A Polynomial Approach to the MIMO LQ Servo and Disturbance Rejection Problems*

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Abstract—A novel matrix fraction solution to the discrete-time LQ stochastic tracking and servo problems is extended to the disturbance rejection problem. Unlike recent contributions, here the resulting control law at time \( t \) is given in terms of an expectation conditioned to the accessible disturbance given up to time \( t + k \). If \( k \) is positive and large enough, the optimal feedforward input can be tightly approximated without requiring any stochastic model for the disturbance. If the latter is known (also for recurrent or predictable disturbance), the optimal feedforward input is obtained by solving a single bilateral Diophantine equation.

1. Introduction

This paper extends recent results on the linear-quadratic (LQ) stochastic tracking and servo problems to the disturbance rejection problem. Unlike recent contributions (Grimble, 1986; Sternard and Söderström, 1988; Hunt and Sebek, 1989a, b), here the disturbance to be rejected at time \( t \) is not assumed to be known up to the same time instant but, on the contrary, up to a generic time \( t + k \).

The reader may wonder whether it can be of any practical interest to consider a time lag or lead \( k \). In fact, for the latter case it could be argued that accessible disturbances acting on a plant are rarely known in advance to the controller. In complex systems there are, however, variables that, though not undesirable, affect other variables as disturbances; e.g. in a drum boiler in a power generation unit, the steam generated by \( \xi \) acts as an accessible disturbance on every link. Another example, though nonlinear, concerns a multilink mechanical manipulator where the desired (future) evolution of the other links acts as an accessible disturbance on every link.

A novel solution is given here in terms of a conditional expectation which allows one, for \( k \) positive and large enough, to tightly approximate the optimal feedforward without requiring any stochastic model for the accessible disturbance. In particular, for the SISO case and \( k > 0 \), the solution given here coincides with the one given in Sternard and Söderström (1988) for a \( k \)-step delayed disturbance affecting the plant, whenever the accessible disturbance model can be used.

The reader is referred to Kučera (1979), whose notations are adopted hereafter as much as possible, for the basic facts on the theory of bilateral Diophantine equations and its use in the context of the LQ stochastic pure regulation problem.

2. Problem formulation and connections with a tracking problem

We deal with the following problem: given a discrete-time multivariable linear time-invariant plant represented by

\[
y(t) = \Psi(d)u(t) + \delta(t)\xi(t) + \zeta(t)\xi(t)
\]

find a nonanticipative control law

\[
u(t) = \sigma(z', u(t-1), \delta(t-1))
\]

\[
z(t) = y(t) + \xi(t)
\]

that stabilizes the plant and in stochastic steady-state (s.s.s.) minimizes the quadratic cost

\[
J = \mathbb{E}\left(\|y(t)\|^2 + \|u(t)\|^2\right).
\]

In the above equations, \( \Psi(d) \in \mathbb{R}^{n \times m} \) denotes the plant output; \( u(t) \in \mathbb{R}^m \) the plant input; \( \{\xi(t) \in \mathbb{R}^n\} \) and \( \{\xi(t) \in \mathbb{R}^n\} \) are mutually independent identically distributed (i.i.d.) vector-valued zero-mean sequences, with covariance matrices \( \Sigma_\xi \) and, respectively, \( \Sigma'_\xi \); \( \Sigma_\xi' = \{\xi(t), \xi(t-1), \ldots\} \); \( \delta(t) \in \mathbb{R}^n \) is the accessible disturbance; \( \sigma(z) \) denotes the set of random variables measurable with respect to the \( \sigma \)-algebra generated by \( z \); \( \psi_\xi = \psi_\xi' = 0 \), \( \psi_u = \psi_u' = 0 \) and \( \|u(t)\|^2 := \sum u^T(t) u(t) \); \( \Psi(d) \), \( \Psi(d) \) and \( \zeta(d) \) denote causal rational transfer matrices in the unit backward-shift operator \( d \). Moreover, it is assumed that:

**Assumption 1.** \( \{\delta(t)\} \) is a second-order stationary process mutually uncorrelated both with \( \{\xi(t)\} \) and \( \{\xi(t)\} \).

**Assumption 2.** \( \{\xi(t)\} \) is strictly causal and its actual realization is free of unstable hidden modes; and every unstable root of the characteristic polynomial (c.p.) of the actual realization of the overall representation (1) is a root of the c.p. of the actual realization of \( \Psi(d) \) with the same multiplicity.

The latter condition in assumption 2 states that all the unstable modes of (1) are due to \( \Psi(d) \). This does not imply that \( \Psi(d) \) and \( \zeta(d) \) in the representation (1) are stable. As shown in Park and Bongiorno (1989), assumption 2 is a necessary and sufficient condition for the stabilizability of (1) via a dynamic linear controller, with inputs \( z(t) \) and \( \delta(t) \), acting on \( u(t) \) only. Write \( P(d) \) and \( Q(d) \) in the form of a left matrix fraction description (l.m.f.d.):

\[
\Psi(d) = A^{-1}(d)B(d) \quad \text{and} \quad \zeta(d) = A^{-1}(d)C(d)
\]

where \( A \in \mathbb{R}^{p \times m} \), \( B \in \mathbb{R}^{p \times m} \) and \( C \in \mathbb{R}^{p \times m} \) is a triple of polynomial matrices, \( \mathcal{B}_\Psi(d) \) denoting the set of \( p \times m \) matrices with elements in \( \mathbb{R}[d] \), the set of polynomials in \( d \), and \( A \) a least common left divisor of \( B \) and \( C \). Define a Hurwitz left spectral factor \( D(d) \in \mathcal{B}_\Psi(d) \) by the relation

\[
A(d)\gamma + C(d)\gamma' = D(d)D^*(d)
\]
where $A^*(d) := A^T(d^{-1})$, the superscript $T$ denoting the transpose. It is also assumed that

**Assumption 3.** $D$ is strictly Hurwitz, i.e. $D^{-1}$ is analytic on the closed unit disk.

Then, problem (1) to (4) can be seen to be equivalent to the following: Find a non-anticipative control law (2) such that it stabilizes the plant represented by

$$z(t) = A^{-1}(d)B(d)u(t) + \Delta(d)\delta(t) + A^{-1}(d)D(d)e(t)$$  \hspace{1cm} (7)

and minimizes in $\mathbb{L}_2$ the quadratic cost

$$J = \mathbb{E}\{\|z(t)\|_{y_i}^2 + \|u(t)\|_{w_i}^2\}$$  \hspace{1cm} (8)

where $\{e(t)\}$ is an i.i.d. vector-valued zero-mean sequence such that

$$A^{-1}(d)D(d)e(t) = \Xi(d)\xi(t) + \zeta(t).$$  \hspace{1cm} (9)

In order to find the solution to the above problem, it is convenient to point out its relationship with the LQ stochastic tracking and servo problem discussed in Mosca and Zappa (1989). In the latter problem, the plant is as in (7) with $\delta(t) = 0$; the admissible control law is

$$u(t) = \sigma(z', u', w').$$  \hspace{1cm} (10)

the quadratic cost to be minimized is

$$J = \mathbb{E}\{\|z(t)\|_{y_i}^2 + \|u(t)\|_{w_i}^2\}$$  \hspace{1cm} (11)

$$\epsilon(t) := z(t) - w(t)$$  \hspace{1cm} (12)

where $w(t)$, a second-order stationary process independent of $\{e(t)\}$, is the output reference to be tracked.

The solution to the problem (10)-(12) was found in Mosca and Zappa (1989) in the following form

$$\mathbb{R}(d)u(t) = -\Xi(d)z(t) + u'(t).$$  \hspace{1cm} (13)

In (13) $u'(t)$ is the feedforward or command input that is given in terms of the following conditional expectation

$$u'(t) = \mathbb{E}\{E^{-1}(d)B_2(d)\psi(d)w(t)|w' + \kappa\}.$$  \hspace{1cm} (14)

where $E(d) \in \mathbb{R}_{m,d}$ is a Hurwitz right spectral factor defined by

$$A_2(d)\psi_2(d)A_2(d) + B_2(d)\psi_2(d)B_2(d) = E(d),$$  \hspace{1cm} (15)

$E^{-1} := (E^*)^{-1}$, and $A_2$, $B_2$ is a right coprime m.f.d. of $\Psi$.

$$\mathbb{R}(d) = B_2(d)A_2(d)^{-1}. \hspace{1cm} (16)$$

Further, in (13) $\mathbb{R}(d)$ in $\mathbb{R}_{m,d}$ and $\Xi(d)$ in $\mathbb{R}_{m,d}$ are the stable causal transfer matrices solving the underlying LQ stochastic pure regulation problem

$$\mathbb{R}(d)u(t) = -\Xi(d)z(t),$$  \hspace{1cm} (17)

viz. (1), (10), (11) with $\delta(t) = 0$ and $w(t) = 0$, and satisfying

$$\mathbb{R}(d)A_2(d) + \Xi(d)B_2(d) = E(d).$$  \hspace{1cm} (18)

In Mosca and Zappa (1989), it was shown that $\{u'(t)\}$ is given by (14) turns out to be a second-order stationary process provided that

**Assumption 4.** $E$ is strictly Hurwitz.

E.g. assumption 4 is fulfilled if $\psi_2 > 0$ and $\psi_3 > 0$ (Casavola et al., 1990). If $z = x + w$ and $\delta(t) = 0$, (7) yields

$$\epsilon(t) = A^{-1}(d)B(d)u(t) + A^{-1}(d)D(d)e(t) - w(t).$$  \hspace{1cm} (18)

Further, (13) can be rewritten as

$$\mathbb{R}(d)u(t) = -\Xi(d)\epsilon(t) + \Xi(d)w(t) + u'(t).$$  \hspace{1cm} (19)

Comparing (18) with (7) and (11) with (8), one sees that the tracking problem is isomorphic to the accessible disturbance problem, provided that (10) is equivalent to

$$u(t) = \sigma(z', u', w').$$  \hspace{1cm} (20)

In such a case, the solution of the accessible disturbance problem can be obtained directly from (19) by substituting $\epsilon(t)$ by $z(t)$, and $w(t)$ by $-\delta(t)$:

$$\mathbb{R}(d)u(t) = -\Xi(d)z(t) + v'(t).$$  \hspace{1cm} (21)

where $v'(t)$ is the accessible disturbance feedforward input

$$v'(t) = \Xi(d)\psi_2(d)\delta(t) - \mathbb{E}\{E^{-1}(d)B_2(d)\psi_2(d)w(t)|w' + \kappa\}. \hspace{1cm} (22)$$

Obviously, (22) does not imply that its two additive components yield separately a stable feedback system. Actually, the opposite is true. Nevertheless, the sum of the two terms in (22) yields a stable system, whenever the equivalence between (10) and (20) holds. This equivalence clearly holds if $k = 0$. However, it need not hold true if $k < 0$. One contribution of this paper is to show that for $k \neq 0$, as given by (22), is a second-order stationary process, and how (22) must be modified so as to cover the general case $k \in \mathbb{R}$.

**Remark 1.** It is to be pointed out that in the present paper Gaussianity is not an issue. For the regulation problem, this aspect was also recognized in Aström and Wittenmark (1984). As shown in Mosca and Zappa (1989), the linear control law (13)-(14) is optimal amongst all possible nonlinear control laws (10) provided that $\{e(t)\}$ is zero-mean i.i.d. and (A.1) is fulfilled.

3. **Main results**

Let $\mathbb{R}$ and $\Xi$ be as in (17), then any admissible control law (2) can be written as

$$\mathbb{R}(d)u(t) = -\Xi(d)z(t) + v(t)$$  \hspace{1cm} (23)

where $v \in \sigma(z', u', \delta' + \kappa)$. Thus, the problem amounts to finding a process $\{v(t)\}$ such that the corresponding control law (23) stabilizes the plant (7) and minimizes the cost (8). Taking into account (23), $z(t)$ and $u(t)$ can be decomposed as follows:

$$z(t) = z_6(t) + z_7(t) + z_8(t)$$

$$u(t) = u_6(t) + u_7(t) + u_8(t)$$  \hspace{1cm} (24)

where

$$z_6(t) = \sum A^{-1}(d)E(d)$$

$$z_7(t) = -\Xi(d)\psi_2(d)\delta(t)$$

$$u_6(t) = -\Xi(d)\psi_2(d)$$

$$u_7(t) = \Xi(d)\psi_2(d)\delta(t)$$

$$u_8(t) = -\Xi(d)\psi_2(d)\delta(t)$$

$$y(t) = \Xi(d)\psi_2(d)\delta(t)$$

where $\Sigma := (\Upsilon + \Xi\psi_2^{-1}\Sigma)^{-1}$ and $\Gamma := -\Xi^{-1}\Sigma\Gamma$.

Then, with reference to (24), the cost functional can be split as follows:

$$J = J_{ee} + J_{hc} + J_{ho} + J_{ob} + J_{oo}$$

where

$$J_{ee} := \mathbb{E}\{\|z_6(t)\|_{y_i}^2 + \|u_6(t)\|_{w_i}^2\};$$

$$J_{hc} := \mathbb{E}\{\|z_7(t)\|_{y_i}^2 + \|u_7(t)\|_{w_i}^2\};$$

$$J_{ho} := \mathbb{E}\{\|z_8(t)\|_{y_i}^2 + \|u_8(t)\|_{w_i}^2\};$$

$$J_{ob} := \mathbb{E}\{\|z_8(t)\|_{y_i}^2 + \|u_8(t)\|_{w_i}^2\};$$

$$J_{oo} := \mathbb{E}\{\|v(t)\|_{y_i}^2\}.$$

It is now convenient to introduce the following two lemmas, whose proofs are given in the Appendix.

**Lemma 1.** Let $\Xi(d), E(d), B_2(d)$ and $\psi_2$ be defined as in (17), (15), (16), and, respectively, (4). Then, under assumption 2,

$$V(d) := \Xi(d) - E^{-1}(d)B_2(d)\psi_2$$

is $\Xi(d)Z(d)A_2(d)\Sigma_2^{-1}(d)$  \hspace{1cm} (26)

where $\Xi(d)$ satisfies the following equation (one of the two
coupled bilateral Diophantine equations that must be used for finding \( \mathbb{S} \) and \( \mathbb{S} \subset \mathbb{S} \) in (17) (Kučera, 1979, p. 127)

\[
\hat{E}(d)Y(d) + Z(d)A_2(d) = \dot{B}_2(d)\psi_2(d). \tag{27}
\]

In the above: \( A_1D_2^{-1} \) is a right coprime m.f.d. of \( D^{-1}A = A_1D_2^{-1} \); \( \hat{E} = E^*e^p \); \( \dot{B}_2 = \dot{B}_2D^p \); \( A_1 = A_1^*D^p \); \( p = \max{\{\delta A_1, \delta B_2, \delta E\}} \) if \( \delta A \) denotes the degree of the polynomial matrix \( A \).

Lemma 2. Let assumptions 1–3 be fulfilled and let \( V(d) \) as in (26). Then,

\[
(g(t)) := \{V(d)\delta(d)\delta(t)\} \tag{28}
\]

is a second-order stationary process.

Next theorem shows how (22) must be modified in order to cover the general case.

Theorem. Let assumptions 1–3 be fulfilled. Then, the optimal control law for the LQ stochastic disturbance rejection problem is given by

\[
\mathbb{S}(d)u(d) = -\mathbb{E}(d)z(d) + v(d) \tag{29}
\]

where \( \mathbb{S} \) and \( \mathbb{S} \) are the transfer matrices in (17) and \( v(d) \) is the accessible disturbance feed-forward input defined by the second-order stationary process

\[
v'(d) := \delta_1E'(d)Z'(d)A_2(d)Z'(d)A_2(d)^* \delta(t) \tag{30}
\]

Proof. Because \( J_{nn} \) and \( J_{~} \) are not affected by \( v \), \( J_{~} = 0 \) as proved in Mosca and Zappa, (1989) and by assumption 2, \( J_{~} = 0 \), the optimal control law is given by (23), with \( v(d) \in \varepsilon\{d, u_1, \delta(t) \} \) minimizing \( \mathbb{J}_{~} + J_{~} = \varepsilon\{\|v(t) + (E^*B_2\psi_2 - \mathbb{E}(d)A(d)\gamma(t))\| \} + \varepsilon\{\|E^*B_2\psi_2 - \mathbb{E}(d)A(d)\gamma(t)\|\} \). Now, the latter expectation equals \( \varepsilon\{|\delta(t)|^2\} \). That this is bounded follows by Lemma 2. The minimum is thus attained at \( v(d) = v'(d) \). That \( v'(d) \) is a second-order stationary process follows by the fact that \( v'(d) \) is the conditional expectation (30) of the second-order stationary process \( g(t) \).

Remark 2. It is to be pointed out, similarly to Remark 1, that Gaussianity is not required: the linear control law (28)–(29) is optimal amongst all possible nonlinear control laws (2) provided that \( \varepsilon(d) \) is zero-mean i.i.d.

As discussed in the proof of Lemma 2, the transfer matrix \( \hat{E}^{-1}Z \) is strictly anticausal and stable, viz., \( \hat{E}^{-1}Z \) is strictly causal and stable. Hence, \( v'(d) \) is a linear combination of conditional expectations of future samples of \( A_1D_2^{-1}\gamma(t) \).

Example 1. Consider a SISO plant with: \( \gamma(t) = y(t); \) \( \mathbb{S} = 1 \).

Further \( \hat{E} = c(1 - b^2 - d^2) \) and \( \dot{B}_2 = (d - b) \). Here the Z polynomial in the \( (X, Y, Z) \) solution of the minimum output-variance control is \( Z = b^2(1 - a b) - b \). According to (30), becomes

\[
v'(t) = e^{c(1 - b^2 - d^2)} f(t) \left\{ \begin{array}{ll}
\frac{b}{b - a} & \delta(t) \beta(t) \\
\frac{1 - ad}{(b - a)} & \delta(t) \beta(t) \end{array} \right. \tag{34}
\]

where \( c_i = a - (1 - a b) \). Then, if \( k > 0 \), the feedforward input can be decomposed as \( v'(t) = \theta'(t) + \tilde{\theta}'(t) \), where

\[
\tilde{\theta}'(t) = e^{-c_i}{c_i\delta(t) + \frac{\delta(t + 1)}{b}} + \sum_{j=0}^{k-1} b^{-j} \delta(t + 1 + j) \left( \sum_{k=0}^{j} b^{-k} \delta(t + j + 2) \right) \tilde{\theta}'(t + k) \tag{35}
\]

where only \( \theta'(t) \) depends on the statistical properties of the disturbance.

Equation (30) can be further elaborated when a stochastic model for the accessible disturbance process is available. In this connection, let us assume that

\[
\theta(d) = G_2(d)F_2^{-1}(d)v(d) \tag{31}
\]

where \( \{v(t)\}, v(t) \in \mathbb{R}^n \), is a sequence of i.i.d. random vectors with zero mean and bounded Covariance, \( G_2F_2^{-1} \) in \( \mathbb{S} \), is a causal rational transfer matrix and \( G_2, F_2 \) in \( \mathbb{S} \). Moreover, it is assumed that:

Assumption 5. \( G_2, F_2 \) are both stable. Consider the following right coprime m.f.d.

\[
A_1dD_2^{-1}d\delta^{-1}(d)D_2(d)G_2(d)F_2^{-1}(d) := N(d)M^{-1}(d) \tag{32}
\]

where \( \delta^{-1}D \) is a left m.f.d. of \( \delta \)

\[
\delta^{-1}(d)D_2(d) = \delta(d) \tag{33}
\]

Let \( Z^* := \delta^{-1} \). Define

\[
\lambda := \max{\{\delta E, \delta Z - (k \& 0)\}} \tag{34}
\]

where \( \delta \) denotes the minimum, and

\[
\hat{E} = d^*E^*, Z = d^*E^*Z^* \tag{35}
\]

Corollary 1. Let assumptions 1–5 be fulfilled and the disturbance modeled as in (31). Then the optimal accessible disturbance feedforward input is given by

\[
v'(t) = \delta(d)M^{-1}(d)F_2(d)G_2(d)\delta(d) \tag{36}
\]

where \( \delta(d) = \delta(d)M^{-1}(d)F_2(d)G_2(d)\delta(d) \) and \( \delta(d) \) is the unique polynomial matrix solution of minimum degree w.r.t. \( F \), i.e., \( \delta E < \delta E \), of the following bilateral Diophantine equation

\[
E(d) = \delta(d) + \Gamma(d)M(d) = d^{k+1}Z\delta d^{k+1} \tag{37}
\]

Proof. See Appendix.

Remark 3. Earlier contributions to the LU accessible disturbance rejection problem have typically required the model (31). For the SISO case, in (Sternard and Söderström, 1988) the problem is solved for \( k = 0 \) via a single scalar Diophantine equation of which (31) is a generalization to any \( k \) and the MIMO case. In the MIMO case and \( k = 0 \), (Hunt and Śebek, 1989a) make use of a pair of matrix Diophantine equations that, as can be shown, are equivalent to (36) for \( k = 0 \). More recently, (Hunt and Śebek, 1989b) discuss how to reduce the two above equations to a single equivalent Diophantine equation.

Example 2. Consider again the plant and the cost functional of Ex.1 and assume a first order autoregressive model for the disturbance.

\[
(1 - fd)\delta(t) = v(t) \tag{38}
\]

W.l.o.g., let \( f \) be different from \( a \). Then, \( N = (1 - ad) \) and \( M = (1 - fd) \). Solving (36), one gets from (35)

\[
v'(t) = \theta'(t) + c_i(1 - ad) \delta(t + 1) - \delta(t + k) \tag{39}
\]

Example 1 and 2 indicate one of the advantages of the more general expression (30) with respect to (35). In fact, when the stochastic model for the disturbance is unavailable, but \( k \)
is positive and large enough, from (30) a tight approximation of the command input can still be obtained simply by replacing $v'(t)$ with $B'(t)$; e.g. in example 2 the increase in the cost functional due to this approximation amounts to

$$J(0'(t)) - J(v'(t)) = e^{-2b(\Delta b)^2}$$

which decays exponentially as $k$ increases.

The last case considered here concerns a recurrent or predictable disturbance. By this we mean a stochastic stationary disturbance whose realizations satisfy the equation:

$$\Psi(d)\delta(t) = \delta_0 \quad (37)$$

where $\Psi(d) \in \mathbb{R}^{m \times n}$ is a proper and marginally stable with all roots of $\det(\Psi(d))$ of unit modulus. Consider the following right coprime m.f.d.

$$A_1(d)D_2A_1^{-1}(d) = N_2(d)M_2^{-1}(d) \quad (38)$$

As for (32), it follows that $M_2$ is stable.

**Corollary 2.** Let (A.1)-(A.4) be fulfilled and the disturbance modeled as in (37). Then the optimal accessible disturbance feedforward input is given by

$$v'(t) = \Delta(d)M_2^{-1}(d)\delta(t) \quad (39)$$

where $\Delta(d)$, $\Gamma(d)$ is the unique solution of minimum degree w.r.t. $\Delta(d)$, i.e. $\Delta \Delta < \Delta^2$, of the following Bilateral Diophantine equation

$$E(d)\Delta(d) + \Gamma(d)\Psi(d)M_2(d) = Z(d)N_2(d) \quad (40)$$

**Proof.** See the Appendix. \hfill \Box

**Example 3.** Let us consider the plant and the cost functional as in examples 1 and 2, $N_2 = C$, $M_2 = 1$, and assume that the disturbance satisfies

$$(1 - d + a^2)\delta(t) = 0.$$ 

Then solving (40), one gets from (39):

$$v'(t) = \frac{b^2}{c(b^2 - b + 1)} \left( \frac{b - 2}{2} \delta(t - 1) \right. \left. + \left( \frac{(a^2b^2 + b(2 - a^2 - a) + (a^2 - a + 1)}{b - a} \right) \delta(t) \right). \hfill \Box

4. Conclusions

An I/O approach to the stochastic LQ accessible disturbance rejection problem for MIMO plants has been dealt with. The solution is an extension of the one in Mosca and Zappa (1989) and is based on the knowledge of the disturbance realizations with a lead or a delay of $k$ steps. This feature has practical advantages in program control and in some adaptive control schemes. In particular, it has been shown how, when $k$ is positive and large enough, the feedforward input can be tightly approximated without requiring any stochastic modeling for the disturbance. If a stochastic model for the disturbance is given, the optimal feedforward input is obtained by solving a bilateral Diophantine equation.

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References


Appendix

**Proof of Lemma 1.** The transfer matrix $\Sigma$, obtained by the unique solution $(X, Y, Z)$ of the LQ stochastic pure regulation problem (Kučera, 1979) has the form:

$$\Xi = YD_2^{-1}. \quad \hfill (41)$$

Taking into account the latter and (27) in (26) we obtain:

$$V = (YD_2^{-1} - \hat{E}^{-1}B\psi_1)$$

$$= \hat{E}^{-1}(\hat{E}Y - B\psi_2)D_2^{-1}$$

$$= \hat{E}^{-1}ZA_1D_2^{-1}. \quad \square$$

**Proof of Lemma 2.** Let $A'$ be a left m.f.d. of $\Sigma$, then $V$ equals $\hat{E}^{-1}ZA_1^{-1}A'^{-1}D$. By virtue of assumption 2 and assumption 3 $A_1D_2^{-1}A'^{-1}D$ is causal and stable. As discussed in Mosca and Zappa (1989), $\hat{E}^{-1}Z$ is anticausal and stable, viz. $(\hat{E}^{-1}Z)^*$ is causal and stable. Now, a causal and stable operation followed in cascade by an anticausal and stable operation yields a two-sided stable operation transforming a second-order stationary process into a second-order stationary process. Then, the conclusion follows by assumption 1. \hfill \Box

**Proof of Corollary 1.** Taking into account assumption 5 and, as discussed in the proof of Lemma 2, stability of $A_1D_2^{-1}A'^{-1}D$, in (32) is stable. From the stability assumption on $E$, the stability of $M$ and properness of $E$, there exists a unique pair of polynomial matrices $E$ and $\Delta$, $\Delta < \Delta^2$, satisfying (36). Next, taking into account (36), (32) and (34), (30) becomes

$$v'(t) = \varepsilon(E^{-1}Z^*NM^{-1}d^*v(t)) \quad (41)$$

We consider separately the case $k \geq 0$ and $k < 0$.

$$(k \geq 0):$$

$$v'(t) = \varepsilon(\hat{E}^{-1}ZNM^{-1}d^*v(t + k)) \quad (42)$$

$$(k < 0):$$

$$v'(t) = \varepsilon(\hat{E}^{-1}ZNM^{-1}v(t + k)) \quad (43)$$
substituting (38) and using (40) in (30):

\[ v'(t) = e^{[\Delta^{-1} Z A_1 D^{-1} A D \delta(t) ]} \]
\[ \Delta^{-1} Z A_1 D^{-1} A D \delta(t) \]
\[ = e^{[\Delta^{-1} (E \Delta + \Gamma \Psi M_2 M^{-1} \delta(t) ) ]} \]
\[ = e^{(\Delta M^{-1}_2 \delta(t) + E^{-1} \Gamma \Psi \delta(t) )} \]
\[ = \Delta M^{-1}_2 \delta(t). \]

That (40) can be uniquely solved as stated, follows from properness of \( \Psi(d) \) and the fact that \( \det E(d) \) and \( \det \Psi(d) \) do not share common roots.