

Non-linear protocols for optimal distributed consensus in networks of dynamic agents[☆]

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Received 18 November 2004; received in revised form 19 April 2006; accepted 10 June 2006

Available online 28 July 2006

Abstract

We consider stationary consensus protocols for networks of dynamic agents with fixed topologies. At each time instant, each agent knows only its and its neighbors' state, but must reach consensus on a group decision value that is function of all the agents' initial state. We show that the agents can reach consensus if the value of such a function is time-invariant when computed over the agents' state trajectories. We use this basic result to introduce a non-linear protocol design rule allowing consensus on a quite general set of values. Such a set includes, e.g., any generalized mean of order p of the agents' initial states. As a second contribution we show that our protocol design is the solution of individual optimizations performed by the agents. This notion suggests a game theoretic interpretation of consensus problems as mechanism design problems. Under this perspective a supervisor entails the agents to reach a consensus by imposing individual objectives. We prove that such objectives can be chosen so that rational agents have a unique optimal protocol, and asymptotically reach consensus on a desired group decision value. We use a Lyapunov approach to prove that the asymptotical consensus can be reached when the communication links between nearby agents define a time-invariant undirected network. Finally we perform a simulation study concerning the vertical alignment maneuver of a team of unmanned air vehicles.

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Keywords: Consensus protocols; Decentralized control; Optimal control; Networks

1. Introduction

Distributed *consensus protocols* are distributed control policies based on local information that allow the coordination of multi-agent systems. Agents implement a consensus protocol to reach *consensus*, that is to (make their states) converge to a same value, called *consensus value*, or *group decision value* [24,28].

Coordination of agents/vehicles is an important task in several applications including autonomous formation flight [2,15], cooperative search of unmanned air vehicles (UAVs) [6], swarms of autonomous vehicles or robots [14,19,30],

multi-retailer inventory control [3,4,17] and congestion/flow control in communication networks [20].

Particularly interesting is the progress in the design and analysis of consensus protocols obtained merging notions and tools from the graph theory and control theory [23]. Actually, a central point in consensus problems is the connection between the graph topology, possibly switching, and delays or distortions in communication links [12]. Switching topology and directional communications are studied in [16,22,24,26,27,31], while cooperation based on the notion of *coordination variable* and *coordination function* in [21,29]. There, coordination variable is referred to as the *minimal amount of information needed to effect a specific coordination objective*, whereas a coordination function *parameterizes the effect of the coordination variable on the myopic objectives of each agent*.

In this paper, n dynamic agents reach consensus on a group decision value by implementing *optimal, distributed and stationary control policies* based on neighbors' state feedback. Here, neighborhood relations are defined by the existence of

[☆] Research supported by PRIN "Robustness optimization techniques for high performance control systems", and PRIN "Analysis, optimization, and coordination of logistic and production systems".

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communication links between nearby agents. We assume that the set of communication links are bidirectional and define a time-invariant connected communication network. In this context, we argue that agents asymptotically reach consensus on the desired group decision value by studying equilibrium properties and stability of the group decision value via Lyapunov theory. Similarly to [2,23,24], our agents follow a first-order dynamics. We restrict the group decision value to be a *permutation invariant* function of the agents' initial states. Permutation invariance means that the value of the function is independent of the agents indexes.

Our contribution to the study on consensus problems is twofold. First, we show that consensus can be reached if the agents' state trajectories satisfy a certain time invariance property. In doing this we consider both linear and non-linear protocols. On the basis of such a result, we prove that the group decision values considered are sufficiently general to include any mean of order p of the agents' initial states, and not only the arithmetic/min/max means usually dealt with in the literature (see, e.g., [23,32]). Second, we show that the distributed/individual optimality of the considered protocols allows interpreting our consensus problem as a *non-cooperative differential game* [1,9] where a supervisor entails the agents to reach a consensus by imposing individual objectives. This perspective reminds the *mechanism design*, or inverse game theory [25]. Indeed, the main topic of the mechanism design is the definition of game rules or incentive schemes that induce self-interested players to cooperate and reach Pareto optimal solutions [1]. The advantage of the mechanism design is that intelligence as well as implementation capabilities are distributed. Reviewing the asymptotically consensus reaching as a *team objective*, the supervisor decomposes this team objective in n *individual objectives*. By doing this, the agents are said “what to aim at” instead of “what to do”, and are free to find the best solution to their subtasks. Obviously, the advantage of distributing intelligence has on one hand a non-indifferent cost due to the necessity of equipping each agent with computational and processing units, but, on the other hand, reduces the monitoring costs. We show that, if the supervisor imposes convex penalty functions, rational agents have a unique optimal protocol. We prove this result through the Pontryagin minimum principle (see, e.g., [7]). From a slightly different point of view, the solution of a mechanism design problem allows to determine whether a set of agents with given individual objective functions will implement the considered protocol.

The present paper states consensus protocol definition and mechanism design as two separate problems for the sake of clarity. However, it must be noted that these two problems may be seen as two faces of the same coin. The consensus protocol definition problem answers to the question of determining the policies that the agents must implement to reach a given consensus. The mechanism design problem answers to the question of which policy is implemented, and hence which consensus value is reached, by selfish agents with given individual objectives.

Unfortunately, solving the mechanism design problem is a difficult task, unless the problem is an *affine quadratic game*

[1]. Our idea is then to translate it into a sequence of more tractable receding horizon problems. At each discrete time t_k , the receding horizon control scheme optimizes over an infinite *planning* horizon $T \rightarrow \infty$, and executes the controls over a one-step *action* horizon $\delta = t_{k+1} - t_k$ [10,11]. The neighbors' state whose evolution is unpredictable, are kept constant over the planning horizon (*naive assumption*) [13,18]. At time t_{k+1} each agent re-optimizes its controls based on the new information on neighbors' state which has become available. To find an approximate solution to the mechanism design problem we then take the limit for $\delta \rightarrow 0$ of the given receding horizon solution.

The paper is organized as follows. In Section 2, we formulate the consensus problem (Problem 1) and the mechanism design problem (Problem 2). Sections 3 and 4 concern the solution to the consensus problem. In particular, in Section 3, we study the time invariance property of the state trajectory. In Section 4, we provide sufficient conditions for the asymptotical stability of the group decision value via Lyapunov theory. Section 5 addresses the mechanism design problem, whose solution is derived starting from the results on the consensus problem. More specifically, we exploit the Pontryagin minimum principle to derive necessary and sufficient optimality conditions. Then, we merge the results on time invariance, stability, and optimality to design a mechanism for the distributed optimal consensus. In Section 6, we simulate the vertical alignment maneuver of a team of UAVs. Finally, in Section 7, we draw some conclusions.

2. Consensus and mechanism design problems

We consider a system of n dynamic agents $\Gamma = \{1, \dots, n\}$ and model the interaction topology among agents through a time-invariant connected network (graph) $G = (\Gamma, E)$. The network is undirected, i.e., if $(i, j) \in E$ then $(j, i) \in E$. The network is connected, that is, for any vertex $i \in \Gamma$ there exists a path, i.e., a sequence of edges in E , $(i, k_1)(k_1, k_2) \dots (k_r, j)$, that connects it with any other vertex $j \in \Gamma$. Finally, the network G is not complete, namely, each vertex i is connected (with one edge) only to a subset of other vertices $N_i = \{j : (i, j) \in E\}$ called *neighborhood* of i .

Each edge (i, j) in the edgeset E means that there is communication from j to i . As (j, i) is also in the edgeset E the communication is bidirectional, namely, if agent i can receive information from agent j then also agent j can receive information from agent i . Also, G not complete means that each agent i exchanges information only with its neighbors.

Each agent i has a (simplified) *first-order dynamics* controlled by a *distributed* and *stationary* control policy

$$\dot{x}_i = u_i(x_i, x^{(i)}) \quad \forall i \in \Gamma, \quad (1)$$

where x_i is the state of agent i and $x^{(i)}$ is the state vector of the agents in N_i with generic component j defined as follows:

$$x_j^{(i)} = \begin{cases} x_j & \text{if } j \in N_i, \\ 0 & \text{otherwise,} \end{cases}$$

and such that (1) has unique solutions. The policy is distributed since, for each agent i , it depends only on the local information

available to it, which is x_i and $x^{(i)}$. No other information on the current or past system state is available to agent i . (We discuss the limitation of this assumption at the beginning of Section 3.) The policy is stationary since it does not depend explicitly on time t . In other words, the policy is a time-invariant and memoryless function of the state. Define the system state vector $x(t) = \{x_i(t), i \in \Gamma\}$, then the system initial state $x(0)$ is the collection of the agents' initial states. Define $u(x) = \{u_i(x_i, x^{(i)}) : i \in \Gamma\}$ as a *distributed stationary protocol* or simply a *protocol*. Let $\hat{\chi} : \mathbb{R}^n \rightarrow \mathbb{R}$ be a generic continuous and differentiable function of n variables x_1, \dots, x_n which is permutation invariant, i.e., $\hat{\chi}(x_1, x_2, \dots, x_n) = \hat{\chi}(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$ for any one to one (permutation) mapping $\sigma(\cdot)$ from the set Γ to the set Γ . Henceforth $\hat{\chi}$ is also called *agreement function*. Putting together slightly different definitions in [24,28,32], we say that a protocol $u(\cdot)$ makes the agents asymptotically reach consensus on a *group decision value* $\hat{\chi}(x(0))$ function of their initial states if $\|x_i - \hat{\chi}(x(0))\| \rightarrow 0$ as $t \rightarrow \infty$. When this happens we also say that the system converges to $\hat{\chi}(x(0))\mathbf{1}$. Here and in the following, $\mathbf{1}$ stands for the vector $(1, 1, \dots, 1)^T$.

Notwithstanding each agent i has only a local information $(x_i, x^{(i)})$ about the system state x , we are interested in making the agents reach consensus on group decision values that are functions of the whole system initial state $x(0)$. In particular, we are interested in agreement functions verifying

$$\min_{i \in \Gamma} \{y_i\} \leq \hat{\chi}(y) \leq \max_{i \in \Gamma} \{y_i\} \quad \text{for all } y \in \mathbb{R}^n. \quad (2)$$

The above condition means that the group decision value must be confined between the minimum and the maximum values of the agents' initial states.

Finally, we define an *individual objective* for an agent i , i.e.,

$$J_i(x_i, x^{(i)}, u_i) = \lim_{T \rightarrow \infty} \int_0^T (F(x_i, x^{(i)}) + \rho u_i^2) dt, \quad (3)$$

where $\rho > 0$ and $F : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a non-negative *penalty function* that measures the deviation of x_i from neighbors' states. We say that a protocol is *optimal* if each u_i optimizes the corresponding individual objective.

In the above context, we face the following problem.

Problem 1 (Consensus problem). Consider a network $G = (\Gamma, E)$ of dynamic agents with first-order dynamics. For any agreement function $\hat{\chi}$ satisfying condition (2), determine a (distributed stationary) protocol, whose components have the feedback form (1), that makes the agents asymptotically reach consensus on $\hat{\chi}(x(0))$ for any initial state $x(0)$.

In the following, a protocol that solves the consensus problem is also referred to as a *consensus protocol*. In addition a consensus protocol is said *optimal*, if its components are the optimal controls $u_i(\cdot)$ corresponding to minimizing (3).

Problem 2 (Mechanism design problem). Consider a network $G = (\Gamma, E)$ of dynamic agents with first-order dynamics. For any agreement function $\hat{\chi}(\cdot)$ determine a penalty function $F(\cdot)$

such that there exists an optimal consensus protocol $u(\cdot)$ with respect to $\hat{\chi}(x(0))$ for any initial state $x(0)$.

Notice that a pair $(F(\cdot), u(\cdot))$ solving Problem 2 must be such that all individual objectives (3) converge to a finite value. Then, it is necessary that the integrand in (3) be null if computed in $\hat{\chi}\mathbf{1}$. We will check later on that this necessary condition is verified by our candidate penalty function $F(\cdot)$.

3. Time invariance of $\hat{\chi}(x(t))$

In this section and in the following one we focus on Problem 1. Initially, we show that if a protocol, which solves a consensus problem, is distributed and stationary then the system state trajectory enjoys the property that $\hat{\chi}(x(t))$ is time-invariant. Then, we find a family of non-trivial protocols that guarantee such a property. We prove that some of such protocols are consensus protocols with respect to $\hat{\chi}(x(0))$ in the next section.

Lemma 1 (Time invariance). Consider a network $G = (\Gamma, E)$ of dynamic agents with first-order dynamics. For an agreement function $\hat{\chi}$ assume there exists a distributed stationary protocol $u(\cdot)$, whose components have the feedback form (1), that makes the agents asymptotically reach consensus on $\hat{\chi}(x(0))$ for any initial state $x(0)$. Then the value of $\hat{\chi}(x(t))$ is time-invariant, i.e., $\hat{\chi}(x(t)) = \hat{\chi}(x(0))$ for all $t > 0$.

Proof. The key idea is that the protocols used are time-invariant (so the system of differential equations (1) is autonomous) and such that the system has unique solutions. Since the system is autonomous and by assumption $x(t) \rightarrow \hat{\chi}(x(0))\mathbf{1}$ as $t \rightarrow \infty$, then $y_s(t) = x(t+s)$ is also a solution (with $y_s(0) = x(s)$) and we also have $y_s(t) \rightarrow \hat{\chi}(y_s(0))\mathbf{1}$ as $t \rightarrow \infty$ for any s . Since $y_s(t)$ and $x(t)$ converge to the same limit we get $\hat{\chi}(y_s(0)) = \hat{\chi}(x(s)) = \hat{\chi}(x(0))$ for all s . \square

From continuity of function $\hat{\chi}$ and supposing that the state trajectory reaches the point $\hat{\chi}(x(0))\mathbf{1}$, the time invariance property stated in Lemma 1 implies also that $\hat{\chi}(x(0)) = \hat{\chi}(\hat{\chi}(x(0))\mathbf{1})$. Note that the last condition satisfies (2). Actually, (2) imposes that function $\hat{\chi}(\cdot)$ must be chosen such that any point $\lambda\mathbf{1}$, for all $\lambda \in \mathbb{R}$, is a *fixed point*, i.e., $\hat{\chi}(\lambda\mathbf{1}) = \lambda$, as it can be trivially derived assuming $y = \lambda\mathbf{1}$.

With this consideration in mind, let us impose the time invariance of $\hat{\chi}(x(t))$ to derive a necessary property for the consensus protocol. As it holds $\hat{\chi}(x(t)) = \text{const}$ then

$$\frac{d\hat{\chi}(x(t))}{dt} = \nabla_x \hat{\chi}(x) \cdot \dot{x} = \sum_{i \in \Gamma} \frac{\partial \hat{\chi}(x)}{\partial x_i} \dot{x}_i = \sum_{i \in \Gamma} \frac{\partial \hat{\chi}(x)}{\partial x_i} u_i = 0. \quad (4)$$

The trivial protocol constantly equal to 0 leaves any value $\hat{\chi}(x(t))$ time-invariant, for any possible $\hat{\chi}(\cdot)$, but obviously does not make the system converge. Consequently, it is no longer considered hereafter.

Some other solutions of Eq. (4) can be obtained easily when $\hat{\chi}(x)$ presents a particular structure. A first possibility is when

Table 1
Means under consideration and their representations in terms of f and g

Mean	$\hat{\chi}(x)$	$f(y)$	$g(z)$
Arithmetic	$\sum_{i \in \Gamma} \frac{1}{n} x_i$	$\frac{1}{n} y$	z
Geometric	$\sqrt[n]{\prod_{i \in \Gamma} x_i}$	$e^{1/ny}$	$\log z$
Harmonic	$\frac{1}{\sum_{i \in \Gamma} \frac{n}{x_i}}$	$\frac{n}{y}$	$\frac{1}{z}$
Mean of order p	$\sqrt[p]{\sum_{i \in \Gamma} \frac{1}{n} x_i^p}$	$\sqrt[p]{\frac{1}{n} y^p}$	z^p

the following condition holds

$$\frac{\partial \hat{\chi}(x)}{\partial x_i} u_i = 0 \quad \forall i \in \Gamma. \quad (5)$$

For example, $\hat{\chi}(x) = \min\{x_i\}$ and $u_i = h(x_i, \min_{j \in N_i}\{x_j\})$ satisfy the above condition, for any $h(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $h(x, y) = 0$ when $x = y$. Actually, $\partial \hat{\chi}(x)/\partial x_i \neq 0$ only for i such that $x_i = \min_{j \in \Gamma}\{x_j\}$, then, by definition of function $h(\cdot)$, it holds $u_i(x_i) = 0$ and hence (5). Note that though $\hat{\chi}(x) = \min\{x_i\}$ is not differentiable, at points where the partial derivative $\partial \hat{\chi}(x)/\partial x_i$ is not defined the protocol $u_i = 0$. The system converges to $\hat{\chi}(x(0))$ if we impose the additional condition $h(x, y) < 0$ when $x > y$. Trivially, analogous argument applies to $\hat{\chi}(x) = \max\{x_i\}$.

We specialize our study considering the following family of agreement function $\hat{\chi}(x)$.

Assumption 1 (Structure of $\hat{\chi}(\cdot)$). Assume that the generic agreement function $\hat{\chi}(\cdot)$ satisfies condition (2) and is such that $\hat{\chi}(x) = f(\sum_{i \in \Gamma} g(x_i))$, for some $f, g : \mathbb{R} \rightarrow \mathbb{R}$ with $dg(x_i)/dx_i \neq 0$ for all x_i .

A point of interest is that the above family of agreement functions is more general than the arithmetic/min/max means already reported in the literature (see, e.g., Table 1). In this sense, observe that the structure of the agreement function is general to the extent that any value in the range between the minimum and the maximum values of the agents' initial states can be chosen as a group decision value. To see this, it is sufficient to consider mean of order p with p varying between $-\infty$ and ∞ .

Theorem 1 (Protocol design rule). For any agreement function $\hat{\chi}(\cdot)$ as in Assumption 1, the non-trivial protocol

$$u_i(x_i, x^{(i)}) = \frac{1}{dg(x_i)/dx_i} \sum_{j \in N_i} \phi(x_j, x_i), \quad \text{for all } i \in \Gamma \quad (6)$$

lets the value $\hat{\chi}(x(t))$ be time-invariant if $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is an antisymmetric function, i.e., $\phi(x_j, x_i) = -\phi(x_i, x_j)$.

Proof. A sufficient condition for $\hat{\chi}(x(t))$ being time-invariant is that its argument $\sum_{i \in \Gamma} g(x_i(t))$ is time-invariant, too. The latter condition means

$$\sum_{i \in \Gamma} \frac{dg(x_i(t))}{dt} = \sum_{i \in \Gamma} \frac{dg(x_i)}{dx_i} \dot{x}_i = \sum_{i \in \Gamma} \frac{dg(x_i)}{dx_i} u_i = 0.$$

It is immediate to verify that protocol (6) satisfies condition $\sum_{i \in \Gamma} (dg(x_i)/dx_i) u_i = 0$. Actually, since ϕ is antisymmetric and the links are bidirectional then we have $\sum_{i \in \Gamma} \sum_{j \in N_i} \phi(x_j, x_i) = 0$. \square

Consider the linear function $\phi(x_j, x_i) = \alpha(x_j - x_i)$ and the different means introduced in Table 1. The arithmetic mean is time-invariant under protocol $u(x_i, x^{(i)}) = \alpha \sum_{j \in N_i} (x_j - x_i)$; the geometric mean under protocol $u(x_i, x^{(i)}) = \alpha x_i \sum_{j \in N_i} (x_j - x_i)$; the harmonic mean under protocol $u(x_i, x^{(i)}) = -\alpha x_i^2 \sum_{j \in N_i} (x_j - x_i)$; the mean of order p under protocol $u(x_i, x^{(i)}) = \alpha (x_i^{1-p}/p) \sum_{j \in N_i} (x_j - x_i)$.

Obviously, due to the time invariance of $\hat{\chi}(x(t))$ if the system converges, it will converge to $\hat{\chi}(x(0))\mathbf{1}$, but it does not necessarily converge. As it turns out at the end of the next section, for the cases in the example, the system converges to $\hat{\chi}(x(0))\mathbf{1}$ only if $\alpha > 0$ (for the harmonic mean only if $\alpha < 0$). In addition, we must also assume that $x_i(0) > 0$ for all $i \in \Gamma$, when we deal with means different from the arithmetic one.

4. Sufficient conditions for convergence

In the previous section, we find a family of protocols as in (6) that guarantees the time invariance of $\hat{\chi}(x(t))$. In this section, we determine sufficient conditions on the structure of functions $g(\cdot)$ and $\phi(\cdot)$ such that a protocol of type (6) makes the system converge to $\hat{\chi}(x(0))\mathbf{1}$ for any agreement function $\hat{\chi}(\cdot)$ and initial state $x(0)$. In particular, we prove that the system converges when the function $g(\cdot)$ is strictly increasing and the function $\phi(\cdot)$ is defined as follows:

$$\phi(x_j, x_i) = \alpha \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)), \quad (7)$$

where $\alpha > 0$, function $\hat{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, locally Lipschitz, odd and strictly increasing, and function $\vartheta : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable with $d\vartheta(x_i)/dx_i$ locally Lipschitz and strictly positive.

Putting together (6) and (7) the resulting protocol is

$$u_i(x_i, x^{(i)}) = \alpha \frac{1}{dg/dx_i} \sum_{j \in N_i} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \quad \text{for all } i \in \Gamma. \quad (8)$$

Initially, we study the stability of the system under protocol (8).

Lemma 2. Consider a network $G = (\Gamma, E)$ of dynamic agents with first-order dynamics and implement a distributed and stationary protocol $u(\cdot)$ whose components have the feedback form (8). Then, all equilibria are of the form $\lambda\mathbf{1}$ and if a trajectory $x(t)$ converges asymptotically to the equilibrium $\lambda_0\mathbf{1}$ then $\lambda_0 = \hat{\chi}(x(0))$, for any initial state $x(0)$.

Proof. First, we show that any equilibrium point x^* must have all its component equal, i.e., $x^* = \lambda\mathbf{1}$ where λ is a constant value. Then we prove that λ is equal to $\hat{\chi}(x(0))$.

Sufficiency: Assume, $x_i = \lambda$, for all $i \in \Gamma$, then $u_i \hat{\phi}(\vartheta(\lambda) - \vartheta(\lambda)) = 0$ since $\hat{\phi}(\cdot)$ is continuous and odd. Thus, the point $x^* = \lambda\mathbf{1}$ is an equilibrium point.

Necessity: Assume that there exists an equilibrium point $x^* \neq \lambda \mathbf{1}$. We prove that such an assumption implies the existence of at least one agent i with $u_i < 0$, and this last result contradicts the definition of equilibrium for x^* . Define $I = \{i \in \Gamma : x_i^* \geq x_j^*, \forall j \in \Gamma\}$ as the set of agents with maximum state value. Trivially, I is included but not equal to Γ , as $x^* \neq \lambda \mathbf{1}$. Then, i and j with $(i, j) \in E$ such that $x_i^* \neq x_j^*$ exist, since the network G is connected. In particular, we can always choose $i \in I$ such that there exists $j \in N_i$ with $x_j^* < x_i^*$. Now observe that, since $\hat{\phi}(\cdot)$ is an odd and strictly increasing and $\vartheta(\cdot)$ is strictly increasing, then $\sum_{j \in N_i} \hat{\phi}(\vartheta(x_j^*) - \vartheta(x_i^*)) < 0$. Actually, all the terms of the sum are non-positive and at least one is strictly negative. Since it also holds that $\alpha \mathbf{1}/dg/dx_i \neq 0$, the contradiction is proved.

Convergence: The above arguments show that $\hat{\chi}(x(0))\mathbf{1}$ is an equilibrium point. Now, we prove by contradiction that $\lambda = \hat{\chi}(x(0))$. Indeed, if $\lambda \neq \hat{\chi}(x(0))$, then $\hat{\chi}(x)$ under protocol (8) is no longer time-invariant. Let us assume that the system actually converges to a different equilibrium point $x^* = \lambda \mathbf{1} \neq \hat{\chi}(x(0))\mathbf{1}$. As $\hat{\chi}(\cdot)$ enjoys the fixed point property, we have $\hat{\chi}(x^*) = \lambda \neq \hat{\chi}(x(0))$. \square

We are now ready to prove that the agents asymptotically reach consensus on $\hat{\chi}(x(0))\mathbf{1}$ when function $g(\cdot)$ is strictly increasing, i.e., $dg(y)/dy > 0$ for all $y \in \mathbb{R}$.

Theorem 2. Consider a network $G=(\Gamma, E)$ of dynamic agents with first-order dynamics and implement a distributed and stationary protocol whose components have the feedback form (8). If function $g(\cdot)$ is strictly increasing, the agents asymptotically reach consensus on $\hat{\chi}(x(0))\mathbf{1}$ for any initial state $x(0)$.

Proof. We follow a line of reasoning similar to the one in [24]. First, observe that consensus reaching corresponds to asymptotic stability of a new variable $\eta = \{\eta_i, i \in \Gamma\}$, where $\eta_i = g(x_i) - g(\hat{\chi}(x(0)))$. The vector η is a bijective function of the system state, since η_i is as strictly increasing as $g(\cdot)$, and $\eta = 0$ corresponds to $x = \hat{\chi}(x(0))\mathbf{1}$. We prove the asymptotic stability (in the quotient space $\mathbb{R}^n/\text{span}\{\mathbf{1}\}$) of the equilibrium point $\eta = 0$ by introducing a candidate Lyapunov function $V(\eta) = \frac{1}{2} \sum_{i \in \Gamma} \eta_i^2$. Trivially, $V(\eta) = 0$ if and only if $\eta = 0$; $V(\eta) > 0$ for all $\eta \neq 0$. It remains to prove that $\dot{V}(\eta) < 0$ for all $\eta \neq 0$.

$$\dot{V}(\eta) = \sum_{i \in \Gamma} \eta_i \dot{\eta}_i = \sum_{i \in \Gamma} \eta_i \frac{dg(x_i)}{dx_i} \dot{x}_i \quad (9a)$$

$$= \sum_{i \in \Gamma} \eta_i \frac{dg(x_i)}{dx_i} u_i = \sum_{i \in \Gamma} \eta_i \frac{dg(x_i)}{dx_i} \times \alpha \frac{1}{dg/dx_i} \sum_{j \in N_i} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \quad (9b)$$

$$= \alpha \sum_{i \in \Gamma} \eta_i \sum_{j \in N_i} \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) \quad (9c)$$

$$= -\alpha \sum_{(i,j) \in E} (g(x_j) - g(x_i)) \hat{\phi}(\vartheta(x_j) - \vartheta(x_i)). \quad (9d)$$

In (9) we simply express η_i and $\dot{\eta}_i$ in terms of the state variables and their derivatives.¹ To get (9d) from (9c), we reorder the terms and exploit the fact that $j \in N_i$ if and only if $i \in N_j$ for each $i, j \in \Gamma$. From (9d) we have $\dot{V}(\eta) \leq 0$ for all η and, in particular, $\dot{V}(\eta) = 0$ only for $\eta = 0$. Actually, as $\alpha > 0$ and $g(\cdot)$, $\hat{\phi}(\cdot)$, and $\vartheta(\cdot)$ are strictly increasing, we have that, for any $(i, j) \in E$, $x_j > x_i$ implies $g(x_j) - g(x_i) > 0$, $\vartheta(x_j) - \vartheta(x_i) > 0$ and $\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) > 0$. Hence, we obtain $\alpha(g(x_j) - g(x_i))\hat{\phi}(\vartheta(x_j) - \vartheta(x_i)) > 0$ if $x_j > x_i$. Trivially, a symmetrical argument holds if $x_j < x_i$. \square

It is possible to partially relax the assumptions of Theorem 2 concerning the monotonicity of function $g(\cdot)$. The reason is evident from the following theorem establishing that all agents' state trajectories are bounded.

Theorem 3. Assume all the conditions in Theorem 2 hold. Then, condition (7) implies that for all $i \in \Gamma$ and $t \geq 0$,

$$\min_{j \in \Gamma} \{x_j(0)\} \leq x_i(t) \leq \max_{j \in \Gamma} \{x_j(0)\}. \quad (10)$$

Proof. Let $\alpha = \min_{j \in \Gamma} \{x_j(0)\}$ and $\beta = \max_{j \in \Gamma} \{x_j(0)\}$. The aim is to prove that all solutions stay inside the hypercube $[\alpha, \beta]^n$. This can be shown by noticing that, for each generic agent i , the state $x_i(t)$ is a continuous. Observe, that on the faces of the polyhedron $[\alpha, \beta]^n$ the corresponding vector field does not point outwards. This translates into $u_i(\alpha, x^{(i)}) \geq 0$ (and $u_i(\beta, x^{(i)}) \leq 0$) which follows immediately from the definition of u_i . \square

Trivially, condition (10) holds even if $g(y)$ is strictly increasing only in the subset of \mathbb{R} defined by $\min_{j \in \Gamma} \{x_j(0)\} \leq y \leq \max_{j \in \Gamma} \{x_j(0)\}$, since the agents' state trajectory values are bounded within the same set. The boundedness of the agents' state trajectories allows us to partially relax the assumptions of Theorem 2 concerning the monotonicity of function $g(\cdot)$. Indeed, Theorem 2 still holds if $g(\cdot)$ is strictly increasing in only a subset $X \in \mathbb{R}$ provided that $x_i(t) \in X$, for all $t \geq 0$ and for all $i \in \Gamma$. Theorem 3 proves that the latter condition is certainly satisfied if X is a connected subset and $x_i(0) \in X$ for all $i \in \Gamma$. As a further trivial generalization of Theorem 2 we observe that it holds even if $g(\cdot)$ is strictly decreasing. However, in this case, α in (7) must be strictly negative instead of positive.

An immediate consequence of the above considerations is the following. Since the means introduced in Table 1 have the component $g(\cdot)$ strictly increasing except the harmonic mean, if we consider the linear function $\phi(x_j, x_i) = \alpha(x_j - x_i)$, the system converges to $\hat{\chi}(x(0))\mathbf{1}$ for $\alpha > 0$ except for the harmonic mean where we need $\alpha < 0$. When we deal with means different from the arithmetic one we also need that $x_i(0) > 0$ for all $i \in \Gamma$, since $g(y)$ may not even be defined for $y \leq 0$.

¹ Let $K = g(\hat{\chi}(x(0)))$, the dynamics of η can be equivalently expressed as

$$\dot{\eta}_i = \alpha \sum_{j \in N_i} \hat{\phi}(\vartheta(g^{-1}(\eta_j + K)) - \vartheta(g^{-1}(\eta_i + K))).$$

The results to communication networks whose topology switches within set of directed graphs that are connected and balanced can be easily extended via a common Lyapunov function approach, see [5].

5. Penalty functions and optimal protocols

In this section, we adopt a mechanism design perspective. We discuss whether we can make the agents asymptotically reach consensus on the group decision value by assigning to each agent an individual objective function to optimize (Problem 2).

Unfortunately, solving Problem 2 is a difficult task. Our idea is then to translate it into a sequence of more tractable problems (Problem 3). At each discrete time t_k , the receding horizon control scheme of Problem 3 optimizes over an infinite *planning* horizon $T \rightarrow \infty$, and executes the controls over a one-step *action* horizon $\delta = t_{k+1} - t_k$. The neighbors' state whose evolution is unpredictable, are kept constant over the planning horizon (Assumption 2) [13,18]. At time t_{k+1} each agent re-optimizes its controls based on the new information on neighbors' state which has become available. To find an approximate solution to Problem 2 we then take the limit for $\delta \rightarrow 0$ of the given receding horizon solution.

The receding horizon update times are $t_k = t_0 + \delta k$, where $k = 0, 1, \dots$. The cost of agent i depends on his state and others' state trajectories as well. Let us denote with $\hat{x}_i(\tau, t_k)$ and $\hat{x}^{(i)}(\tau, t_k)$, $\tau \geq t_k$, respectively, the predicted state of agent i and of his neighbors.

Problem 3 (Receding horizon). For all agents $i \in \Gamma$ and times t_k , $k = 0, 1, \dots$, given the initial state $x_i(t_k)$, and $x^{(i)}(t_k)$ find

$$\hat{u}_i^*(\tau, t_k) = \arg \min \mathcal{J}_i(x_i(t_k), x^{(i)}(t_k), \hat{u}_i(\tau, t_k)),$$

where

$$\begin{aligned} \mathcal{J}_i(x_i(t_k), x^{(i)}(t_k), \hat{u}_i(\tau, t_k)) \\ = \lim_{T \rightarrow \infty} \int_{t_k}^T (\mathcal{F}(\hat{x}_i(\tau, t_k), \hat{x}^{(i)}(\tau, t_k)) + \rho \hat{u}_i^2(\tau, t_k)) d\tau \end{aligned} \quad (11)$$

subject to

$$\dot{\hat{x}}_i(\tau, t_k) = \hat{u}_i(\tau, t_k), \quad (12a)$$

$$\dot{\hat{x}}_j(\tau, t_k) = \hat{u}_j(\tau, t_k) := 0 \quad \forall j \in N_i, \quad (12b)$$

$$\hat{x}_i(t_k, t_k) = x_i(t_k), \quad (12c)$$

$$\hat{x}_j(t_k, t_k) = x_j(t_k) \quad \forall j \in N_i. \quad (12d)$$

In the above problem, Eqs. (12a) and (12b) predict, respectively, the evolution of the state of agent i and of his neighbors, and conditions (12c) and (12d) represent the initial state at time t_k . Note that whereas agent i may predict with a certain approximation the evolution of its state as described by (12a), nothing can he know about the evolution of the states of his neighbors (12b). In this context, agent i at time t_k , $k = 0, 1, \dots$ assumes

that the states of his neighbors are constant over the planning horizon, that is $\hat{x}^{(i)}(\tau, t_k) = x^{(i)}(t_k)$, $\forall \tau > t_k$ (*naive assumption*).

According to the standard receding horizon scheme, the agents update the receding horizon control policy when a new initial state update $x^{(i)}(t_{k+1})$ is available. As a result, for all $i \in \Gamma$, we have the following *closed-loop* system:

$$\dot{\hat{x}}_i = u_{iRH}(\tau), \quad \tau \geq t_0,$$

where the applied receding horizon control law $u_{iRH}(\tau)$ satisfies

$$u_{iRH}(\tau) = \hat{u}_i^*(\tau, t_k), \quad \tau \in [t_k, t_{k+1}).$$

Once we assume (12b) we have $\hat{x}^{(i)}$ constant in (11) and the receding horizon problem (Problem 3) reduces to one dimension (in fact, n one-dimensional problems). This is evident, if we explicit dependence of $\mathcal{F}(\cdot)$ only on the state $\hat{x}_i(\tau, t_k)$, and simplify the expression of the individual objective function (11) as

$$J_i = \lim_{T \rightarrow \infty} \int_{t_k}^T (\mathcal{F}(\hat{x}_i(\tau, t_k)) + \rho \hat{u}_i^2(\tau, t_k)) d\tau. \quad (13)$$

Hence, the problem reduces to determine the control $\hat{u}_i(\tau, t_k)$ that optimizes (13).

Now, to verify that a given control $\hat{u}_i(\tau, t_k)$ is optimal, we use the Pontryagin minimum principle. To do this, first, we must construct the Hamiltonian function (for sake of simplicity dependence on τ and t_k is dropped):

$$H(\hat{x}_i, \hat{u}_i, p_i) = (\mathcal{F}(\hat{x}_i) + \rho \hat{u}_i^2) + p_i \hat{u}_i. \quad (14)$$

Second, we must impose the Pontryagin necessary conditions:

$$\text{Optimality condition: } \frac{\partial H(\hat{x}_i, \hat{u}_i, p_i)}{\partial \hat{u}_i} = 0 \Rightarrow p_i = -2\rho \hat{u}_i. \quad (15)$$

$$\text{Multiplier condition: } \dot{p}_i = -\frac{\partial H(\hat{x}_i, \hat{u}_i, p_i)}{\partial \hat{x}_i}. \quad (16)$$

$$\text{State equation: } \dot{\hat{x}}_i = \frac{\partial H(\hat{x}_i, \hat{u}_i, p_i)}{\partial p_i} \Rightarrow \dot{\hat{x}}_i = \hat{u}_i. \quad (17)$$

$$\begin{aligned} \text{Minimality condition: } \frac{\partial^2 H(\hat{x}_i, \hat{u}_i, p_i)}{\partial \hat{u}_i^2} \Bigg|_{\hat{x}_i = \hat{x}_i^*, \hat{u}_i = \hat{u}_i^*, p_i = p_i^*} \\ \geq 0 \Rightarrow \rho \geq 0. \end{aligned} \quad (18)$$

$$\text{Boundary condition: } H(\hat{x}_i^*, \hat{u}_i^*, p_i^*) = 0. \quad (19)$$

This last condition requires that the Hamiltonian must be null along any optimal path $\{\hat{x}_i^*(t), \forall t \geq 0\}$ (see, e.g., [7], Section 3.4.3).

We recall that the Pontryagin minimum principle provides *necessary* but not sufficient optimality conditions [7]. The above conditions become also sufficient to identify a unique optimal solution if also the following condition holds [7].

$$\text{Uniqueness condition: } \mathcal{F}(x_i) \text{ is convex.} \quad (20)$$

The following theorem states sufficient conditions on the structure of $\mathcal{F}(x_i)$ that allow us to determine analytically a unique optimal control policy $\hat{u}_i(\cdot)$.

Theorem 4. Consider an agent i with first-order dynamics, at times $t_k = 0, 1, \dots$, assign it an objective function as (11) whose penalty function is

$$\mathcal{F}(\hat{x}_i(\tau, t_k)) = \rho \left(\frac{1}{dg/dx_i} \sum_{j \in N_i} (\vartheta(x_j(t_k)) - \vartheta(\hat{x}_i(\tau, t_k))) \right)^2, \quad (21)$$

where $g(\cdot)$ is increasing, $\vartheta(\cdot)$ is concave, and $1/dg(y)/dy$ is convex. Then the following control policy is the unique optimal solution to Problem 3,

$$\begin{aligned} \hat{u}_i^*(\tau, t_k) &= u_i(x_i(\tau)) \\ &= \alpha \frac{1}{dg/dx_i(\tau)} \sum_{j \in N_i} (\vartheta(x_j(t_k)) - \vartheta(x_i(\tau))), \quad \alpha = 1. \end{aligned} \quad (22)$$

Proof. We first show that the problem of minimizing (13) is well posed by proving that, given a penalty function as in (21), there exists at least a control policy for which objective (13) converges. Second we certificate the optimality of control policy (22) via Pontryagin minimum principle and, finally, we prove the uniqueness of the optimal solution by showing that $\mathcal{F}(\cdot)$ is convex.

To prove that the problem is well posed, let us start by showing that there exists at least one reachable state x_i^* under a stationary control policy function only of local information $(x_i, x^{(i)})$ in which both the penalty (21) and the control itself are null. Now, the penalty (21) is null in a state \hat{x}_i^* such that $\sum_{j \in N_i} (\vartheta(x_j(t_k)) - \vartheta(\hat{x}_i^*)) = 0$. From the latter we also have $x_i^* = \vartheta^{-1}(\sum_{j \in N_i} \vartheta(x_j(t_k))/|N_i|)$ which means that x_i^* can be determined on the basis of the local information $(\hat{x}_i(\tau, t_k), x^{(i)}(t_k))$ available to agent i . Therefore, there trivially exists a control policy that is null in x_i^* and makes the objective function (13) converge.

Now, to certificate the optimality of control policy (22) with $\alpha = 1$, let us show that it satisfies conditions (15)–(20) imposed by the Pontryagin minimum principle. Our hypothesis on the agent dynamics and on the structure of the agent objective function trivially satisfy (17) and (18). By computing \dot{p}_i from (15) and substituting the obtained value in (16) we have

$$2\rho \dot{\hat{u}}_i = \frac{\partial H(\hat{x}_i, \hat{u}_i, p_i)}{\partial \hat{x}_i}. \quad (23)$$

In (17), we can write $\dot{\hat{u}}_i = (\partial \hat{u}_i / \partial \hat{x}_i) \dot{\hat{x}}_i = (\partial \hat{u}_i / \partial \hat{x}_i) \hat{u}_i$. Hence condition (23) becomes $2\rho (\partial \hat{u}_i / \partial \hat{x}_i) \hat{u}_i = \partial H(\hat{x}_i, \hat{u}_i, p_i) / \partial \hat{x}_i$. Integrating and imposing condition (19) we obtain that a possible solution of (23) must satisfy

$$\rho \hat{u}_i^2 = \mathcal{F}(\hat{x}_i). \quad (24)$$

It is immediate to verify that $\hat{u}_i(\tau, t_k) = 1/dg/d\hat{x}_i \sum_{j \in N_i} (\vartheta(x_j(t_k)) - \vartheta(\hat{x}_i(\tau, t_k)))$ satisfies the above condition.

Finally, to prove that control policy (22) is the unique optimal solution, let us show that $\mathcal{F}(\hat{x}_i)$ is convex. To do this, let us write $\mathcal{F} = \mathcal{F}_3(\mathcal{F}_1(\hat{x}_i), \mathcal{F}_2(\hat{x}_i))$ where function $\mathcal{F}_1(\hat{x}_i) = (dg/dx_i)^{-1}$, function $\mathcal{F}_2(\hat{x}_i) = \sum_{j \in N_i} (\vartheta(x_j(t_k)) - \vartheta(\hat{x}_i))$ and $\mathcal{F}_3 = (\mathcal{F}_1(\hat{x}_i) \cdot \mathcal{F}_2(\hat{x}_i))^2$. With $\mathcal{F}_3(\cdot)$ being non-decreasing in each argument, function $\mathcal{F}_3(\cdot)$ is convex if both functions $\mathcal{F}_1(\cdot)$ and $\mathcal{F}_2(\cdot)$ are also convex [8]. Function $\mathcal{F}_1(\cdot)$ is convex as $(dg/d\hat{x}_i)^{-1}$ is convex by hypothesis. Analogously, $\mathcal{F}_2(\cdot)$ is convex as $\vartheta(\cdot)$ is concave. \square

The above theorem holds trivially for $\alpha = -1$ if $dg/dx_i < 0$ for all $x_i(0)$.

Note that the optimal control policy (22) is a feedback policy with respect to the only state \hat{x}_i , whereas is an open-loop one with respect to the neighbors' states.

Then, an immediate consequence of Theorem 4 and of the above assumption is the following corollary.

Corollary 1. Consider a network $G = (\Gamma, E)$ of dynamic agents with first-order dynamics, at times $t_k = 0, 1, \dots$, assign it an objective function as (11) whose penalty function is

$$\mathcal{F}(\hat{x}_i(\tau, t_k)) = \rho \left(\frac{1}{dg/dx_i} \sum_{j \in N_i} (\vartheta(x_j(t_k)) - \vartheta(\hat{x}_i(\tau, t_k))) \right)^2, \quad (25)$$

where $g(\cdot)$ is increasing, $\vartheta(\cdot)$ is concave, and $1/(dg(y)/dy)$ is convex. If the update time interval $\delta \rightarrow 0$ then the following conditions hold:

(i) the penalty function:

$$\begin{aligned} \mathcal{F}(x_i(\tau, t_k)) &\rightarrow F(x_i, x^{(i)}) \\ &= \rho \left(\frac{1}{dg/dx_i} \sum_{j \in N_i} (\vartheta(x_j) - \vartheta(x_i)) \right)^2, \end{aligned} \quad (26)$$

(ii) the applied receding horizon control law:

$$u_{iRH}^*(\tau) \rightarrow u_i(x_i, x^{(i)}) = \frac{1}{dg/dx_i} \sum_{j \in N_i} (\vartheta(x_j) - \vartheta(x_i)). \quad (27)$$

From the above corollary it is straightforward to derive a solution to the mechanism design problem (Problem 2). Indeed, a supervisor can make the agents asymptotically reach consensus on the group decision value $\hat{\gamma}(x) = f(\sum_{i \in \Gamma} g(x_i))$ by assigning them an individual objective function (3) with penalty function (26) to optimize, provided that $g(\cdot)$ is increasing, $1/dg(y)/dy$ is convex and the update time interval δ is "sufficiently" small in comparison with the speed of variation of state $x(\cdot)$.

It is worth to observe that, in general, control (27) cannot be proved to be optimal for (3) if the naive expectation assumption is dropped. However, to the best of authors' knowledge, even in presence of more information about the evolution of $x^{(i)}(t)$, it would be difficult to determine the optimal control for the generic agent i as conditions (15)–(20) are in general difficult to

solve. A notable exception is when $dg/dx_i = 1$ and $\vartheta(x_i) = x_i$ for all $i \in \Gamma$. In this case, agents asymptotically reach consensus on the arithmetic mean of the values of their initial states and it is immediate to verify that control (27) is optimal even when the naive expectation assumption is dropped (see, e.g., *linear quadratic differential games* in [1, chapter 6]).

6. Simulation studies: vertical alignment maneuver for UAVs

We consider a team of four UAVs in longitudinal flight and initially at different heights. Each UAV controls the vertical rate without knowing the relative position of all UAVs but only of neighbors according to the communication network topology depicted in Fig. 1. For instance, the fourth UAV knows the position of only the first UAV, and the first UAV knows the position of the fourth and second UAV and so on. In the above partial information context, we are interested in determining a suitable distributed vertical rate control strategy that allows the UAVs to align their paths according to the formation center at time 0 that we assume expressed as the generic agreement function of the initial UAVs heights. In particular, in the four simulated vertical alignment maneuvers, the position of the formation center is defined, respectively, as the (i) arithmetic mean, (ii) geometric mean, (iii) harmonic mean, (iv) mean of order 2 of the initial positions of all UAVs. The initial height is $x(0) = (5, 5, 10, 20)^T$. We stress once again that the challenging aspect is that the UAVs know the heights of only their neighbors and are required to align their paths according to the path of

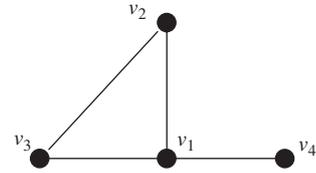


Fig. 1. The information flow in a network of four agents.

the formation center, which in turns depend on the unknown position of all UAVs.

In case (i), (see e.g., [24,2,23]) the UAVs are given an individual objective (3) with penalty $F(x_i, x^{(i)}) = (\sum_{j \in N_i} (x_j - x_i))^2$ and consequently implement the optimal linear protocol

$$u(x_i, x^{(i)}) = \sum_{j \in N_i} (x_j - x_i) \tag{28}$$

to asymptotically align on the arithmetic mean of $x(0)$. Notice that, the approximate objective function (13) converges even if the neighbors are not aligned, since both the above integrand penalty $F(x_i, \bar{x}^{(i)})$ and control (28) are null when x_i is equal to the arithmetic mean computed over the only neighbors' states. Fig. 2(a) shows the simulation of the longitudinal flight dynamics.

In case (ii) the UAVs are given an individual objective $F(x_i, x^{(i)}) = (x_i \sum_{j \in N_i} (x_j - x_i))^2$ and implement the optimal protocol

$$u(x_i, x^{(i)}) = x_i \sum_{j \in N_i} (x_j - x_i) \tag{29}$$

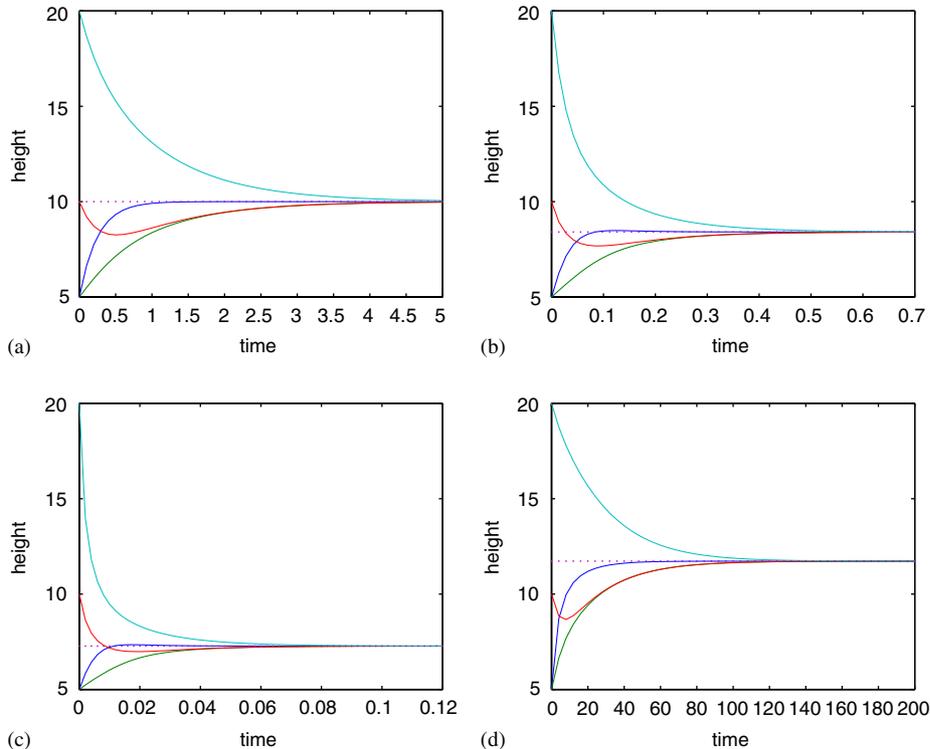


Fig. 2. Longitudinal flight dynamics converging to: (a) the arithmetic mean under protocol (28); (b) the geometric mean under protocol (29); (c) the harmonic mean under protocol (30); and (d) the mean of order 2 under protocol (31).

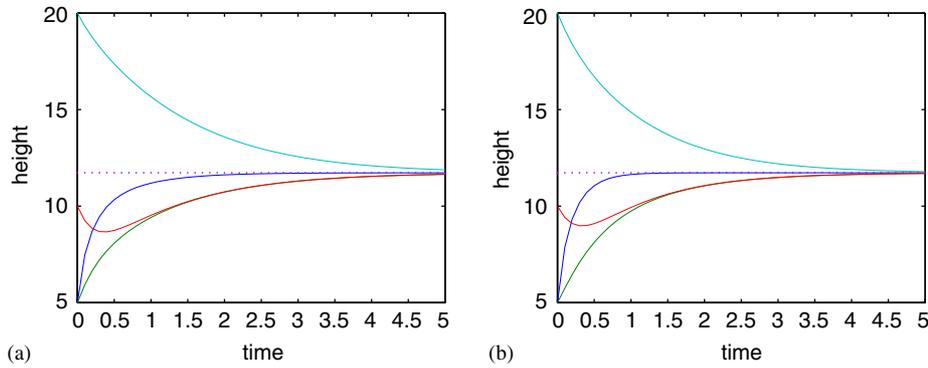


Fig. 3. Longitudinal flight dynamics converging to the mean of order 2: (a) under protocol (32); (b) under protocol (33).

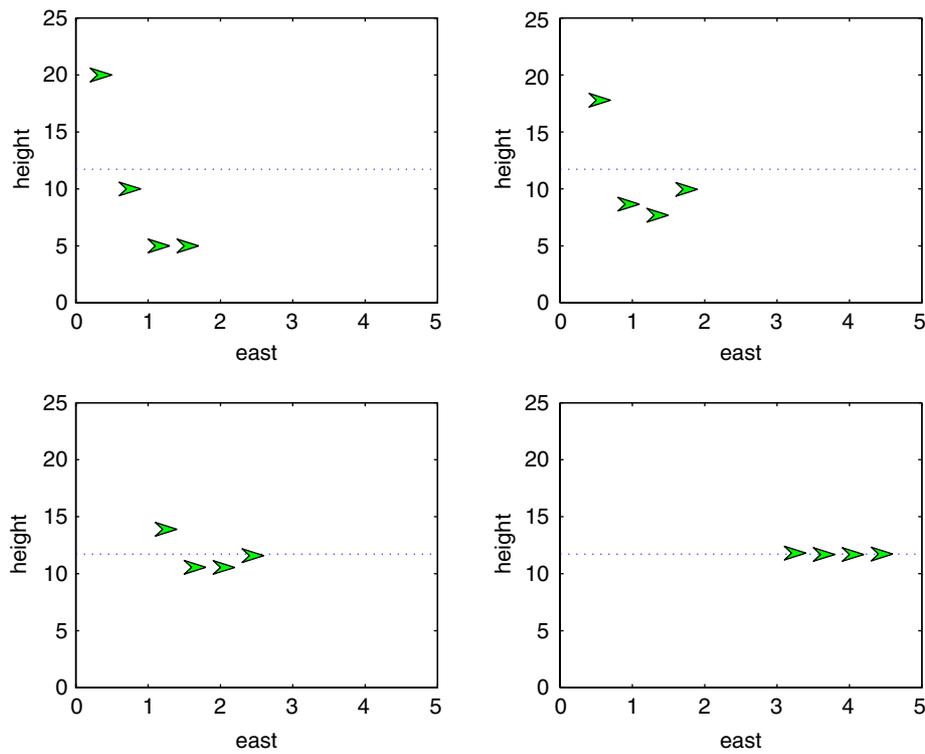


Fig. 4. Vertical alignment to the mean of order 2 on the vertical plane.

to asymptotically align on the geometric mean of $x(0)$. Fig. 2(b) shows the simulation of the longitudinal flight dynamics.

In case (iii) the UAVs are given an objective function $F(x_i, x^{(i)}) = (x_i^2 \sum_{j \in N_i} (x_j - x_i))^2$ and implement the optimal protocol

$$u(x_i, x^{(i)}) = -x_i^2 \sum_{j \in N_i} (x_j - x_i) \tag{30}$$

to asymptotically align on the harmonic mean of $x(0)$. Fig. 2(c) shows the simulation of the longitudinal flight dynamics.

Finally, in case (iv) the UAVs are given a function $F(x_i, x^{(i)}) = ((1/2x_i) \sum_{j \in N_i} (x_j - x_i))^2$ and implement the

optimal protocol

$$u(x_i, x^{(i)}) = \frac{1}{2x_i} \sum_{j \in N_i} (x_j - x_i) \tag{31}$$

to asymptotically align on the mean of order 2 of $x(0)$. Fig. 2(d) shows the simulation of the longitudinal flight dynamics.

Protocols (28)–(31) are characterized by different converging times (see Fig. 2). These differences are due to the fact that the protocols multiply the common term $\sum_{j \in N_i} (x_j - x_i)$ for different powers of x_i , respectively $1, x_i, -x_i^2$ and $\frac{1}{2}x_i^{-1}$. Being $x_i \geq 1$ for all $i \in \Gamma$ and $t \geq 0$, the lower the power, the higher the converging time. Consider the vertical alignment to the mean of power 2. To obtain a converging time comparable with the

one of the vertical alignment to the arithmetic mean, we modify the protocol so that it turns to be a ratio between polynomials whose numerator is of an order greater than the denominator as in the arithmetic mean case. As an example, in Fig. 3(a) results are reported with protocol (31) modified as

$$u(x_i, x^{(i)}) = \frac{1}{2x_i} \sum_{j \in N_i} (x_j^2 - x_i^2). \quad (32)$$

An analogous result can be obtained if we multiply protocol (31) by twice an upper bound of $\max_{i \in \Gamma} \{x_i(0)\}$. The resulting scaled protocol is

$$u(x_i, x^{(i)}) = \frac{\max_{i \in \Gamma} \{x_i(0)\}}{2x_i} \sum_{j \in N_i} (x_j - x_i) \quad (33)$$

and the corresponding longitudinal dynamics is displayed in Fig. 3(b). Unfortunately both protocols (32) and (33) have some drawbacks. In (32), the function $\vartheta(x_i) = x_i^2$ is not concave, whereas to implement protocol (33) the UAVs must have an a priori knowledge or at least a bound of $\max_{i \in \Gamma} \{x_i(0)\}$.

An example of vertical alignment maneuver under protocol (33) is displayed in Fig. 4.

7. Conclusions and discussion

In this paper we have considered a set of agents with a simple first-order dynamics and we have faced the problem of making the agents' states reach consensus on a group decision value of interest. For group decision values with a quite general structure as established in Assumption 1, we have shown that the agents can reach consensus using a distributed and stationary linear or non-linear protocol, provided that the networks defined by the communication links between agents is time-invariant and connected. Also, we have proposed a game theoretic approach to solve consensus problems. Under this perspective, consensus is the result of a mechanism design. A supervisor imposes individual objectives. Then, the agents reach asymptotically consensus as a side effect of the optimization of their own individual objectives on a local basis.

Among the limitations of Assumption 1 on the structure of function $\hat{\chi}(x) = f(\sum_{i \in \Gamma} g(x_i))$, we stress the requirements regarding the permutation invariance of functions $g(\cdot)$ and their definition on only each agent state x_i and not on the states of the neighboring agents $x^{(i)}$. Such limitations on function $g(\cdot)$ turn useful in proving the negativity of the derivative of Lyapunov function $V(\eta)$ considered in Theorem 2. In turn the non-positivity of $\dot{V}(\eta)$ allows us to show that protocol (8) makes the agents asymptotically converge to the desired group decision value.

If we consider a more general function $g_i(x, x^{(i)})$ for each agent, following a line of reasoning as in Section 3, we can prove that, e.g., a protocol with component $u_i(x, x^{(i)}) = 1/(\sum_{j \in N_i \cup \{i\}} \partial g_i / \partial x_j) \sum_{j \in N_i} (x_j - x_i)$, for all $i \in \Gamma$, leaves the value $\hat{\chi}(x)$ time-invariant. Also, we can show that the same protocol makes the agents asymptotically converge to the group decision value if $\partial g_i / \partial x_j > 0$, for all $j \in N_i \cup \{i\}$. However, in this case, we are able to prove the

analogous of Theorem 2 only making use of a Lyapunov function that takes into account of the structure of the network G . Hence, the proof of convergence is not generalizable to networks with switching topology. This last negative result should not surprise as, in case of a switching topology, functions $g_i(\cdot)$ should be redefined at each switching instant as they are defined not only on the value of x_i but also on the value of $x^{(i)}$ and hence they depend on the topology of the network G .

As an example, consider a system with $x_i(0) > 0$, for all $i \in \Gamma$, for which the desired consensus value is $\hat{\chi}(x(0)) = 1/n \sum_{i \in \Gamma} \sum_{j \in N_i} a_{ij} \sqrt{x_i(0)x_j(0)}$ where $a_{ij} > 0$ and $\sum_{j \in N_i} a_{ij} = 1$, for all $i \in \Gamma$, $j \in N_i$. Then a possible consensus protocol on a fixed network is $u_i = 2 \sum_{j \in N_i} (x_j - x_i) / \sum_{j \in N_i} \sqrt{x_i/x_j + x_j/x_i}$.

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