq 0$ for all $i \in \Gamma$ and for any $x \in P(d^\xi, E)$. The latter means that the state x_i of each agent i must not be less than the average state of its neighbors in N_i, and this situation occurs only if all the agents have the same state.

(iii) The vector $\mathbf{1}$ is a ray as it is immediate to verify that if $x \in P(d, E)$, then $x + \pi\mathbf{1} \in P(d, E)$ for any $\pi \in \mathbb{R}$. To prove that a vector $\mathbf{1}$ is the unique ray, up to multiplication by a nonzero scalar, consider a vector v not parallel to $\mathbf{1}$. We note that $0 \in P(d, E)$ and we prove that for some $\pi \in \mathbb{R}$ the point $0 + \pi v \notin P(d, E)$. As $-\frac{\sum_{j\in N_i} d_{ij}}{|N_i|} - \xi$ and $-\frac{\sum_{j\in N_i} d_{ij}}{|N_i|} + \xi$ are fixed values, we have that $\pi v \in P(d, E)$ for any $\pi \in \mathbb{R}$ if and only if $v_i - \frac{\sum_{j\in N_i} v_j}{|N_i|} = 0$ for all $i \in \Gamma$. The latter conditions define a linear system with $n - 1$ independent conditions (provided that G is connected), and the solutions are of type $v = \eta\mathbf{1}$ for $\eta \in \mathbb{R}$, contradicting the hypothesis that v is not parallel to $\mathbf{1}$. ◻

The above lemma implies that $P(d, E)$ is an unbounded polyhedron which includes $span\{\mathbf{1}\}$ and whose rays are parallel to $span\{\mathbf{1}\}$.

In the proof of the previous theorem, we have observed that $-\frac{\sum_{j\in N_i} d_{ij}}{|N_i|} - \xi \leq 0 \leq -\frac{\sum_{j\in N_i} d_{ij}}{|N_i|} + \xi$ for all $i \in \Gamma$. When such inequalities hold strictly, $P(d, E)$ is a full-dimensional polyhedron. Actually, any x of type $(0, \dots, 0, \delta, 0, \dots, 0)$ belongs to $P(d, E)$ if we choose $\delta > 0$ sufficiently small. However, not all the polyhedra $P(d, E)$ are full-dimensional, as is apparent by recalling that $P(d^\xi, E) = span\{\mathbf{1}\}$.

In the following we generalize the results of Lemma 1 to the case in which the disturbance is not constant over time. In other words, we are concerned with the study of the equilibrium points for generic disturbance realizations $\{d(t) \in D,\ t \geq 0\}$. First, we can say that only the points in $span\{\mathbf{1}\}$ are equilibrium points for all the possible disturbance realizations $\{d(t) \in D,\ t \geq 0\}$. To see this, observe that (i) they are the only equilibrium points if $d(t) = \xi$ for all t and (ii) condition (6) implies $u(x_i, y^{(i)}) = 0$ for all $i \in \Gamma$ for any realization $\{d(t) \in D,\ t \geq 0\}$.

Before introducing the following lemma we need to generalize definition (8). Consider the generic box $Q = \{d \in D : d^- \leq d \leq d^+\} \subseteq D$, where d^- and d^+ are in D and $d^- \leq d^+$ componentwise. Then define

$$P(Q, E) = \left\{ x \in \mathbb{R}^n : -\frac{\sum_{j\in N_i} d^+_{ij}}{|N_i|} - \xi \leq \frac{\sum_{j\in N_i} x_j}{|N_i|} - x_i \leq -\frac{\sum_{j\in N_i} d^-_{ij}}{|N_i|} + \xi \quad \forall i \in \Gamma \right\}. \tag{10}$$

We prove that $P(Q, E) = \bigcup_{d\in Q} P(d, E)$. Actually, both $P(Q, E) \supseteq \bigcup_{d\in Q} P(d, E)$ and $P(Q, E) \subseteq \bigcup_{d\in Q} P(d, E)$ hold. For any $d \in Q$, if $d^- \leq d \leq d^+$, then $P(d, E) \subseteq P(Q, E)$; hence $P(Q, E) \supseteq \bigcup_{d\in Q} P(d, E)$. Also, to prove $P(Q, E) \subseteq \bigcup_{d\in Q} P(d, E)$, consider a generic point $\hat{x} \in P(Q, E)$. It belongs to $P(\hat{d}, E)$, where for any $i \in \Gamma$ we set

$$\hat{d}_{ij} = \begin{cases} d_{ij}^- & \text{if } j \in N_i,\ \frac{\sum_{j\in N_i} \hat{x}_j}{|N_i|} - \hat{x}_i \geq 0, \\ d_{ij}^+ & \text{if } j \in N_i,\ \frac{\sum_{j\in N_i} \hat{x}_j}{|N_i|} - \hat{x}_i < 0, \\ d_{ij}^+ & \text{otherwise.} \end{cases} \tag{11}$$

As $\hat{d} \in Q$ by construction, we have $P(\hat{d}, E) \subseteq \bigcup_{d\in Q} P(d,E)$ which implies $P(Q,E) \subseteq \bigcup_{d\in Q} P(d,E)$. Then, we can conclude that $P(Q,E) = \bigcup_{d\in Q} P(d,E)$ holds true.

In particular,

$$P(D,E) = \left\{ -2\xi \leq \frac{\sum_{j\in N_i} x_j}{|N_i|} - x_i \leq 2\xi \quad \forall i \in \Gamma \right\} = \bigcup_{d\in D} P(d,E). \tag{12}$$

Let us define, for a given realization $d(t)$ and a subset Q of D, the set

$$\mu(Q) := \{t \in \mathbb{R} : d(t) \in Q\}. \tag{13}$$

Also assume that such a set is unbounded, namely, that

$$\mu_{\bar{t}}(Q) := \{t \in \mathbb{R} : d(t) \in Q,\ t \geq \bar{t}\} \neq \emptyset \quad \forall \bar{t} \geq 0. \tag{14}$$

LEMMA 2. *Given system* (7)*, consider a disturbance realization* $\{d(t) \in D,\ t \geq 0\}$ *and box* $Q = \{d \in D : d^- \leq d \leq d^+\} \subseteq D$*. Assume that* $\mu(Q)$ *is unbounded as expressed by* (14)*. Then, equilibrium points* x *exist and belong to* $P(Q,E)$*.*

Proof. We first observe that the points in *span*$\{\mathbf{1}\}$ are equilibrium points for a given disturbance realization $d(t)$ and also that they belong to $P(Q,E)$. We then prove by contradiction that $x \notin P(Q,E)$ cannot be an equilibrium point. If $x \notin P(Q,E)$, at least for one of its components, call it i, it holds that either

$$\frac{\sum_{j\in N_i} x_j}{|N_i|} - x_i < -\frac{\sum_{j\in N_i} d_{ij}^+}{|N_i|} - \xi$$

or

$$\frac{\sum_{j\in N_i} x_j}{|N_i|} - x_i > -\frac{\sum_{j\in N_i} d_{ij}^-}{|N_i|} + \xi.$$

The previous conditions imply that the value of $u(x_i, y^{(i)})$ is either strictly less than zero or strictly greater than zero for all $d \in Q$. Then, for any $\bar{t} \geq 0$, there exists a time $t \geq \bar{t}$ such that $u(x(t), y^{(i)})$ is always either strictly greater than 0 or less than 0. From $u_i(x_i, y^{(i)}) = \mathcal{D}_{|N_i|\xi}\big(\sum_{j\in N_i}(x_j + d_{ij} - x_i)\big)$ and the continuity of $d(t)$ we have that $u(x(t), y^{(i)})$ is continuous as well. Then, $u(x(t), y^{(i)})$ is always either strictly greater than 0 or less than 0 in a finite neighborhood of $\bar{t}$. Hence x cannot be an equilibrium point. ∎

An immediate consequence of the above lemma is the following corollary.

COROLLARY 1. *Given system* (7)*, consider a disturbance realization* $\{d(t) \in D,\ t \geq 0\}$ *and a finite set* $\mathcal{Q} = \{Q_1, Q_2, \ldots\}$ *of boxes of* D*. Assume that the set* $\mu(Q_r)$ *is unbounded for all* $Q_r \in \mathcal{Q}$*. Then, equilibrium points* x *for* $u(.)$ *exist and belong to* $\bigcap_{Q_r\in\mathcal{Q}} P(Q_r, E)$*.*

In addition, Corollary 1 gives us hope that if the disturbance realization enjoys some general properties the system can reach an equilibrium point close to the set *span*$\{\mathbf{1}\}$. As an example, consider a disturbance realization in Corollary 1 characterized, at least, by $\mathcal{Q} = \{Q_1, Q_2\}$, with $Q_1 = \{d \in D : -\xi \leq d \leq -\hat{d}\}$ and $Q_2 = \{d \in D : \hat{d} \leq d \leq \xi\}$ with $0 < \hat{d} \leq \xi$. We obtain that the only equilibrium points x are in

$$P(Q_1, E) \cap P(Q_2, E) \tag{15}$$
$$= \left\{ \frac{\sum_{j\in N_i} \hat{d}_{ij}}{|N_i|} - \xi \leq \frac{\sum_{j\in N_i} x_j}{|N_i|} - x_i \leq -\frac{\sum_{j\in N_i} \hat{d}_{ij}}{|N_i|} + \xi \quad \forall i \in \Gamma \right\}.$$

The above set obviously defines a neighborhood of the set *span*$\{\mathbf{1}\}$, as *span*$\{\mathbf{1}\} \subseteq P(Q, E)$ for any possible subset Q of D. Of interest is that the radius of the neighborhood becomes smaller and smaller as $\hat{d} \to \xi$ and that $P(Q_1, E) \cap P(Q_2, E) =$ *span*$\{\mathbf{1}\}$ if $\hat{d} = \xi$. The same results hold for all situations in which we can guarantee the disturbance realizations characterized by $\mathcal{Q} = \{Q_1, Q_2\}$ such that $P(Q_1, E) \cap P(Q_2, E)$ is equal to a neighborhood of *span*$\{\mathbf{1}\}$ with a small radius.

The following corollary asserts that $P(D, E)$ is the minimal set including all the possible equilibrium points for a policy $u(.)$ given a UBB disturbance in D.

COROLLARY 2. *Given system* (7), *if the disturbance is UBB in D, then*

(i) *given any $x \in P(D, E)$, there exists a disturbance realization $\{d(t) \in D,\ t \geq 0\}$ that has x as an equilibrium point;*

(ii) *given any disturbance realization $\{d(t) \in D,\ t \geq 0\}$, all its equilibrium points belong to $P(D, E)$.*

Proof. (i) Any generic point $\hat{x} \in P(D, E)$ is trivially an equilibrium point for the corresponding realization $d(t) = \hat{d}$, $t \geq 0$, where $\hat{d}$ is defined as in (11).

(ii) If any disturbance realization $\{d(t) \in D : t \geq 0\}$ has $\mu_{\tilde{t}}(D) \neq \emptyset$ for all $\tilde{t} > 0$, then all its equilibrium points for $u(.)$ belong to $P(D, E)$ by Lemma 2. ☐

5. Stability. In this subsection we prove the asymptotic stability of the equilibrium points. To this end, we introduce a basic property of the stationary protocol $u(.)$ whose components have the feedback form (4). Also, we denote by sign : $\mathbb{R} \to \{-1, 0, 1\}$ the function that returns 1 if its argument is positive, -1 if its argument is negative, and 0 if its argument is null.

LEMMA 3. *Given system* (7),

(i) *either* $\text{sign}(u_i(x_i, y^{(i)})) = \text{sign}(\sum_{j\in N_i}(x_j - x_i))$ *or* $\text{sign}(u_i(x_i, y^{(i)})) = 0$ *for each $i \in \Gamma$, for each $t \geq 0$;*

(ii) $|u_i(x_i, y^{(i)})| \leq |\sum_{j\in N_i}(x_j - x_i)|$.

Proof. Let us start by proving thesis (i). Keeping in mind the definition of dead-zone function used in (7), we have

$$u_i(x_i, y^{(i)}) = \begin{cases} 0 & \text{if } |\sum_{j\in N_i}(x_j + d_{ij} - x_i)| \leq |N_i|\xi, \\ \sum_{j\in N_i}(x_j + d_{ij} - x_i) - |N_i|\xi & \text{if } \sum_{j\in N_i}(x_j + d_{ij} - x_i) \geq |N_i|\xi, \\ \sum_{j\in N_i}(x_j + d_{ij} - x_i) + |N_i|\xi & \text{if } \sum_{j\in N_i}(x_j + d_{ij} - x_i) \leq -|N_i|\xi. \end{cases}$$

In the first case, when $|\sum_{j\in N_i}(x_j + d_{ij} - x_i)| \leq |N_i|\xi$, both theses (i) and (ii) are trivially satisfied, since $u_i(x_i, y^{(i)}) = 0$.

In the second case, when $\sum_{j\in N_i}(x_j + d_{ij} - x_i) \geq |N_i|\xi$, it is easy to see that the two following conditions hold true: $u_i(x_i, y^{(i)}) \geq 0$ and $\sum_{j\in N_i}(x_j - x_i) \geq 0$. Indeed, the former derives straightforwardly from the "if" condition, i.e., $u_i(x_i, y^{(i)}) = \sum_{j\in N_i}(x_j + d_{ij} - x_i) - |N_i|\xi \geq 0$. To prove the latter condition, $\sum_{j\in N_i}(x_j - x_i) \geq 0$, we can use the fact that $|N_i|\xi - \sum_{j\in N_i} d_{ij} \geq 0$, as $-\xi \leq d_{ij} \leq \xi$ for all $(i,j) \in \Gamma \times \Gamma$. From this last inequality we can derive $\sum_{j\in N_i}(x_j - x_i) \geq |N_i|\xi - \sum_{j\in N_i} d_{ij} \geq 0$, and then thesis (i) is proved. It remains to prove thesis (ii). We can use again the fact that $|N_i|\xi - \sum_{j\in N_i} d_{ij} \geq 0$ to obtain the inequalities $0 \leq u_i(x_i, y^{(i)}) = \sum_{j\in N_i}(x_j - x_i) - (|N_i|\xi - \sum_{j\in N_i} d_{ij}) \leq \sum_{j\in N_i}(x_j - x_i)$, and so thesis (ii) is proved as well.

Finally, in the third case, when $\sum_{j\in N_i}(x_j + d_{ij} - x_i) \leq -|N_i|\xi$, a symmetric argument holds. $\square$

THEOREM 1. *Given system* (7)*, there is an equilibrium point* $x^* \in P(D,E)$ *such that* $x(t) \to x^*$.

Proof. We start by observing that any equilibrium x^* must belong to $P(D,E)$ from Corollary 2 and therefore we need only prove that $x(t) \to x^*$. We prove the convergence to equilibrium points x^* by showing that the distance in the 2-norm between two successive sampled points of the state trajectory is upper bounded by the integral of the control between the two sampling times and that such an integral tends to zero for increasing time. Now, to prove that the integral of the control tends to zero it suffices to prove that any upper bound of it tends to zero. We exploit the results of Lemma 3 to find that an upper bound can be obtained by differentiating the values of a candidate Lyapunov function in the two successive sampled states.

More specifically, let us consider the candidate Lyapunov function $V(x(t)) = \frac{1}{2}\sum_{i\in\Gamma}\sum_{j\in N_i}(x_j(t) - x_i(t))^2$ and show that its limit $\lim_{t\to\infty} V(x(t))$ is bounded. To show the boundedness of its limit, in the following, we first show that such a function is positive semidefinite and then that it is also nonincreasing. For the first part, trivially, $V(x(t)) = 0$ if and only if $x(t) \in span\{\mathbf{1}\}$; also, $V(x(t)) > 0$ for all $x(t) \notin span\{\mathbf{1}\}$, and we can conclude that $V(x(t))$ is positive semidefinite.

For the second part, we need to study the behavior of $\dot{V}(x(t))$ for $t \to \infty$. With this purpose, for $\dot{V}(x(t))$ we can write

$$
\begin{aligned}
(16)\qquad \dot{V}(x(t)) &= \sum_{i\in\Gamma}\sum_{j\in N_i}(x_j(t) - x_i(t))\left(u_j(x(t), y^{(i)}) - u_i(x(t), y^{(i)})\right) \\
&= -2\sum_{i\in\Gamma} u_i(x(t), y^{(i)}) \sum_{j\in N_i}(x_j(t) - x_i(t)) \\
&= -2\sum_{i\in\Gamma} \operatorname{sign}(u_i(x(t), y^{(i)}))\,\operatorname{sign}\left(\sum_{j\in N_i}(x_j(t) - x_i(t))\right) \\
&\quad \cdot \left|u_i(x(t), y^{(i)})\right| \left|\sum_{j\in N_i}(x_j(t) - x_i(t))\right|.
\end{aligned}
$$

From the above conditions it is apparent that $V(x(t))$ is not increasing as $\dot{V}(x(t)) \leq 0$ for all $x(t) \in \mathbb{R}^n$. Now, we can use the proved fact that $V(x(t))$ is positive semidefinite and not increasing to simply conclude that $\lim_{t\to\infty} V(x(t))$ is bounded. We now differentiate the values of the candidate Lyapunov function in two successive sampled states.

For any sequence $\{t_k\}_0^\infty$ with $t_{k+1} \geq t_k$ for all k, $t_0 = 0$ and $\lim_{k\to\infty} t_k = \infty$, we can write

$$\lim_{t\to\infty} V(x(t)) - V(x(0)) = \int_0^\infty \dot V(x(t))dt = \sum_{k=0}^\infty \int_{t_k}^{t_{k+1}} \dot V(x(\tau))d\tau. \tag{17}$$

We observe that $\dot V(x(t)) \le 0$ and therefore also $S(k,r) = \int_{t_k}^{t_{k+r}} \dot V(x(\tau))d\tau \le 0$ for all $k, r \ge 1$. Also, as the right-hand side (RHS) of (17) is bounded we must have $\lim_{k\to\infty} S(k,r) = 0$ for all $r \ge 1$. Note that $S(k,r)$ is the difference of the values of the candidate Lyapunov function in two successive sampled states. In other words, these conditions on $S(k,r)$ tell us that such a difference is nonpositive and tends to zero for increasing time. Now, we use Lemma 3 to show that $S(k,r)$ bounds the integral of the control.

We observe that in (16) we have $\operatorname{sign}(u_i(x(t), y^{(i)}))\operatorname{sign}\big(\sum_{j\in N_i}(x_j(t) - x_i(t))\big) = |\operatorname{sign}(u_i(x(t), y^{(i)}))|$ by Lemma 3(i) and $\big|\sum_{j\in N_i}(x_j(t) - x_i(t))\big| \ge \big|u_i(x(t), y^{(i)})\big|$ by Lemma 3(ii). Then, $\dot V(x(t)) \le -2\sum_{i\in\Gamma}\big|u_i(x(t), y^{(i)})\big|^2 = -2\|u(t)\|_2^2$, where $u(t) = [u_i(x_i(t), y^{(i)})]_{i\in\Gamma}$ and $\|.\|_2$ is the 2-norm. Then $-S(k,r) \ge 2\int_{t_k}^{t_{k+r}} \|u(\tau)\|_2^2 d\tau$, and $\lim_{k\to\infty}\int_{t_k}^{t_{k+r}} \|u(\tau)\|_2^2 d\tau = 0$ for all $r \ge 1$. Now

$$\begin{aligned} \|x(t_{k+r}) - x(t_k)\|_2^2 &= \left\| \int_{t_k}^{t_{k+r}} u(\tau)d\tau \right\|_2^2 \\ &\le \int_{t_k}^{t_{k+r}} \|u(\tau)\|_2^2 d\tau \to 0 \quad \text{for } k\to\infty, \quad \forall\, r \ge 1. \end{aligned} \tag{18}$$

The above condition implies that any sequence $\{x(t_k)\}_0^\infty$ is a Cauchy sequence. Then it follows that $\lim_{t\to\infty} x(t)$ exists and the sequence converges to a finite value. Also, the condition $\int_{t_k}^{t_{k+1}} \|u(\tau)\|_2^2 d\tau \to 0$ guarantees that the converging point is an equilibrium point. ☐

An immediate consequence of the above theorem are the following corollaries.

COROLLARY 3. *Given system* (7)*, consider a disturbance realization* $\{d(t) \in D,\ t \ge 0\}$ *and a box* $Q = \{d \in D : d^- \le d \le d^+\} \subseteq D$*. Assume that the set* $\mu(Q)$ *is unbounded as expressed by* (14)*. Then the system trajectory converges to equilibrium points in* $P(Q, E)$*.*

COROLLARY 4. *Given system* (7)*, consider a disturbance realization* $\{d(t) \in D,\ t \ge 0\}$ *and a finite set* $\mathcal{Q} = \{Q_1, Q_2, \ldots\}$ *of boxes of* D*. Assume that the set* $\mu(Q_r)$ *is unbounded for all* $Q_r \in \mathcal{Q}$*. Then the system trajectory converges to* $\bigcap_{Q_r\in\mathcal{Q}} P(Q_r, E)$*.*

Finally, we can conclude that for a disturbance realization which in Corollaries 1 and 4 is characterized, at least, by $\mathcal{Q} = \{Q_1, Q_2\}$, with $Q_1 = \{d \in D : -\xi \le d \le -\hat d\}$ and $Q_2 = \{d \in D : -\hat d \le d \le \xi\}$, with $0 < \hat d \le \xi$, the system trajectory converges to a neighborhood of the set $span\{\mathbf{1}\}$ with the ray of the neighborhood that becomes smaller and smaller as $\hat d \to \xi$ and $P(Q_1, E) \cap P(Q_2, E) = span\{\mathbf{1}\}$ if $\hat d = \xi$.

6. Problem feasibility. In this section, we discuss when the consensus problem is feasible, that is, when system (7) converges to the target set $T = T_1 \cap T_2$ from the initial state $x(0)$. In particular, we first prove that the system trajectory $x(t)$ is always bounded in $T_2 = \{x \in \mathbb{R}^n : \min_{j\in\Gamma} x_j(0) \le x_i \le \max_{j\in\Gamma} x_j(0) \text{ for all } i \in \Gamma\}$ for all $t \ge 0$. Then, we show that $x(t)$ cannot converge to $T_1 = \{x \in \mathbb{R}^n : |x_i - x_j| \le 2\epsilon$ for

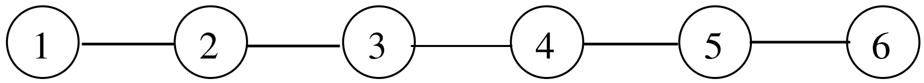

FIG. 2. *Chain network for the proof of Lemma* 4 *and for Example* 1.

all $i, j \in \Gamma\}$ for any value of the parameter ϵ. Finally, we provide some bounds on ϵ for which the convergence of $x(t)$ to T_1 and hence to T is guaranteed.

Let us denote by $i_1(t)$ any agent with the maximal state at time t, formally $i_1(t) = \arg\max_{j\in\Gamma}\{x_j(t)\}$ for all $t \geq 0$. Obviously, $i_1(t)$ depends on time t. The dynamics (7) implies $\dot{x}_{i_1} \leq 0$ for each possible agent $i_1(t)$ and for all $t \geq 0$. Analogously, we define $i_n(t) = \arg\min_{j\in\Gamma}\{x_j(t)\}$ for all $t \geq 0$ and observe that $\dot{x}_{i_n} \geq 0$ for each possible agent $i_n(t)$ and for all $t \geq 0$. Hence, for all $t \geq 0$ and $\Delta t > 0$, we have $\min_{j\in\Gamma} x_j(0) \leq \min_{j\in\Gamma} x_j(t) \leq \min_{j\in\Gamma} x_j(t+\Delta t) \leq \max_{j\in\Gamma} x_j(t+\Delta t) \leq \max_{j\in\Gamma} x_j(t) \leq \max_{j\in\Gamma} x_j(0)$, which is equivalent to saying that $x(t) \in T_2$ for all $t \geq 0$ and that the difference between the maximum and the minimum agent states may not increase over time.

We define as a chain network $G = (\Gamma, E)$ of n nodes, a network with Γ equal to $\{1, 2, \dots, n\}$ and E made of the edges connecting i with $i+1$ for $i = 1, 2, \dots, n-1$ (see, e.g., Figure 2). The next lemma shows that, at least for some systems, the value of the parameter ϵ defining target set T in (3) cannot be chosen arbitrarily small.

LEMMA 4. *Given system* (7) *define a chain network* $G = (\Gamma, E)$*, with* n *even. Let* $d = 0$ *and* $x(0)$ *be defined as follows:* $x_n(0) = 0$*,* $x_{n-1}(0) = \xi$*, then recursively,* $x_{n-i}(0) = x_{n-(i-1)}(0) + (2i-1)\xi$ *for* $i = 1, \dots, n/2$ *and* $x_i(0) = x_{i+1}(0) + (2i-1)\xi$ *for* $i = 1, \dots, n/2$*. Then protocol* (4) *cannot guarantee the convergence to a target set* T *if* $2\epsilon < \frac{n^2-2n+2}{2}\xi$.

Proof. It can be directly verified that $x(0) \in P(0, E)$ and $x_1(0) = \frac{n^2-2n+2}{2}\xi$. Then

$$\max_{i,j\in\Gamma} \max_{x\in P(D,E)} \{x_i - x_j\} \geq x_1 - x_n = \frac{n^2-2n+2}{2}\xi.$$

Then we can conclude that protocol (4) cannot guarantee the convergence to a target set T if $2\epsilon < \frac{n^2-2n+2}{2}\xi$. ◻

On the other hand, the following lemma states that, if ϵ is big enough, then protocol (4) guarantees the convergence of any system (7) to the target set T.

LEMMA 5. *Given system* (7) *defining an undirected network, protocol* (4) *guarantees the convergence of any system* (7) *to the target set* T *if* $2\epsilon \geq \frac{n^3}{2}\xi$.

Proof. Without loss of generality, throughout this proof we assume that the nodes of Γ are sorted such that $\max_{i\in\Gamma}\{x_i\} = x_1 \geq x_2 \geq \cdots \geq x_n = \min_{i\in\Gamma}\{x_i\}$.

Given a generic network, we note that $x_1 - x_n$ can be expressed as $\sum_{k=1}^{n-1} x_k - x_{k+1}$. Then we determine an upper bound for each term $x_k - x_{k+1}$.

Given the generic node k, we consider the cut $(S_k, \bar{S}_k)$, with $S_k = \{1, 2, \dots, k\}$ and $\bar{S}_k = \{k+1, \dots, n\}$.

Lemma 1(i) states that if $x \in P(d, E)$, we have $-\sum_{j\in N_i} d_{ij} - |N_i|\xi \leq \sum_{j\in N_i}(x_j - x_i) \leq -\sum_{j\in N_i} d_{ij} + |N_i|\xi$ for all $i \in \Gamma$. Consequently, as $-\xi \leq d \leq \xi$ for all $d \in D$, if $x \in P(D, E)$, we have $-2|N_i|\xi \leq \sum_{j\in N_i}(x_j - x_i) \leq 2|N_i|\xi$ for all $i \in \Gamma$.

We observe that $\sum_{i\in S_k}\sum_{j\in N_i}(x_j - x_i) = \sum_{i\in S_k}\sum_{j\in N_i\cap\bar{S}_k}(x_j - x_i)$, as the left-hand side (LHS) of this equality includes both $(x_i - x_j)$ and $(x_j - x_i)$ among its terms when both i and j belong to S_k. Hence,

$$-2\sum_{i\in S_k}|N_i|\xi \le \sum_{i\in S_k}\sum_{j\in N_i\cap\bar{S}_k}(x_i-x_j) \le 2\sum_{i\in S_k}|N_i|\xi.$$

Keeping in mind that network G is undirected, we note that $\sum_{i\in S_k}\sum_{j\in N_i\cap\bar{S}_k}(x_j-x_i) = -\sum_{j\in\bar{S}_k}\sum_{i\in N_j\cap S_k}(x_i-x_j)$ as both the LHS and the RHS of this equality include a term (x_i-x_j) for each edge belonging to the cut $(S_k,\bar{S}_k)$, that is, with the extreme $i\in S_k$ and the extreme $j\in\bar{S}_k$. Hence, it also holds that

$$-2\sum_{i\in \bar{S}_k}|N_i|\xi \le \sum_{i\in S_k}\sum_{j\in N_i\cap\bar{S}_k}(x_i-x_j) \le 2\sum_{i\in \bar{S}_k}|N_i|\xi.$$

The above inequalities, together with the assumption $x_i \ge x_j$ if $i<j$, imply

$$0 \le \sum_{i\in S_k}\sum_{j\in N_i\cap\bar{S}_k}(x_i-x_j) \le \min\left\{2\sum_{i\in S_k}|N_i|\xi, 2\sum_{i\in\bar{S}_k}|N_i|\xi\right\}.$$

Denote by m_k the number of the edges belonging to the cut $(S_k,\bar{S}_k)$. As $x_k - x_{k+1} \le x_i - x_j$ for each $i\in S_k,\ j\in N_i\cap\bar{S}_k$, then $(x_k-x_{k+1}) \le \frac{2\xi}{m_k}\min\{\sum_{i\in S_k}|N_i|, \sum_{i\in\bar{S}_k}|N_i|\} \le 2\xi$. The latter inequality holds as $\sum_{i\in S_k}|N_i|/m_k$ is maximal and equal to $k(k-1)+1$ if $m_k=1$ and the vertexes in S_k define a clique. As $\min\{k(k-1)+1,(n-k)(n-k-1)+1\} \le \frac{n^2-2n+2}{4} \le \xi\frac{n^2}{4}$, we finally have

$$(x_1-x_n) \le \sum_{k=1}^{n-1}\frac{1}{m_k}\min\left\{2\sum_{i\in S_k}|N_i|\xi, 2\sum_{i\in\bar{S}_k}|N_i|\xi\right\} \le \frac{n^3}{2}\xi.$$

Finally, we can conclude that, as

$$\max_{i,j\in\Gamma}\max_{x,E:x\in P(D,E)}\{x_i-x_j\} \le (x_1-x_n) \le \frac{n^3}{2}\xi, \tag{19}$$

the system converges to the target set T if $2\epsilon \ge \frac{n^3}{2}\xi$. □

We note that the upper bound in (19) is probably not tight, since the least connected undirected networks (that is, chain networks such as the one in Figure 2) have an $O(n^2)$ bound. For fixed E, a strict upper bound $\bar{\epsilon}$ for ϵ can always be determined numerically in polynomial time. We have

$$\bar{\epsilon} = \max_{i,j\in\Gamma}\max_{x\in P(D,E)}\{x_i-x_j\}, \tag{20}$$

whose brute force computation requires the solution of $n(n-1)$ linear programming problems of type $\max_{x\in P(D,E)}\{x_i-x_j\}$.

Example 1. Consider a network of six agents with chain topology depicted in Figure 2. The initial state is $x(0)=[100,100,100,0,0,0]^T$, $\xi=1$, and disturbances are $d_{12}=d_{21}=d_{23}=d_{32}=d_{34}=1$ and $d_{43}=d_{45}=d_{54}=d_{56}=d_{65}=-1$. Figure 3(a) shows the time plot of the evolution of the state $x(t)$ for $0\le t\le 20$, and as can be seen, trajectories converge to the equilibrium $x^*=[63,61,55,45,39,37]^T$ with $\epsilon=26$ (note that the initial deviation between maximum and minimum values of the state is 100). Figure 3(b) displays a zoom of the trajectories for $0\le t\le 3$, pointing out that $\min_{j\in\Gamma}x_j(0)=\min_{j\in\Gamma}x_j(t)<\max_{j\in\Gamma}x_j(t)=\max_{j\in\Gamma}x_j(0)$ for $0\le t\le 0.5$.

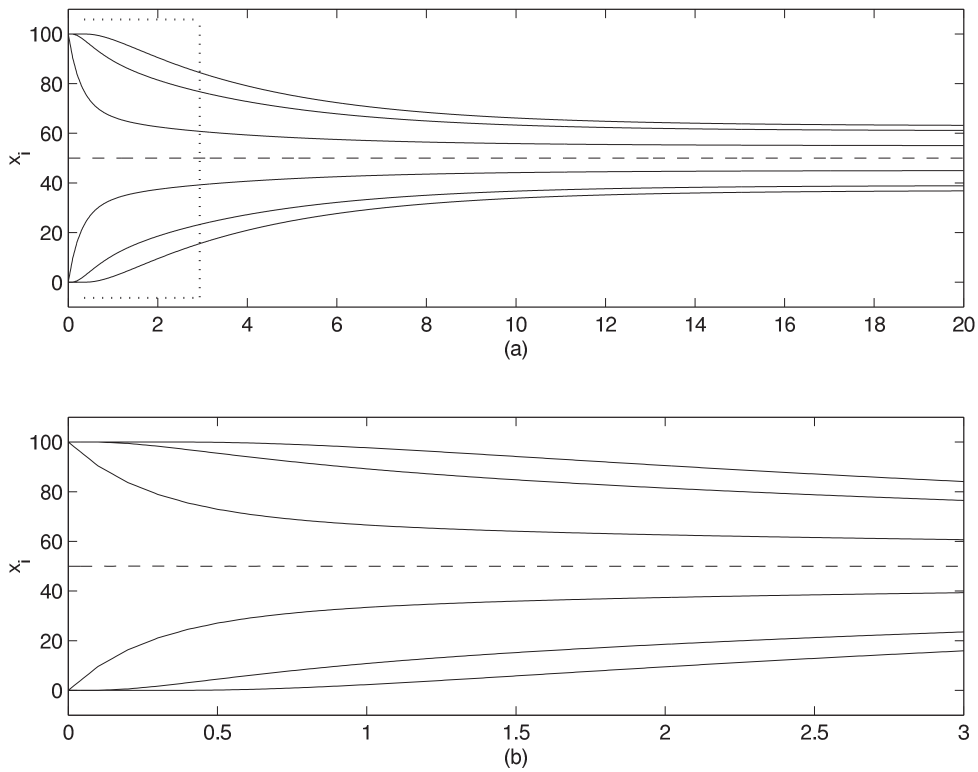

FIG. 3. (a) *Time plot of the state x for an initial state $x(0) = [100, 100, 100, 0, 0, 0]^T$. Trajectories converge to the equilibrium $x^* = [63, 61, 55, 45, 39, 37]^T$ with $\epsilon = 26$;* (b) *zoom of the first time instants (dotted rectangle in* (a)*), which highlights* $\min_{j\in\Gamma} x_j(0) = \min_{j\in\Gamma} x_j(t) < \max_{j\in\Gamma} x_j(t) \leq \max_{j\in\Gamma} x_j(0)$ *for* $0 \leq t \leq 0.5$.

7. Conclusions. Despite the literature on consensus that is now becoming extensive, only a few approaches have considered a disturbance affecting the measurements. In our approach we have assumed a UBB noise in the neighbors' state feedback as it requires the least amount of a priori knowledge of the disturbance. Only the knowledge of a bound on the realization is assumed, and no statistical properties need to be satisfied. Because of the presence of UBB disturbances, convergence to equilibria with all equal components is, in general, not possible. Therefore, the main contribution has been the introduction and solution of the ϵ-consensus problem, where the states converge in a target set of radius ϵ asymptotically or in finite time. In solving the ϵ-consensus problem we have focused on linear protocols and presented a rule for estimating the average from a compact set of candidate points.

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