

Infrared asymptotic dynamics of gauge invariant charged fields: QED versus QCD

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The freedom one has in constructing locally gauge invariant charged fields in gauge theories is analyzed in full detail and exploited to construct, in QED, an electron field whose two-point function $W(p)$, up to the fourth order in the coupling constant, is normalized with on-shell normalization conditions and is, nonetheless, infrared finite; as a consequence the radiative corrections vanish on the mass shell $p^2 = \mu^2$ and the free field singularity is dominant, although, in contrast with quantum field theories with mass gap, the eigenvalue μ^2 of the mass operator is not isolated. The same construction, carried out for the quark in QCD, is not sufficient for cancellation of infrared divergences to take place in the fourth order. The latter divergences, however, satisfy a simple factorization equation. We speculate on the scenario that could be drawn about infrared asymptotic dynamics of QCD, should this factorization equation be true in any order of perturbation theory.

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I. INTRODUCTION AND MAIN RESULTS

In ordinary quantum field theory (QFT) with a mass gap the notion of a particle is recovered from that of an interacting local field as a consequence of infrared (IR) asymptotic dynamics: a near-mass-shell pole singularity in each of the momenta incoming any Green function (guaranteed in Lagrangian QFT's by the possibility of imposing on-shell normalization conditions on both mass and wave function renormalizations) ensures the existence of the Lehmann-Symanzik-Zimmermann (LSZ) asymptotic limit of the field [1]. One is thus provided with an ordinary free Fock field, by means of which an irreducible Wigner-type representation of the Poincaré group, sitting on an isolated mass hyperboloid, is in turn constructed. In this context the fact that the field or particle may or may not carry quantum numbers associated with some unbroken global internal symmetry is irrelevant.

In gauge theories (we will always have in mind QED and QCD in continuum Minkowski four-dimensional space-time with unbroken electric and color charges) things go in a different way. Indeed, the issue is one about which, as yet, there is no general consensus.

On the one side QED—with the exception of its zero charge sector—still is only a theory of inclusive cross sections, in which all the theoretical setup of quantum mechanics (states, observables, representation of symmetries, and the like) has no satisfactory *explicit* representation, in spite of the general model-independent investigations [2,3] that have delimited, so to speak, a possible battlefield: the battle is not yet won and one could, in a provocative way, summarize the situation by saying that the question: “what is an electron in QED” is still open.

On the other side there is, in QCD, the problem of confinement of colored gluons and quarks, about which there is

even less to say. Many mechanisms and criteria have been proposed over the years: some (as, e.g., the Wilson loop area behavior [4], or the fundamental role of topology leading to the dual Meissner effect [5]) are so suggestive that have become common language; others [the $1/(k^2)^2$ IR behavior of the full gluon propagator [6–9], the quartet mechanism [10] and the metric confinement [11] both based on the existence of LSZ asymptotic limits for color fields, violation of asymptotic completeness [12], the obstruction in the IR dressing due to Gribov ambiguities [13], and so many others that it would be impossible—and nonsensical—to quote them all here] do not share the same popularity, but time and again are reconsidered and revived. However, so far none of these criteria has led to a systematic and generally accepted description of what confinement is.

Prudentially we regard confinement as a delicate, multifaceted subject one can look at from different standpoints. We try here just to offer a further standpoint, not necessarily in conflict with others, but endowed with the possibility of a sound mathematical verification based on the only input of implementing in QCD the symmetries that we believe relevant: local gauge invariance and Poincaré.

It is convenient to state the terms of the problem of the particle content of charged sectors in gauge theories within the framework of the Lagrangian approach. We shall also assume that all the fields entering the Lagrangian are local fields. These will be referred to as the basic fields of the model. Reference [14] gives in detail the local covariant formulation of the theory we shall rely on in the sequel. In particular the adjective “physical” will be referred to the fields that commute with—or to states that are annihilated by—the Becchi-Rouet-Stora-Tyutin generator (the choice of the local covariant formulation deserves a further comment: the fact that manifest covariance is necessary to implement the renormalization procedure may be regarded upon as a technical complication; to our knowledge, however, a proof of renormalizability is given only in this context [15]: that is why we stick to it).

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In this framework it is convenient to distinguish four steps, all relevant in designing the relationship between field and particle. We will try to keep these steps as nonoverlapping as possible:

- (i) form of physical (composite) charged fields,
- (ii) IR asymptotic dynamics,
- (iii) existence of asymptotic limits and on-shell particle states,
- (iv)₁ S matrix.

In this paper we will be mainly concerned with only (i) and (ii). It should be clear, however, that the study of these points is relevant not *per se*, but only in the perspective of either going through the entire program (as possible for QED) or spotting where the chain breaks down, as one expects to happen in the case of QCD. The following discussion should help clarifying why only (i) and (ii) need to be considered.

In the case of QFT with mass gap (i) is simply non-necessary: the LSZ program, consisting of (ii), (iii), and (iv)₁, is fully exhaustive. In particular (iii) provides free fields whose two-point function is the usual one with an isolated pole at $p^2 = \mu^2$. A Hilbert space is reconstructed from this two-point function via the well known Fock procedure and the scattering matrix S_{LSZ} , provided by step (iv)₁, is regarded upon as a unitary operator that maps the Fock space onto itself. The historical alternative to this program is provided by the use of the interaction picture: this skips point (ii), assumes the existence of on-shell states, i.e. (iii), and the outcome is Dyson's scattering matrix S_D . In comparing the two strategies one ends up with $S_{\text{LSZ}} = S_D$, but, from a conceptual point of view, there is, in our opinion, no point about the superiority of the LSZ scheme: deriving the scattering matrix from a set of off-shell T -ordered correlations functions brings the model closer to the rails provided by Wightman reconstruction theorem, and, in turn, helps to have an explicit representation of all the theoretical set up of quantum mechanics.

In the case of QED, barring the problem of deriving the scattering matrix from a set of gauge invariant correlation functions [this is precisely one of the problems (i) aims at], it has been clear from the very outset that S_D is plagued with IR divergences and efforts have been made to overcome this drawback: the Block-Nordsieck prescription for inclusive cross sections [16], culminated in the Yennie-Frautschi-Suura paper [17] is well known.

It is important to mention that Zwanziger [18] has shown the possibility of replacing the logarithms of the ‘‘vanishing photon mass’’ with the logarithms of an arbitrary finite mass (thus averting the infrared catastrophe) and that the dependence on this unphysical parameter drops out in the computation of the Block-Nordsieck cross sections. So, by this way, one is left with a technically defensible (i.e., IR finite) S matrix, although the impossibility of comparing the associated probability transitions with experimental data makes its utility rather doubtful.

There has been instead another direction pioneered by Chung' work [19]: this has been the first attempt to implement the Block-Nordsieck prescription by re-defining the asymptotic on-shell states in such a way to get rid of IR

divergences at the level of the scattering matrix: i.e., it has been the first attempt to propose an alternative to the last of the above steps:

- (iv)₂ improved S matrix.

The work of Kulish and Faddeev [20] represents, to our knowledge, the most refined version of how to effect this step. But, for it too, the assumption of a Fock space of on-shell particle states is the starting point and it must, therefore, be regarded as a refinement of the Dyson interaction picture approach. The main points of this work can be summarized as follows. First the origin of IR divergences is identified in the splitting of the total Hamiltonian in the free and interaction part, $H = H_0 + V$, showing that the time evolution factor associated to V is not well defined in perturbation theory; it is then argued that a different splitting $H = H_0 + V_{\text{as}} + (V - V_{\text{as}}) = H_{\text{as}} + (V - V_{\text{as}})$, operated on the basis of physical intuition, gives rise to a consistent asymptotic well defined time evolution operator $W_{\text{as}}(t)$ associated to H_{as} . It is then shown that the asymptotic states obtained either as $\mathcal{H}_{\text{as}} = W^\dagger(t)\mathcal{H}_{\text{Fock}}$ for $t \rightarrow -\infty$, or equivalently as $\mathcal{H}_{\text{as}} = W(t)\mathcal{H}_{\text{Fock}}$ for $t \rightarrow +\infty$, are suitable in and out states giving rise to a IR finite S matrix and that the associated transition probabilities exactly coincide with those of Block and Nordsieck. It is correctly emphasized that the price for this achievement is to realize that the operators $W^\dagger(-\infty)$ and $W(+\infty)$ must be thought as defined on a space larger than $\mathcal{H}_{\text{Fock}}$ and that the space of generalized coherent asymptotic states \mathcal{H}_{as} is not included in $\mathcal{H}_{\text{Fock}}$. In connection with this, the authors also explicitly mention that, from the point of view of representing Poincaré group, the physical subspace of \mathcal{H}_{as} does not contain discrete irreducible terms with non-vanishing mass.

So the reason why we investigate (i) in QED is to provide a set of gauge invariant and Poincaré covariant correlation functions whose IR asymptotic dynamics should provide positive evidence in favor of point (iii). There remains the fact that, even succeeding in giving such evidence, the possible outcomes are either Zwanziger's (S matrix perfectly defined, but depending on an unphysical mass parameter) or Kulish-Faddeev's (the ‘‘softness’’ of undetected photons associated to Block-Nordsieck inclusive cross sections enters the definition of charged particle states, thus compromising the representability of Lorentz boosts).

For QCD there exists no analogue to our knowledge, of the Kulish-Faddeev work. We like to think that one reason for this is the implausibility of assuming on-shell quark and gluon states, even in the sense of (iii). We will indeed shortly offer evidence that the above chain breaks down exactly in going from (ii) to (iii) and the problem of defining the scattering matrix on color states, according to either (iv)₁ or (iv)₂, becomes ill posed. This concludes the discussion about the possibility of restricting our investigations to only (i) and (ii).

Going into more detail for (i), it is well known that physical fields that are localized functions of the basic fields transform trivially (i.e., have zero charge) under any charge operator associated to a current obeying a Gauss law: $j_\nu = \partial^\mu F_{\mu\nu}$. Indeed, in intuitive terms, thanks to the latter, the action of the charge on any field Φ takes the form

$$\delta\Phi = \lim_{R \rightarrow \infty} \left[\int_{S_R} dS_i F_{0i}, \Phi \right], \quad (1.1)$$

where S_R is the surface of the sphere of radius R in three-space. Therefore, if Φ is (or the fields in terms of which it is constructed are) smeared with functions of compact support, thanks to locality, $\delta\Phi$ vanishes for R large enough. To avoid this the field Φ must have at least a “tail” through the sphere at infinity in Minkowski space (whether only in spacelike or even in timelike directions is a subject to be taken up in the next section). In this sense, as long as one is interested in physical nontrivially charged fields, only nonlocalized functions of the basic fields ought to be considered. Since the above statement has been given the status of a theorem [21], there is little to add and there is general agreement about it.

The theorem gives no hint, however, about the explicit form of such fields. According to the terminology also recently used in Ref. [13], such nonlocalized functions will be shortly referred to as “dressed” fields: a physical, *interacting* (i.e., Heisenberg) electron field should be dressed with a cloud of photons, as well as with its own Coulomb field.

Dirac [22] was the first to show, in an explicit way, how the dressing could be done in order to endow an electron with its own Coulomb field. His aim was a quantization of QED that would involve only those degrees of freedom that actually contribute to the dynamic evolution of the system. In retrospect, it does not sound as a surprise that he gave up the manifest covariance properties of the physical fields under Lorentz transformations: it was well known, after the Gupta-Bleuler formulation, that, even when restricting to the zero charge sector, such manifestly covariant formulations do involve indefinite metric, i.e., extra degrees of freedom irrelevant to the dynamic evolution.

After Dirac other authors have investigated different ways of dressing the basic fields, with different motivations and with different aims. The list given by Refs. [23–32] only gives some references that are closer in spirit to the present article and, in any event, has no pretension to completeness. Reference [13] provides a much more comprehensive bibliography, whereas Ref. [32] provides its updating.

On the same footing as Dirac, covariance is given up also in the model investigations of Steinmann [26], who has the same aim as Dirac, and of Ref. [13] and other works by the same group, who instead think of the dressed fields as composite operators within the usual formulation of the gauge theory.

The nonimplementability of Lorentz boosts in the charged sectors of QED is indeed, after the model independent investigations of Refs. [2,3], taken for granted to the point that, once the symmetry is broken by hand from the very beginning of the construction, no attempt is made to restore it.

We understand that this breaking of Lorentz symmetry put in by hand in model investigations is different from the breaking claimed in Refs. [2,3]. The latter refers to the impossibility of having a unitary representation of Lorentz boosts on \mathcal{H}_{as} , the space of asymptotic generalized coherent states, i.e., it is a statement that finds its collocation within (iv)₂. In this context it is far from being clear whether Lorentz symmetry must be given up right away from step (i). One of us and his coworkers [25,28] have endorsed the statement that it should be possible to go from (i) to (iii) without

the need of breaking Lorentz symmetry. It has also been argued in Ref. [25] that, for QED, one expects to recover the usual IR divergent (or alternatively, Zwanziger’s [18]) S matrix, so that the undertaking of the Kulish-Faddeev program remains, if attractive, necessary. In the present paper, instead, we will continue in making the effort of enforcing Lorentz symmetry from (i) down to (iii), because in this way the comparison between QED and QCD turns out to be much more suggestive. We admit, our results only concern some two-point functions and are, for now, incomplete. It is true, on the other hand, that the results of the present and the following papers [33] open the possibility of performing systematic model calculations that could help putting this comparison on a sounder basis: this is one of the several questions to be discussed in the conclusions.

Among the references we have cited, the work by Steinmann deserves a special mention, for not only it has been close in spirit to ours along the years, but it has been constantly inspiring. We feel it is not by chance that another part of Steinmann’s and collaborator’s work, not immediately connected with the problems specific to gauge theories, is invaluable to the approach presented here. Indeed, it turns out that the usual Dyson expansion formula for the calculations of vacuum expectation values (VEV) of the type $\langle T(\dots) \rangle$ is not sufficient for our purposes. The composite fields we will introduce, will themselves be T^\pm -ordered formal power series. So the calculation of their correlation functions will demand the ability at computing—in perturbation theory (PT)—both Wightman functions and, more in general, multi-time-ordered VEV’s of the type $\langle T^\pm(\dots) \cdots T^\pm(\dots) \rangle$. References [34–36] exactly provide the algorithm for doing all this.

Our attitude in the present paper is that we do not want to make any *a priori* assumption about IR asymptotic dynamics, with the exception of enforcing symmetries: local gauge, translations and Lorentz in particular. IR asymptotic dynamics should, hopefully, emerge by itself, i.e., only by our ability at calculating the near-mass-shell behavior of correlation functions, once a particular gauge invariant charged field has been selected within the framework of step (i) above. In other words the main point is that (i) leaves a remarkable freedom and evidently any choice made in selecting the form of physical charged fields may, and indeed does, affect the outcomes of (ii). Our work will, as a consequence, consist in exploiting all the freedom (i) leaves to see whether there exists a field with a near-mass-shell behavior mild enough to enable one to face point (iii) and eventually (iv). In the case the motivations about the necessity of having fields with a mild near-mass-shell behavior should be recapitulated in more intuitive and physical terms, we have found the discussion given in Ref. [30] about QED particularly sound.

So, in order to directly compare QED and QCD, we will construct “dressed electron” $e(x)$ and “quark” $q(x)$ fields (we could also construct the “gluon” [28], but the investigation of its behavior in higher orders is better postponed to future work, for a comparison with its QED analogue would be less stringent: the photon has no charge) whose two-point functions up the fourth order in the coupling constant—the simplest place where a difference between QED and QCD

may emerge—have the following properties: (1) they are independent of the gauge-fixing parameter, (2) ultraviolet divergences brought about by the compositeness of the dressing are cured by a single renormalization constant introduced in the definition, and (3) on-shell normalization conditions can be imposed, in the IR regularized theory, on the single IR divergent graphs with two different outcomes.

(3a) In QED a complete cancellation of IR divergences takes place, and the two-point function is given by

$$W(p) = \int d^4x e^{ip \cdot (x-y)} \langle e(x) \bar{e}(y) \rangle, \quad (1.2)$$

$$W(p) = W_0(p) + \frac{\alpha}{\pi} W_1 + \left(\frac{\alpha}{\pi} \right)^2 W_2 + \dots, \quad (1.3)$$

where

$$W_0(p) = (\not{p} + \mu)(2\pi) \theta(p^0) \delta(p^2 - \mu^2) \quad (1.4)$$

is the Wightman function of the free spinor field, whereas the higher order terms, described by the two invariant functions $a_i(p^2/\mu^2)$ and $b_i(p^2/\mu^2)$, $i \geq 1$:

$$W_i(p) = \theta(p^0) \theta(p^2 - \mu^2) \frac{1}{\mu^2} (a_i \not{p} + b_i \mu) \quad (1.5)$$

are given, to the first order, by

$$a_1 = \frac{\mu^2}{2p^2} \left(1 - \frac{\mu^2}{p^2} \right), \quad b_1 = 0, \quad (1.6)$$

and, to the second order (whose full form is given in Sec. V) have the near-mass-shell asymptotic form

$$\begin{aligned} a_2 &\simeq \frac{5}{9} r + \frac{1}{6} r^2 \ln r - \frac{7}{4} r^2 + \dots, \\ b_2 &\simeq -\frac{7}{36} r - \frac{1}{6} r^2 \ln r + \frac{5}{24} r^2 + \dots, \end{aligned} \quad (1.7)$$

with $r = p^2/\mu^2 - 1 \rightarrow 0$.

(3b) In QCD, assuming dimensional regularization for IR divergences—i.e., $D = 4 \rightarrow 4 + 2\epsilon$ [37,38]—the latter do not cancel, but obey the factorization equation

$$\epsilon \frac{\partial}{\partial \epsilon} w_2(p, \epsilon) = + \left(\frac{1}{2\epsilon} \right) \frac{11}{6} C_A w_1(p, \epsilon), \quad (1.8)$$

where $w = \Sigma(\alpha/\pi)^n w_n$ is defined by the amputation of the interacting part

$$W_{ij}(p, \epsilon) = \int d^4x e^{ip \cdot (x-y)} \langle q_i(x) \bar{q}_j(y) \rangle, \quad (1.9)$$

$$W_{ij}(p, \epsilon) = \delta_{ij} W_0(p) + \frac{i}{\not{p} - \mu + i0} \delta_{ij} w(p, \epsilon) \frac{-i}{\not{p} - \mu - i0}. \quad (1.10)$$

The first evident comment about the above results is that the game, played twice with the same rules, gives two qualitatively different results. It is sufficient, *per se*, to state that the IR asymptotic dynamics of the two models is different (this was expected), even in perturbation theory (possibly, this is a less widespread belief).

Concerning Eq. (1.3), although the result is that the singularity of the free field theory is not altered by the radiative corrections that vanish on the mass shell, it is true that the mass hyperboloid $p^2 = \mu^2$ is not isolated as in the mass gap case. This result, expected on the basis of simple physical intuition, does not seem, on the one hand, to invalidate the possibility of effecting the Fock construction. On the other hand a careful comparison with the observations made in Ref. [39] (where it has been pointed out that, in the case of gauge theories, the particle content might be recovered at the cost of abandoning Wigner notion of an irreducible representation of Poincaré group sitting on an isolated mass hyperboloid) would certainly be instructive. The investigation of this point pertains however the step (iii) above, so we will not pursue it in this article.

Concerning the second result (1.8), we find it intriguing for two reasons. The first is that it is simple—we mean the factorization. The second is the occurrence of the celebrated $\frac{11}{6} C_A$ factor, *with the plus sign*.

We cannot therefore resist the temptation of commenting on the consequences (1.8) would have, were it true in any order of PT. In the latter case its integration would yield

$$w(p, \epsilon) = e^{-(\alpha/\pi)\Delta(\epsilon)} w(p) \xrightarrow{\epsilon \rightarrow 0^+} 0, \quad (1.11)$$

$$\Delta(\epsilon) = \frac{11}{6} C_A \frac{1}{2\epsilon}, \quad (1.12)$$

with $w(p)$ IR finite. This hints at a different scenario, in which the Heisenberg “quark” field, *as a result of IR asymptotic dynamics*, is a free field not asymptotically, but at any momentum p .

It may be useful to recall the example of the Faddeev-Popov (FP) ghost in QED: in that case the field is free by construction and there is a factorization of correlation functions involving the ghost into a bunch of free ghost two-point-functions times a connected correlation function only involving fields with zero ghost number. Could one say that the ghost number is confined?

In QCD, even if Eq. (1.11) were true, one could not immediately conclude, as in the case of local fields [40] (we remind that dressed fields do not share the locality property), that the $q_i(x)$ is a free field. Nonetheless a working hypothesis could be to check whether, as a consequence of IR asymptotic dynamics, the factorization of quark and gluon free two-point functions, out of connected correlation functions only involving color singlets, does indeed take place. If all this happens the quark field may exist only as a free field: the problem of the S matrix would regard (as has long since been the case in practice) only color singlets.

Of course, it is not necessary for the above scenario to take really place that the function $\Delta(\epsilon)$ preserves, on possibly going from Eq. (1.8) to an exact result, the specific form

given by Eq. (1.12) suggested by our fourth order calculation. It might dress up even as a full series in α , provided that $\Delta \rightarrow +\infty$ for $\epsilon \rightarrow 0^+$ carried on holding.

We are aware that, on extrapolating our result (1.8) to Eq. (1.11), we have raised more questions (all orders, gluon, IR asymptotics of many-point functions) than we will answer in this article. But, in the framework we will set up, these questions do not seem to us prohibitively out of the range of traditional and well established tools of QFT.

The paper is organized as follows. In Sec. II the freedom one has in dressing the basic fields is analyzed in detail on the level of classical fields. Section III sets the stage for the calculation of quantum correlation functions: it is argued that an algorithm for computing VEV with several time orderings, i.e., of the type $\langle T^\pm(\dots) \dots T^\pm(\dots) \rangle$, is needed and the exhaustive work of Ostendorff and Steinmann [34–36], giving such an algorithm, is summarized. Section IV systematically explores in PT the lowest order of the two-point functions relative to the fields constructed in Sec. II, and the full form of W_1 , Eq. (1.6), is established. Section V gives a concise outlook of the fourth order calculations: the full form of W_2 , Eq. (1.7), is given together with a description of the way we follow to calculate it and to obtain Eq. (1.8). The full derivation of the latter results, as well as the proofs of their properties (1)-(3) above, are left for forthcoming papers. In Sec. VI we give a retrospective of the construction we have done and pinpoint the open problems that, in our opinion, most urgently should be faced in order to give the further, necessary support to such a construction.

II. CLASSICAL FIELDS

Let $\psi(x)$ denote a multiplet of Dirac fields transforming as the fundamental representation \mathcal{R} of the color group $SU(N)$ (the extension to whatever compact semisimple Lie group being trivial). We shall denote by $\mathbf{A}_\mu(x) = t^a A_\mu^a(x)$ the Yang-Mills potentials. Here t^a , $a = 1, \dots, N^2 - 1$, are the Hermitian generators in \mathcal{R} , satisfying the commutation relations $[t^a, t^b] = i f^{abc} t^c$, $t_{il}^a t_{lj}^a = C_F \delta_{ij}$, $C_F = (N^2 - 1)/(2N)$; whereas the structure constants f^{abc} are real, completely antisymmetric and obey $f^{abcd} f^{abcd} = C_A \delta^{ab}$, $C_A = N$. The scalar and wedge products in \mathcal{R} are accordingly defined by $\mathbf{A} \cdot \mathbf{B} = 2 \text{Tr}(\mathbf{A}\mathbf{B})$, $\mathbf{A} \wedge \mathbf{B} = -i[\mathbf{A}, \mathbf{B}]$.

It will be understood that the dynamics of the above fields is defined by the Lagrangian \mathcal{L} given, e.g., in Ref. [14], in which the gauge-fixing term $-\xi/2(\partial\mathbf{A}) \cdot (\partial\mathbf{A})$ as well as the Faddeev-Popov ghosts have been introduced and the Becchi-Rouet-Stora-Tyutin symmetry is at work. All the fields in \mathcal{L} are assumed to be local fields.

Let $\mathbf{C}(x) = t^a C^a(x) \in \mathcal{R}$ be the FP ghost field, satisfying

$$\mathbf{C}(t, \mathbf{x}) \rightarrow 0, \quad \text{for } |\mathbf{x}| \rightarrow \infty. \quad (2.1)$$

We shall call local gauge transformations of ψ , $\bar{\psi}$ and \mathbf{A}_μ the following:

$$\begin{aligned} \delta \mathbf{A}_\mu &= \partial_\mu \mathbf{C} + g \mathbf{A}_\mu \wedge \mathbf{C}, \\ \delta \psi &= +ig \mathbf{C} \psi, \quad \delta \bar{\psi} = -ig \bar{\psi} \mathbf{C}. \end{aligned} \quad (2.2)$$

Consider now the formal power series [27]

$$\begin{aligned} V(y; f) &= \sum_{N=0}^{+\infty} (+ig)^N \int d^4 \eta_1 \cdots \int d^4 \eta_N \\ &\quad \times f_N^{\nu_1 \cdots \nu_N}(y - \eta_1, \dots, y - \eta_N) \\ &\quad \times \mathbf{A}_{\nu_1}(\eta_1) \cdots \mathbf{A}_{\nu_N}(\eta_N), \end{aligned} \quad (2.3)$$

$$\begin{aligned} V^\dagger(x; f) &= \sum_{M=0}^{+\infty} (-ig)^M \int d^4 \xi_1 \cdots \int d^4 \xi_M \\ &\quad \times f_M^{\mu_1 \cdots \mu_M}(x - \xi_1, \dots, x - \xi_M) \\ &\quad \times \mathbf{A}_{\mu_M}(\xi_M) \cdots \mathbf{A}_{\mu_1}(\xi_1), \end{aligned} \quad (2.4)$$

where the terms $M, N=0$ are by definition 1.

We claim that one can choose *real* kernel functions f 's such that V and V^\dagger transform under Eq. (2.2) according to

$$\delta V = +ig \mathbf{C} V, \quad \delta V^\dagger = -ig V^\dagger \mathbf{C}. \quad (2.5)$$

Before we proceed to enforce the transformation properties (2.5), two comments are in order about the multiple convolutions displayed in Eqs. (2.3) and (2.4). (i) The first is that they are mandatory if one is interested, as we are, in obtaining translation covariant solutions to Eq. (2.5). (ii) The second is that the convolutions extending to the whole Minkowski space explicitly expose the fact that V and V^\dagger may be nonlocalized functions of the basic local fields \mathbf{A} , provided the support of the f 's is suitably chosen. In view of the discussion about Eq. (1.1), this is quite welcome because we are aiming at constructing locally gauge invariant fields that carry nontrivial global color numbers: indeed, concerning local gauge transformations, once Eq. (2.5) are satisfied, the spinor fields

$$\Psi_f(x) = V^\dagger(x; f) \psi(x), \quad \bar{\Psi}_f(y) = \bar{\psi}(y) V(y; f) \quad (2.6)$$

are obviously invariant under Eq. (2.2) while they transform as \mathcal{R} and $\bar{\mathcal{R}}$ when \mathbf{C} is not chosen according to Eq. (2.1), but is constant with respect to x .

Let us go back to enforcing Eq. (2.5). Steinmann has faced this problem in Ref. [27]. He assumes that, on introducing Eq. (2.2) into Eqs. (2.3) and (2.4), the derivatives can be reversed by parts. While this can be justified for space derivatives, thanks to the boundary conditions (2.1) on the ghost, the thing is less justifiable for the time derivatives, as one has no *a priori* control on asymptotic behavior in time. In electrodynamics there is a way out: since the ghost is free, one can choose suitable solutions of the d'Alambert equation [25] that justify the neglect of boundary terms. In the non-abelian case the problem is there: we shall, as in Ref. [27], just ignore it, recalling however the statement (1) of the introduction that, in the case of quantum fields, we will be able to prove the ξ independence of correlation functions.

With this *proviso*, Steinmann has shown that the requirement that Eq. (2.5) be satisfied by Eqs. (2.3) and (2.4) order by order in g leads to a linear inhomogeneous recursive system for the f 's. The Fourier transforms of the first of the equations he gives is

$$k_\nu \hat{f}_1^\nu(k) = i, \quad (2.7)$$

whereas the f with $N > 1$ arguments is determined in terms of the f with $N - 1$ arguments by

$$\begin{aligned} (k_1)_{\nu_1} \hat{f}_N^{\nu_1 \cdots \nu_N}(k_1, \dots, k_N) &= i[\hat{f}_{N-1}^{\nu_2 \cdots \nu_N}(k_2, \dots, k_N) - \hat{f}_{N-1}^{\nu_2 \cdots \nu_N}(k_1 + k_2, k_3, \dots, k_N)], \\ (k_\alpha)_{\nu_\alpha} \hat{f}_N^{\nu_1 \cdots \nu_\alpha \cdots \nu_N}(k_1, \dots, k_\alpha, \dots, k_N) &= +i[\hat{f}_{N-1}^{\nu_1 \cdots \nu_{\alpha-1} \nu_{\alpha+1} \cdots \nu_N}(k_1, \dots, k_{\alpha-2}, k_{\alpha-1} + k_\alpha, k_{\alpha+1}, \dots, k_N) \\ &\quad - \hat{f}_{N-1}^{\nu_1 \cdots \nu_{\alpha-1} \nu_{\alpha+1} \cdots \nu_N}(k_1, \dots, k_{\alpha-1}, k_\alpha + k_{\alpha+1}, k_{\alpha+2}, \dots, k_N)], \\ (k_N)_{\nu_N} \hat{f}_N^{\nu_1 \cdots \nu_N}(k_1, \dots, k_N) &= i \hat{f}_{N-1}^{\nu_1 \cdots \nu_{N-1}}(k_1, \dots, k_{N-2}, k_{N-1} + k_N), \end{aligned} \quad (2.8)$$

with $2 \leq \alpha \leq N - 1$. We also take from Ref. [27] that the solutions of Eqs. (2.7) and (2.8), that for any integer N satisfy

$$\sum_{J=0}^N (-1)^J \hat{f}_J^{\nu_1 \cdots \nu_J}(k_1, \dots, k_J) \hat{f}_{N-J}^{\nu_{J+1} \cdots \nu_N}(k_{J+1}, \dots, k_N) = 0, \quad (2.9)$$

$$\begin{aligned} \sum_{J=0}^N (-1)^{N-J} \hat{f}_J^{\nu_1 \cdots \nu_J}(k_J, \dots, k_1) \hat{f}_{N-J}^{\nu_{J+1} \cdots \nu_N}(k_{J+1}, \dots, k_N) \\ = 0, \end{aligned} \quad (2.10)$$

give rise to unitary series $V(x;f)V^\dagger(x;f) = V^\dagger(x;f)V(x;f) = 1$.

Let us first focus on Eq. (2.7). A family of solutions to this equation that also satisfies Eqs. (2.9) and (2.10) is

$$\hat{f}_1^\nu(k; c) = in^\nu \frac{1}{2} \left(\frac{1+c}{n \cdot k - i0} + \frac{1-c}{n \cdot k + i0} \right), \quad (2.11)$$

where c is a real parameter and n^ν is a 4-vector that we leave, for the moment, unspecified. Two particular solutions from Eq. (2.11) are

$$\hat{f}_{+1}^\nu(k; n) = \frac{in^\nu}{n \cdot k - i0}, \quad (2.12)$$

$$\hat{f}_{-1}^\nu(k; n) = \frac{in^\nu}{n \cdot k + i0}. \quad (2.13)$$

It can be verified that the two following sets of functions \hat{f}_{+N} and \hat{f}_{-N} , given by

$$\begin{aligned} \hat{f}_{\pm N}^{\nu_1 \cdots \nu_N}(k_1, \dots, k_N; n) \\ = \frac{in^{\nu_1}}{n \cdot (k_1 + \dots + k_N) \mp i0} \cdots \frac{in^{\nu_N}}{n \cdot k_N \mp i0} \end{aligned} \quad (2.14)$$

separately satisfy all Eqs. (2.8)–(2.10). These solutions also fulfil the factorization property

$$\sum_{\text{perm}} \hat{f}_{\pm N}^{\nu_1 \cdots \nu_N}(k_1, \dots, k_N; n) = \frac{1}{N!} \hat{f}_{\pm 1}^{\nu_1}(k_1; n) \cdots \hat{f}_{\pm 1}^{\nu_N}(k_N; n) \quad (2.15)$$

well known as eikonal identity, as well as the n -reflection exchange relation

$$\hat{f}_{\pm N}^{\nu_1 \cdots \nu_N}(k_1, \dots, k_N; n) = \hat{f}_{\mp N}^{\nu_1 \cdots \nu_N}(k_1, \dots, k_N; -n). \quad (2.16)$$

We will also need a third set of solutions, that extend to higher orders the lowest order solution obtained by setting $c = 0$ in Eq. (2.11):

$$\hat{f}_{01}^\nu(k; n) = \frac{in^\nu}{2} \left(\frac{1}{n \cdot k - i0} + \frac{1}{n \cdot k + i0} \right) \quad (2.17)$$

with the principal value (PV) prescription for the $n \cdot k$ denominator. This is evidently connected with the problem of exposing a family of solutions that interpolates between \hat{f}_{+N} and \hat{f}_{-N} . We have found that, with n^ν kept fixed and even after imposing the unitarity constraints (2.9) and (2.10), the higher the N the higher the number of complex parameters due to the occurrence of Poincaré-Bertrand terms. However, if also the eikonal identity (2.15) is enforced, the interpolating family only depends on the real parameter c appearing in Eq. (2.11). Just to give a flavor of the thing, it is found that

$$\begin{aligned} \hat{f}_2^{\nu_1 \nu_2}(k_1, k_2; c) &= \hat{f}_1^{\nu_1}(k_1 + k_2; c) \hat{f}_1^{\nu_2}(k_2; c) \\ &\quad + \frac{\pi^2}{2} (1 - c^2) n^{\nu_1} n^{\nu_2} \delta(n \cdot k_1) \delta(n \cdot k_2). \end{aligned} \quad (2.18)$$

We have explicitly found up to $f_4(k_1, \dots, k_4; c)$ and we also have a guess about $f_N(k_1, \dots, k_N; c)$ for generic N .

But, for the sake of conciseness we will no longer elaborate on this topic, also because higher orders will not be needed in the perturbative calculations we will perform in later sections. The important for the sequel is that there exists a solution, denoted by $\hat{f}_{0_N}^{\nu_1 \dots \nu_N}(k_1, \dots, k_N; n)$, that extends Eq. (2.17) to any order N . In connection with Eq. (2.16), note that the solution \hat{f}_0 , in addition to satisfying the eikonal identity, is also invariant under n reflection

$$\hat{f}_{0_N}^{\nu_1 \dots \nu_N}(k_1, \dots, k_N; n) = \hat{f}_{0_N}^{\nu_1 \dots \nu_N}(k_1, \dots, k_N; -n). \quad (2.19)$$

The relationship between the present approach and other ones [13,23,26,27], can now be clarified.

Consider, to this purpose, $V_-(y; n)$, i.e., the V obtained by inserting the solution \hat{f}_{-N} [i.e., Eqs. (2.13) and (2.14) with the minus sign] into Eq. (2.3). It is useful to represent all the denominators in \hat{f}_{-N} by means of the one-parameter integral representation $(b + i0)^{-1} = -i \int_0^{+\infty} d\omega \exp[i\omega(b + i0)]$. In this way it is possible to explicitly perform the $d_4 k_j$ integrations in the anti-Fourier transform of the \hat{f} 's. These integrations give rise to $\delta_4(x - \eta_j + \Sigma_i \omega_i)$ that allow, in turn, for the elimination of the $d^4 \eta_j$ integrations in Eq. (2.3). Some further obvious manipulations convert Eq. (2.3) into

$$\begin{aligned} V_-(y; n) &= \sum_{N=0}^{\infty} (ig)^N \int_0^{+\infty} d\omega_1 \dots \int_{\omega_{N-1}}^{+\infty} d\omega_N \\ &\quad \times n \cdot \mathbf{A}(y - n\omega_1) \dots n \cdot \mathbf{A}(y - n\omega_N) \\ &= \mathcal{P}^+ \exp \left[ig \int_0^{+\infty} d\omega n \cdot \mathbf{A}(y - n\omega) \right]. \end{aligned} \quad (2.20)$$

The right-hand side of the above formula is the usual definition of the path-ordering symbol \mathcal{P}^+ . If n is chosen to be a spacelike vector, the above representation clarifies that V_- is nothing but a rectilinear string operator in the manner of Mandelstam [23] extending to spacelike infinity. The case of n spacelike may serve also to accommodate Buchholz case [41]. For this reason we will generically refer to all the V and V^\dagger operators as to the string operators, regardless of whether n is spacelike or timelike.

In the same way one finds that

$$V_+^\dagger(x; n) = \mathcal{P}^+ \exp \left[-ig \int_0^{+\infty} d\omega n \cdot \mathbf{A}(x + n\omega) \right] \quad (2.21)$$

(again a \mathcal{P}^+ for the order of the $n \cdot \mathbf{A}$ factors in V^\dagger is reversed with respect to V). It is now convenient to introduce the decomposition of the Minkowski four-space \mathcal{M}_4 into the future and past light cones, and their complement:

$$\mathcal{M}_4 = \mathcal{C}_+ \cup \mathcal{C}_0 \cup \mathcal{C}_- \quad (2.22)$$

and in the sequel, referring to the above decomposition, the indices σ and τ will always take the values $\pm, 0$.

Suppose now that, we choose $n \in \mathcal{C}_\pm$, i.e., in the future or past light cone. Then the statement that, respectively, $n_0 \geq 0$ is Lorentz invariant, and also $x^0 - \omega_i n^0 = \tau_i \leq \tau_{i-1} \leq \dots \leq \tau^0 = x^0$. In view of this, Eq. (2.20) can be written using the T^\pm chronological ordering symbols: for $n \in \mathcal{C}_\pm$

$$V_-(x; n) = T^\pm \exp \left[+ig \int_0^{+\infty} d\omega n \cdot \mathbf{A}(x - n\omega) \right], \quad (2.23)$$

and likewise for Eq. (2.21): for $n \in \mathcal{C}_\pm$

$$V_+^\dagger(x; n) = T^\pm \exp \left[-ig \int_0^{+\infty} d\omega n \cdot \mathbf{A}(x + n\omega) \right]. \quad (2.24)$$

So far this is no big difference: the ordering operators, either \mathcal{P}^\pm or T^\pm , only order the color matrices t^{a_1}, \dots, t^{a_N} in the N th term of the above series, whereas the fields $A_{\nu_1}^{a_1}, \dots, A_{\nu_N}^{a_N}$ inasmuch as classical fields, are not sensitive to this ordering. In the case of classical electrodynamics— $t^a \rightarrow 1$ —such operators are simply useless. The role of the T^\pm ordering will instead become crucial when we will keep it in the definition of the quantum Heisenberg operators.

We have also to consider string operators in which the string vector n is chosen spacelike. In this case the difference between the arguments of two neighboring A 's is spacelike, so only the color matrices are sensitive to the ordering, whereas even the Heisenberg fields of the quantum case commute with one another, due to locality. The solution we will consider for $n \in \mathcal{C}_0$ are $V_0(x; n)$ and $V_0^\dagger(x; n)$, i.e., the ones corresponding to the solution \hat{f}_{0_N} that extends Eq. (2.17) and fulfils the n -reflection invariance property (2.19).

Let us now introduce the characteristic functions

$$\chi_\sigma(n) = \begin{cases} 1 & \text{if } n \in \mathcal{C}_\sigma, \quad \sigma = \pm 1, 0, \\ 0 & \text{otherwise,} \end{cases} \quad (2.25)$$

and correspondingly the fields

$$\Psi_\pm(x; n) = \chi_\pm(n) \quad T^\pm [V_\pm^\dagger(x; n) \psi(x)], \quad (2.26)$$

$$\Psi_0(x; n) = \chi_0(n) \quad V_0^\dagger(x; n) \psi(x),$$

$$\bar{\Psi}_\pm(x; n) = \chi_\pm(n) \quad T^\pm [\bar{\psi}(x) V_\pm(x; n)], \quad (2.27)$$

$$\bar{\Psi}_0(x; n) = \chi_0(n) \quad \bar{\psi}(x) V_0(x; n).$$

These fields fulfil the Dirac conjugation properties

$$\overline{\Psi_\pm(x; n)} = \bar{\Psi}_\mp(x; -n), \quad (2.28)$$

$$\overline{\Psi_0(x; n)} = \bar{\Psi}_0(x, -n),$$

that follow from the n -reflection properties (2.16) and (2.19). As a consequence the composite fields

$$\Psi(x; n) = z_+ \Psi_+ + z_- \Psi_- + z_0 \Psi_0, \quad (2.29)$$

$$\bar{\Psi}(x;n) = z_- \bar{\Psi}_+ + z_+ \bar{\Psi}_- + z_0 \bar{\Psi}_0$$

(with the complex constants z 's satisfying $\bar{z}_\pm = z_\mp$, $\bar{z}_0 = z_0$, and to be specified later, for the quantum fields, when effecting renormalization) satisfy the Dirac conjugation relation

$$\bar{\Psi}(x;n) = \Psi(x;-n)^\dagger \gamma_0 \quad (2.30)$$

which, in Fourier transform, takes the form

$$\bar{\tilde{\Psi}}(p;n) = \hat{\tilde{\Psi}}(-p;-n). \quad (2.31)$$

All the constructions done so far, to go from ψ to Ψ , can be crudely summarized in this way: one has traded the gauge variance of ψ for the dependence of Ψ on the string vector n . We will refer to this fact as a breaking, put in by hand, of the original Lorentz symmetry—an unpleasant feature one would like to get rid of. We dedicate the rest of this section to give an heuristic description of how we will try to accomplish this task.

The Dirac equation for the ordinary ψ in linear covariant gauges is first converted into the equation of motion for $\Psi(x;n)$. We write it in momentum representation

$$\begin{aligned} (\not{p} - \mu) \hat{\Psi}(p;n) \\ = g \gamma_\alpha t^a \int \frac{d^4 k}{(2\pi)^4} \sum_\sigma z_\sigma T_\sigma^{\alpha\beta}(k;n) A_\beta^a(k) \hat{\Psi}_\sigma(p-k;n) \\ = g \gamma_\alpha t^a Q^{\alpha\alpha}(p;n), \end{aligned} \quad (2.32)$$

where the index σ refers to the decomposition of Ψ with respect to the light cone of n , Eq. (2.26). Accordingly, the projectors T are given by

$$\begin{aligned} T_\mp^{\alpha\beta}(k;n) &= g^{\alpha\beta} - \frac{k^\alpha n^\beta}{n \cdot k \pm i0}, \\ T_0^{\alpha\beta}(k;n) &= g^{\alpha\beta} - \text{PV} \frac{k^\alpha n^\beta}{n \cdot k}, \end{aligned} \quad (2.33)$$

and satisfy

$$n_\alpha T_\sigma^{\alpha\beta}(k;n) = 0, \quad T_\sigma^{\alpha\beta}(k;n) k_\beta = 0. \quad (2.34)$$

Thanks to the second of Eqs. (2.34), the longitudinal degrees of freedom of A_β^a are expected to decouple. Thanks to the first of Eqs. (2.34), the vector field to which Ψ is coupled is $\mathcal{A}^{\alpha\alpha} = T^{\alpha\beta} A_\beta^a$ that satisfies $n \cdot \mathcal{A}^a = 0$.

Were it not for the subtleties due to the $\pm i0$ prescriptions (i.e., to the light-cone decomposition of the field with respect to n), this formally is the equation satisfied by the Dirac field in the axial gauge. One could try to take this as a substitute of the ordinary Dirac equation in linear covariant gauges and $\Psi(x;n)$ (with n , as in a gauge fixing, chosen once for all) as the variable substituting ψ and in terms of which to attempt a gauge-invariant formulation of the theory—much in the spirit of Refs. [22,23,26].

We will not take this attitude. We will continue to think of $\Psi(x;n)$ as a composite field in a theory where ψ and A_α^a play the role of basic dynamic variables.

This point of view leaves open the possibility of choosing different n 's for different Ψ 's. More clearly, we want to leave open the possibility of computing quantum correlation functions of the type $\langle \Psi(x;m) \bar{\Psi}(y;n) \rangle$, in which any field has its own string and with no restriction on whether both m and n are taken either timelike or spacelike.

This also is the point where we can explain how we will recover the lost Lorentz symmetry. We will discuss about the possibility of taking the limit

$$n \rightarrow p \quad (2.35)$$

in Eq. (2.32).

A serious warning about this limit is that its very existence is far from being trivial: we will give some positive evidence in favor of it only in the case of quantum fields in Sec. IV. For now we will just forget about any mathematical rigor and assume its existence: this enables us to draw some conclusions and formulate some expectations about quantum fields.

The first consideration about the limit (2.35) is that it does not mess up the Dirac conjugation properties of Ψ , as evident from Eq. (2.31). Let us then call

$$\hat{q}(p) = \hat{\Psi}(p;p). \quad (2.36)$$

Then, by setting $n=p$ in Eq. (2.32), one obtains

$$\begin{aligned} (\not{p} - \mu) \hat{q}(p) &= g \gamma_\alpha t^a \int \frac{d^4 k}{(2\pi)^4} \\ &\times \sum_\sigma z_\sigma T_\sigma^{\alpha\beta}(k;p) \hat{A}_\beta^a(k) \hat{\Psi}_\sigma(p-k;p) \end{aligned} \quad (2.37)$$

that makes evident why we have kept our point of view: differently from $\hat{\Psi}(p;n)$, the field \hat{q} may exist only as a composite field: in the right-hand side (RHS) of the above equation the $\hat{\Psi}$ appears with two different values of its arguments, so the field \hat{q} does not satisfy a closed equation. For \hat{q} , as already for $\hat{\Psi}(p;n)$, it is expected that the unphysical degrees of freedom of A_α^a decouple: the second of Eqs. (2.34) still applies.

But this is not the end of the story. If, according to a well known argument, the near-mass-shell behavior of the field is driven by the classical currents responsible for the interaction with soft gluons or photons, we can make a guess about it by operating the replacement $\gamma^\alpha \rightarrow \mu p^\alpha / p \cdot k$ within the integration in the RHS of Eq. (2.37). It is then seen that, thanks now to the first of Eqs. (2.34) with $n=p$, also the classical currents decouple and no longer drive the asymptotic IR dynamics of q . As a result, the near-mass-shell behavior of the field q we have defined should be at least milder than that of both the gauge-variant ψ and the n -dependent Ψ .

The observation above, finally, clarifies why we have constructed strings allowing for the choice of a timelike vector:

in the classical currents the momentum is close to the mass shell $p^2 \approx \mu^2 > 0$. All these expectations for the quantum fields will find confirmation in the following sections. This means that we will give meaning, to some extent, to the heuristic formula

$$q(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \left[\int d^4 y e^{ip \cdot y} \Psi(y; n) \right]_{n=p} \quad (2.38)$$

with $\Psi(y; n)$ given by Eq. (2.29). The utility of this formula is to clarify that the kind of delocalization involved in $q(x)$ is by far more complicated than that, recalled in connection with Eq. (1.1), of a field with a ‘‘tail’’ going to infinity along a string that is rectilinear in coordinate representation, as is

the case for $\Psi(y; n)$. In pictorial terms the strings contributing to $q(x)$ are spread out all over x space: this happens when a string, rectilinear in p space, is integrated upon with $\exp(-ip \cdot x)$ as weighting factor. The field $q(x)$ thus rather resembles a kind of space-time candy-sugar cloud centered at x .

III. PERTURBATION THEORY FOR QUANTUM FIELDS

The present section is devoted to set up diagrammatic rules for the calculation, in perturbation theory, of the correlation functions of the quantum gauge invariant charged fields we have sketched in Sec. II.

We define the quantum field corresponding to Eq. (2.29) in the following way:

$$\begin{aligned} \Psi(x; m) = & \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \left\{ \sum_{M=0}^{\infty} (-ig)^M t^{a_M} \dots t^{a_1} \prod_{j=1}^M \int \frac{d^4 k_j}{(2\pi)^4} \right. \\ & \left. \times \sum_{\sigma=\pm, 0} \chi_{\sigma}(m) \zeta_{\sigma}^{1/2} \langle V_{\sigma}^{\dagger} \rangle^{-1} \hat{f}_{\sigma_M}^{\mu_1} \dots \mu_M(k_1, \dots, k_M; m) T^{\sigma} \left[\hat{A}_{\mu_M}^{a_M}(k_M) \dots \hat{A}_{\mu_1}^{a_1}(k_1) \hat{\psi} \left(p - \sum_{j=1}^M k_j \right) \right] \right\}. \quad (3.1) \end{aligned}$$

In the above formula the time-ordering operators T^{\pm} and the identity operator $T^0 = 1$ act on the Heisenberg fields in the square bracket. Moreover $\zeta_{+} = \zeta_{-}$ and ζ_0 will play the role of real renormalization constants, introduced to take care of the compositeness of Ψ . In addition, also the factors $\langle V_{\sigma}^{\dagger} \rangle^{-1}$ are constants whose values will be fixed later, when their necessity to avert some ill-defined one particle reducible graphs will be realized. For now, what is needed to know is that the ζ_{σ} and the $\langle V_{\sigma}^{\dagger} \rangle^{-1}$ have the right conjugation properties such that $\zeta_{\sigma} \langle V_{\sigma}^{\dagger} \rangle^{-1}$ can be identified with the z_{σ} of Eq.

(2.29): in this way $\Psi(x; m)$ is, in turn, obtained by taking the straightforward Dirac conjugate of Eq. (3.1).

It should be finally noted that the structure of formula (3.1) is slightly different from Eq. (2.29). In fact, in the latter case one can recognize the time-ordering of the fields only after performing, as we have done in Sec. II, the $d^4 \xi_j$ integrations of Eq. (2.4). Here, instead, the $d^4 k_j$ integration involving the f 's, that are in turn responsible for this ordering, are indicated but not yet performed: the $T^{\pm, 0}$ are there simply by definition.

The light-cone decomposition of the field with respect to m —the second line of the above formula—makes it evident that, depending on the choice of the string vector m relative to any single field, one must be able to compute VEV's of the type $\langle T^{\sigma_n}(\dots) \dots T^{\sigma_1}(\dots) \rangle$, with $\sigma_i = \pm, 0$. This observation entails that the usual Dyson perturbation theory formula for the development of one single T -ordered product is not sufficient to our purposes. An extension of the Dyson algorithm is therefore needed and, fortunately for us, such an extension is already available, thanks to the work of Osterdorff [34] and Steinmann [35]. We recapitulate their results

for the reader's convenience (reporting more or less *verbatim* the content of the appendix of Ref. [36]).

Let us denote by $X = \{x_1, \dots, x_r\}$ a set of four-vectors and Φ stand for any basic field (A_{α}^a, ψ etc.) of interest for us. Let also $T^{\sigma}(X)$ denote the corresponding product of the fields $\Phi(x_1) \dots \Phi(x_r)$. In the multi-time-ordered vacuum expectation value

$$W(X_n, \sigma_n | \dots | X_2, \sigma_2 | X_1, \sigma_1) = \langle T^{\sigma_n}(X_n) \dots T^{\sigma_1}(X_1) \rangle \quad (3.2)$$

any σ_i may take the value \pm only (the case $T^0 = 1$ of no ordering will be included later). The perturbative contribution to order g^N to W is obtained as follows.

Graphs. All the graphs with $\sum r_i$ external points and a number of internal points suitable to match the order N in PT are drawn.

Partitions. Each of the above graphs is partitioned in non-overlapping subgraphs—the ‘‘sectors’’—such that all the external points of X_i belong to the same sector, called an external sector. In general, there exist sectors not containing external points, called internal sectors. Internal points may belong to external as well as to internal sectors, depending on the partition considered.

Sector numbers. To any sector S , a number $s(S)$ is assigned according to the following rules: (i) For the sector containing the external points X_i : $s = i$, (ii) for an internal sector S , $s(S)$ is a noninteger number between the maximum and the minimum sector numbers relative to the neighboring sectors (i.e., the sectors connected to S by at least by one line of the graph), (iii) if $\sigma_i \neq \sigma_{i+1}$ there is no internal sector with $i < s(S) < i + 1$.

Equivalence. If two partitions only differ in the numbering of the sectors—not in their topology—they are inequivalent if, for at least one pair of neighbouring sectors S' , S'' , one has $s(S') > s(S'')$ in the first partition, $s(S') < s(S'')$ in the second.

Type. The sectors are either T^+ or T^- sectors in the following way: the external sector with number i is a T^{σ_i} sector; the internal sector with $i < s(S) < i+1$ and $\sigma_i = \sigma_{i+1}$ is a T^{σ_i} sector as well.

Diagrammatic rules. Any partition is converted into an analytical expression according to the following: (i) Inside a T^+ sector ordinary Feynman rules for propagator and vertices apply, (ii) inside a T^- sector the complex conjugate of Feynman rules hold, (iii) any internal sector contributes a (-1) factor, (iv) finally, a line connecting two different sectors S' and S'' corresponds, in momentum space, to a factor

$$\delta_{ij}(\not{p} + \mu) 2\pi\theta(\pm p_0) \delta(p^2 - \mu^2) \quad \text{quarks,} \quad (3.3)$$

$$\delta_{ab}[-g_{\mu\nu} 2\pi\theta(\pm k_0) \delta(k^2) + k_\mu k_\nu \dots] \quad \text{gluons,} \quad (3.4)$$

where the dots in the second stand for gauge terms that decouple in all the W functions we will calculate and the \pm applies according to whether the number sectors satisfy $s(S') \leq s(S'')$.

Sum. The contribution of order g^N to W is obtained by summing the contribution of all inequivalent partitions so obtained and multiplying the result for the appropriate combinatorial factor.

The inclusion of the case $T^0 = 1$ of no ordering is taken into account by the following observation. Single fields $\Phi(x)$ are included in the above scheme by allowing external sectors with only one field as argument: $\Phi(x) = T^\pm[\Phi(x)]$. In this way the single partitions of a graph do depend on the choice of the sign, but the sum, expectedly, does not. This, in particular, provides the algorithm for computing Wightman functions in PT.

Some comments about the above Steinmann rules are in order. The iterative derivation of the above rules is based on the following inputs: (i) the equations of motion of the model, (ii) Wightman axioms for the Wightman functions (including locality, but excluding positivity), (iii) on-shell normalization conditions.

Within these assumptions the solution provided by the above rules is shown to be unique. Concerning the last point we emphasize that, whenever needed, an IR regulator must be at work (which one is suitable for the models considered here will be discussed later).

Moreover Steinmann himself emphasizes that no use of the asymptotic condition is ever made. This is quite welcome for us for, in the contrary case, this would imply some assumption on the IR asymptotic dynamics: this is exactly what we do not want to do.

The above rules provide the tool necessary for computing in PT, at least in principle, all the correlation functions of the gauge invariant charged fields, as the ‘‘quark’’ (3.1): this algorithm provides us immediately with the ‘‘quantum part’’ of the calculation, i.e., that part that only involves the quantum fields in the RHS of (3.1). About this part one should also observe that all the degrees of freedom, physical as well as unphysical, are associated to local fields that propagate in causal way.

However, there remains the ‘‘classical part’’ of the calculation, consisting in checking whether the d^4k_j integrations involving both the VEV's and the \hat{f} 's we have chosen (that should provide the decoupling of the unphysical degrees of freedom) are well defined. We face this problem in the next section where we only consider two-point functions, because the rules we have reported above are somewhat unusual and more complicated than the Feynman rules everybody is used to: we better start learning the new game in the simplest case.

IV. TWO-POINT FUNCTIONS

Our aim is to see how the algorithm given in the preceding section works in the case of the two-point function

$$\begin{aligned} & \int d^4x e^{ip \cdot x} \int d^4y e^{iq \cdot y} \langle \Psi(x; m) \bar{\Psi}(y; n) \rangle \\ &= (2\pi)^4 \delta_4(p+q) W(p, m; q, n) \\ &= \sum_{M, N=0}^{\infty} \sum_{\sigma, \tau = \pm, 0} (-ig)^M (ig)^N \prod_{i=1}^M \int \frac{d^4k_i}{(2\pi)^4} \prod_{j=1}^N \int \frac{d^4l_j}{(2\pi)^4} \\ & \quad \times \zeta_\sigma^{1/2} \langle V_\sigma^\dagger \rangle^{-1} \chi_\sigma(m) \hat{f}_{\sigma M}^{\mu_1 \dots \mu_M}(k_1, \dots, k_M; m) \zeta_\tau^{1/2} \langle V_\tau \rangle^{-1} \chi_\tau(n) \hat{f}_{\tau N}^{\nu_1 \dots \nu_N}(l_1, \dots, l_N; n) \\ & \quad \times \left\langle T^\sigma \left[\hat{\mathbf{A}}_{\mu_M}(k_M) \dots \hat{\mathbf{A}}_{\mu_1}(k_1) \hat{\psi} \left(p - \sum k_i \right) \right] T^\tau \left[\hat{\psi} \left(q - \sum l_j \right) \hat{\mathbf{A}}_{\nu_1}(l_1) \dots \hat{\mathbf{A}}_{\nu_N}(l_N) \right] \right\rangle. \end{aligned} \quad (4.1)$$

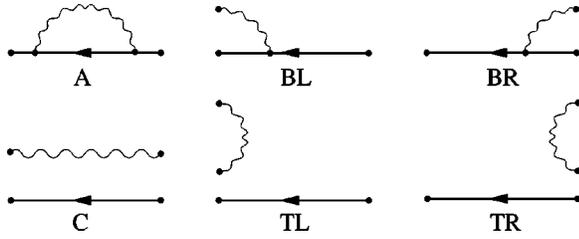


FIG. 1. The graphs associated to the VEV's of (4.1) contributing to the $\mathcal{O}(g^2)$ of $W(p,m;q,n)$.

Due to the presence of the string vectors m and n , this two-point function extends Eq. (1.9) that will be recovered in the end of this section. In analogy to Eq. (1.10) we will denote the amputation of Eq. (4.1) by

$$(2\pi)^4 \delta_4(p+q) w(p,m;q,n) = \gamma_a t^a \langle Q^{\alpha a}(p,m) \bar{Q}^{b\beta}(q,n) \rangle \gamma_{\beta t}^b, \quad (4.2)$$

where the Q 's are the currents defined in Eq. (2.32).

Consistently with Steinmann assumptions, we assume that the QCD Lagrangian [14] has been IR regulated and renormalized with on-shell normalization conditions. Up to order g^2 the calculation is essentially abelian: the color matrices in the two vertices contract to C_F ($\rightarrow 1$ for QED) and there is no three-gluon vertex. To this order, therefore, one can think of regularizing IR divergences by giving a mass λ to the photon or gluon and UV divergences by dimensional regularization $4 \rightarrow 4 - 2\epsilon$, $\epsilon > 0$.

As a matter of fact, on going to order g^4 it will be seen in [33] that the mass regularization is not adequate and we shall use dimensional regularization $4 \rightarrow 4 + 2\epsilon$, $\epsilon > 0$ for the IR [37,38] (this IR ϵ should not be confused with the UV ϵ , anyway they will never be simultaneously used) and non-Lagrangian Pauli-Villars regularization [43] for UV. Details about the problems connected with the choice of the regularizations are given in Sec. V.

It is convenient to group the graphs contributing to the VEV in Eq. (4.1) in the following way. (1) Usual or local graphs: those with the $M=N=0$ in the above double series, i.e., the graphs contributing to the Wightman function $\langle \psi(x) \bar{\psi}(y) \rangle$. (2L) Left graphs: $M>0, N=0$. (2R) Right graphs: $M=0, N>0$, specular to the left graphs. (3) Left and right graphs: both $M>0$ and $N>0$. This is exemplified by the six graphs in Fig. 1, that gives the graphs contributing to order g^2 .

The sector partitions of the above graphs depend on whether either m or n are chosen in \mathcal{C}_{\pm} or in the complement \mathcal{C}_0 of the light cone. To cover all the nine possibilities, it

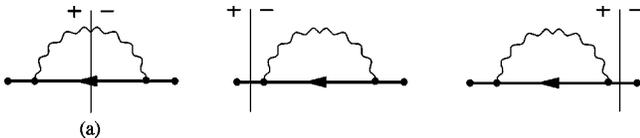


FIG. 2. The sector partitions of the ‘local’ graph A.

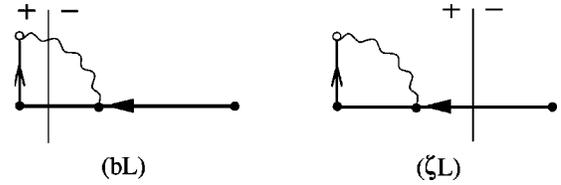


FIG. 3. The contributions to $W(p,m;q,n)$ originated by the sector partitions of graph BL.

would be sufficient to consider only five cases, thanks to the Dirac conjugation properties of the fermion field, Eq. (2.30). However, we will furtherly restrict ourselves only to the three cases that are more interesting for our purposes: (A) $m \in \mathcal{C}_+, n \in \mathcal{C}_-$, (B) $m \in \mathcal{C}_+, n \in \mathcal{C}_0$, (C) $m, n \in \mathcal{C}_0$.

The discussion of the remaining cases is, after these, a simple exercise. In any event, the lowest order graph, common to all cases, contributes the free two-point Wightman function of the spinor field, Eq. (1.4).

A. $m \in \mathcal{C}_+, n \in \mathcal{C}_-$

Only the term of Eq. (4.1) with $\sigma = +, \tau = -$ contributes, there are only two external sectors: sector 1, in the right, that is a T^- sector, and sector 2 in the left, that is a T^+ sector. Since the two sectors are of different type, there can be no internal sectors. The partitions of the above six graphs are thus obtained by drawing a cutting vertical line in all possible positions. In the cut lines we convene that momentum always flows from right to left, i.e., from sector 1 to sector 2 so that the replacement rules (3.3) and (3.4) are always taken with the plus sign.

All this resembles, and is nothing else but, the familiar Cutkosky-Veltman cutting rules. It should be noted that this regards only the VEV in the last line of Eq. (4.1). The \hat{f} vertices contributed by the string operators, not even drawn in Fig. 1, are not touched upon by Steinmann rules: their denominators are instead prescribed by our definition (3.1).

In addition, this identification of Steinmann rules with Cutkosky-Veltman rules happens only thanks to the choice made for m and n . Different choices, as well as VEV's with more than two external sectors, are covered only by Steinmann rules.

The partitions drawn in Fig. 2–4 refer to Eq. (4.1), i.e., to the whole $\langle \Psi \bar{\Psi} \rangle$, not only to the VEV in it: the vertical lines represent the string denominators of the \hat{f} 's, whereas each vertex on a vertical line—an empty circle—contributes a factor proportional to either gm_{μ} or gn_{ν} ; there also is a four-dimensional integration for the loop.

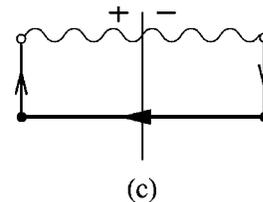


FIG. 4. The contribution to $W(p,m;q,n)$ originated by the only sector partition of graph C.

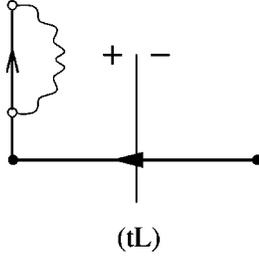


FIG. 5. The contribution to $W(p,m;q,n)$ originated by the only sector partition of graph TL.

Concerning graph A, its partitions are given in Fig. 2. Only the one marked (a) is nonzero, the other two vanish thanks to both mass and wave function on-shell normalization conditions.

Graph BL has the two partitions given in Fig. 3 and named (bL) and (ζ L). There are the two specular and complex conjugate partitions (bR) and (ζ R) from BR. Graph C has the only partition (c) given in Fig. 4. Graph TL too has only the partition (tL) given in Fig. 5. There also is the partition (tR) complex conjugate of the above.

We start with discussing the last graph. It is ill defined because its contribution to $W_1(p,m;q,n)$ is proportional to the integral

$$\int d_4k \frac{gm^\mu}{m \cdot k - i0} \frac{gm^\nu}{m \cdot (k-k) - i0} \frac{-ig_{\mu\nu} + \dots}{k^2 - \lambda^2 + i0} \quad (4.3)$$

that is not defined. Even in QED, where, due to absence of color matrices, one could take for \hat{f}_2 the symmetrized form

$$\hat{f}_{+2}^{\mu\nu}(k_1, k_2; m) = \frac{1}{2!} \frac{im_\mu}{m \cdot k_1 - i0} \frac{im_\nu}{m \cdot k_2 - i0},$$

the momentum conservation $k_1 = -k_2$ from the photon propagator would yield the integral

$$\int d_4k \frac{1}{k^2 - \lambda^2 + i0} \frac{1}{m \cdot k - i0} \frac{1}{m \cdot k + i0}$$

plagued with a pinch singularity. So one has to get rid of it. This is exactly the task of the factors $\langle V_\sigma^\dagger \rangle^{-1}$ in Eq. (4.1), as we now explain.

The initial observation is that, thanks to translation invariance, the VEV of $V(x;m)$ cannot depend on x . So it may only be a (ill defined) constant times the identity matrix in color space. Imagine now that the theory has been provisionally regularized by defining it on a space-time of finite volume Ω : translation invariance is temporarily broken and momentum conservation does not hold, so that Eq. (4.3) is now well defined: all the graphs depend on Ω and tend to the expression that the above rules provide for them in the limit $\Omega \rightarrow \infty$. However, before the limit is taken and up to order g^2 , the factor $\langle V_+^\dagger \rangle^{-1}$ times $W_0(p)$ provides exactly the partition (tL), but with opposite sign.

Independently of any heuristic explanation, the factors $\langle V_\sigma^\dagger \rangle^{-1}$ are the instruction for the neglect of all the graphs including self interaction of the strings, as that given in Fig.

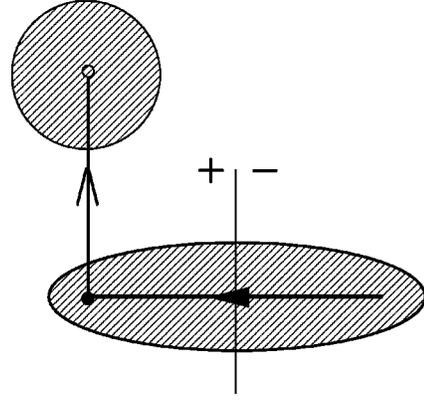


FIG. 6. An example of One String Reducible graph.

6, i.e. of the graphs that can be disconnected with one only cut in the string associated with the field $\Psi(x;m)$ —one could call them one string reducible graphs.

Likewise, $\langle V_\sigma \rangle^{-1}$ operates on the one string reducible graphs associated with the string of the field $\bar{\Psi}(y;n)$. We thus arrive to the conclusion that to order g^2 only the six partitions (a), (bL), (bR), (c), and (ζ L), (ζ R) survive, as well as the counterterms coming from the expansion of $\zeta_+ = \zeta_- \simeq 1 + \alpha/\pi\zeta_1$. For example the partition (ζ L) can be parametrized in the form

$$(\zeta\text{L}) = \frac{\alpha}{\pi} C_F \left[\gamma(\beta(m,p); \text{UV}, \text{IR}) + \delta(\beta(m,p)) \frac{m\mu}{m \cdot p} \right] W_0(p), \quad (4.4)$$

where, in terms of the ultraviolet and infrared cutoffs

$$\text{UV} = \frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\kappa^2}{\mu^2}, \quad (4.5)$$

$$\text{IR} = \ln \frac{\lambda}{\mu} \quad (4.6)$$

(dimensional regularization and mass term for the vector meson, respectively) and of the functions

$$\beta(m,p) = \sqrt{1 - \frac{m^2 p^2}{(m \cdot p)^2}} \quad (4.7)$$

(in which $m, p \in \mathcal{C}_\pm \Rightarrow 0 < \beta < 1$ and $m \rightarrow p \Rightarrow \beta \rightarrow 0$)

$$B(\beta) = \frac{1}{2\beta} \ln \left| \frac{1+\beta}{1-\beta} \right|, \quad (4.8)$$

$$\Xi(\beta) = \left[\frac{1}{\beta} \text{Li}_2(\beta) + \frac{1}{2\beta} \text{Li}_2\left(-\frac{1+\beta}{1-\beta}\right) \right] + [\beta \rightarrow -\beta] \quad (4.9)$$

the calculation of the invariant functions γ and δ gives the result

$$\begin{aligned} \gamma(\beta; \text{UV}, \text{IR}) &= \frac{1}{2} \left\{ \frac{1}{2} \text{UV} + 1 - \Xi(\beta) \right. \\ &\quad + B(\beta) \left[2\text{IR} + \ln \frac{1 - \beta^2}{4} + 1 \right] \\ &\quad + \left(\frac{1}{\xi} - 1 \right) \left[\frac{1}{4} \text{UV} + \text{IR} + \frac{7}{4} \right. \\ &\quad \left. \left. - \frac{1}{2} \frac{\ln \xi}{1 - \xi} \right] \right\} W_0(p), \end{aligned} \quad (4.10)$$

$$\delta(\beta) = -\frac{1}{2} B(\beta). \quad (4.11)$$

In Eq. (4.10) the contribution of the $g_{\mu\nu}$ and of the longitudinal terms of the vector meson propagator are the first and the second couple of lines, respectively. Likewise, the calculation of (ζR) is obtained by Eq. (4.4) with the replacement

$$(\zeta\text{R}) = \frac{\alpha}{\pi} C_F W_0(p) [m \rightarrow n]. \quad (4.12)$$

Obviously, the choice of ζ_1 can only modify the invariant function γ .

The first thing to note about the above graphs is that the coefficient of UV in γ does not depend either on p or on m, n . Therefore, this dependence (as well as the dependence on ξ) can be renormalized away.

The second thing to note is that the coefficient of IR—proportional to $B[\beta(m, p)] + B[\beta(n, p)]$ —does depend on p : the infrared divergence cannot be eliminated by renormalization.

We choose

$$\begin{aligned} \frac{\alpha}{\pi} \zeta_1 &= \frac{\alpha}{\pi} C_F \left\{ -\frac{1}{2} \text{UV} - 2\text{IR} + 1 - \left(\frac{1}{\xi} - 1 \right) \right. \\ &\quad \left. \times \left(\frac{1}{4} \text{UV} + \text{IR} + \frac{7}{4} - \frac{1}{2} \frac{\ln \xi}{1 - \xi} \right) \right\}, \end{aligned} \quad (4.13)$$

in which the finite part of ζ_1 has been chosen in such a way that when both the limits $m \rightarrow p, n \rightarrow -p$ are taken in Eqs. (4.4) and (4.12), respectively, one obtains

$$(\zeta\text{L}) + (\zeta\text{R}) + \frac{\alpha}{\pi} \zeta_1 W_0(p) \xrightarrow{m, -n \rightarrow p} 0. \quad (4.14)$$

We have now to discuss the sector partitions (a), (bL), (bR), (c). They have in common the two-body phase space

$$\begin{aligned} \Gamma_2(p) &= \int d\Gamma_2 \\ &= \int d^4k \theta(k_0) \delta(k^2) \theta(p_0 - k_0) \delta((p - k)^2 - \mu^2) \\ &= \frac{\pi}{2} \theta(p_0) \theta(p^2 - \mu^2) \left(1 - \frac{\mu^2}{p^2} \right) \end{aligned} \quad (4.15)$$

and their contribution to the amputated two-point function (1.10) is

$$w_1(p, m; -p, n) = \frac{g^2}{(2\pi)^2} C_F \int d\Gamma_2 N(p; m, n), \quad (4.16)$$

where from

$$\begin{aligned} N(p; m, n) &= (-g_{\mu\nu}) \left[\gamma^\mu - (\not{p} - \mu) \frac{m^\mu}{m \cdot k} \right] (\not{p} - \not{k} + \mu) \\ &\quad \times \left[\gamma^\nu - (\not{p} - \mu) \frac{n^\nu}{n \cdot k} \right], \end{aligned} \quad (4.17)$$

the contribution of each sector partition is clearly identifiable.

The following comments should help. (a) The factors $(\not{p} - \mu)$ in the square brackets of Eq. (4.17) are due to the amputation. (b) The contribution of the spurious degrees of freedom in the gluon propagator is obtained by the replacements

$$\delta(k^2) \rightarrow \delta(k^2 - \lambda^2) - \delta(k^2 - \lambda^2/\xi)$$

in the two-body phase-space (4.15) and

$$-g_{\mu\nu} \rightarrow k_\mu k_\nu / \lambda^2$$

in Eq. (4.17). The latter converts each of the square brackets into $(\not{p} - \not{k} - \mu)$ that in turn, on multiplying the factor $(\not{p} - \not{k} + \mu)$, gives zero—thanks to the fermion delta function in the two-body phase-space. (c) The prescriptions $\pm i0$ in the string vertex denominators have been omitted inasmuch as irrelevant to Eq. (4.17): indeed, $m, -n$, and also k , thanks to Eq. (4.15), belong to C_+ , so that both $m \cdot k$ and $-n \cdot k$ are strictly positive on the two-body phase space. (d) For the same reason there is no need of IR regularization in Eq. (4.16).

Use of covariance shows that $w_1(p, m; -p, n)$ can be expressed in terms of three integrals: one is $\Gamma_2(p)$, Eq. (4.15). As for the others, it is convenient for later use to define

$$I(p; m) = \int d\Gamma_2 \frac{\Pi(m)}{(m \cdot k)}, \quad (4.18)$$

where $\Pi(m)$ is a prescription:

$$\Pi(m) = \begin{cases} \pm 1 & \text{if } m \in C_\pm, \\ \text{PV} & \text{if } m \in C_0, \end{cases} \quad (4.19)$$

and likewise

$$J(p; m, n) = \int d\Gamma_2 \frac{\Pi(m)}{(m \cdot k)} \frac{\Pi(n)}{(n \cdot k)}. \quad (4.20)$$

The results of the calculations, for any m and n , are

$$I(p, m) = \Gamma_2(p) \frac{2}{p^2 - \mu^2} \frac{p^2}{|p \cdot m|} B(\beta(m, p)), \quad (4.21)$$

$$J(p; m, n) = \Gamma_2(p) \frac{4}{(p^2 - \mu^2)^2} \frac{p^2}{|m \cdot n|} B(\beta(m, n)) \tag{4.22}$$

with B given by Eq. (4.8).

Only one observation is relevant about the above integrals, namely that the limits $m \rightarrow -n$, $m \rightarrow p$, $n \rightarrow -p$ exist, commute with one another and commute with the phase-space integration to give

$$I(p; p) = \Gamma_2(p) \frac{2}{p^2 - \mu^2}, \tag{4.23}$$

$$J(p; m, -m) = J(p; p, -p) = \Gamma_2(p) \frac{4}{(p^2 - \mu^2)^2}. \tag{4.24}$$

As long as the contribution of sector partitions (a), (bL), (bR) and (c) is infrared finite, the lesson to be learned from adding this to the contribution of partitions (ζ L) and (ζ R) [given by Eqs. (4.4), (4.12)] is that the perturbative theory for the field $\Psi(x; m)$, $m \in \mathcal{C}_+$, and its Dirac conjugate is plagued with the same IR pathology as for the gauge dependent $\psi(x)$. Should one stop here, nothing would have been gained.

The only way to get rid of the IR divergence given by sector partitions (ζ L) and (ζ R) is to take both the limits $m \rightarrow p$ and $n \rightarrow -p$. In this case, due to the last two formulas, the contribution of sector partitions (a), (bL), (bR), (c) simplifies to

$$w_1(p) = C_F \theta(p_0) \theta(1 - \varrho) (1 - \varrho) \left(\not{p} \frac{1 + \varrho}{2} - \mu \right), \tag{4.25}$$

where

$$\varrho = \mu^2/p^2, \tag{4.26}$$

whence, on reinserting the external propagators omitted for the amputation, taking into account Eq. (4.14) and setting $C_F = 1$, one obtains the $W_1(p)$ appearing in Eq. (1.3) and given by Eqs. (1.5), (1.6). This is the piece of evidence that we can give in this paper, working to order g^2 , about the existence—and, to some extent, the necessity—of taking the limit (2.35), discussed in Sec. II. The extension of Eq. (4.25) to the region $0 < p^2 < \mu^2$ is legitimate and trivial.

Also the extension of Eq. (4.25) to the region $p^2 < 0$ is trivial—it also gives zero. But in this case there is a problem of consistency between this extension, on the one side, and the Steinmann rules and the limits m , $-n \rightarrow p$ on the other side. In this region infact the taking of the limits requires that m and/or n be spacelike from the outset and this, in turn, changes the sector partitions contributing to w_1 . It is however plausible to expect that the naive extrapolation of Eq. (4.25) to $p^2 < 0$ is correct: indeed all the sector partitions, even when calculated with the Steinmann rules suited for m and/or n spacelike, should display either a $\Gamma_2(p)$ or a $W_0(p)$ factor, as encountered in the present section. If really so,

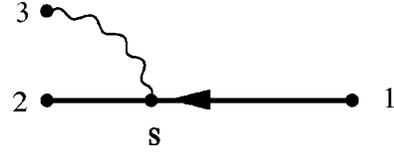


FIG. 7. The sector partitions (labeled by s) of graph BL that occurs when m is spacelike.

setting $p^2 < 0$ gives zero, due to the support properties of these factors, and the limits m , $-n \rightarrow p$ are quite safe.

We feel, however, that in order to check the abovementioned consistency, the exposing of the results of the explicit calculation is more convincing. Also because, should one be interested in the perturbative theory of the fields with $m \neq p$, there arise some difficulties connected with renormalizability that are better explicitly inspected. This is dealt with in the next subsections.

B. $m \in \mathcal{C}_0$, $n \in \mathcal{C}_-$

There is again the contribution of local graphs, namely, those contributing to the ordinary Wightman function $\langle \psi(x) \bar{\psi}(y) \rangle$, i.e., graph A of Fig. 1. This is expected to be the same as in the previous section, as independent of the string vectors m and n . Indeed, as commented after the last Steinmann rule in Sec. III, we have the freedom to assign a time-ordering label to each field, being sure that the final result does not depend on the assignment. We choose to write $\langle \psi(x) \bar{\psi}(y) \rangle = \langle T^+[\psi(x)] T^-[\bar{\psi}(y)] \rangle$, that takes us back to the case discussed in the previous section: only sector partition (a) of Fig. 2 gives a nonvanishing contribution.

Let us now discuss the sector partitions of graph BL of Fig. 1. Now the three external vertices must be given sector numbers as in Fig. 7 and the number s can be given values $1 \leq s \leq 3$. So in principle there are five inequivalent partitions. The two partitions in which s is noninteger have three on-shell lines joining in the same vertex, so their contribution is zero. There remain the three sector partition labeled by $s = 1, 2, 3$.

The first— $s = 1$ —is again (bL) in Fig. 3, so its contribution is easily recovered from Eqs. (4.16) and (4.17), provided the integral (4.18) is taken, according to Eqs. (4.1) and (2.17), with the PV prescription. In fact, in this case the denominator $m \cdot k$ is no longer positive on the two-body phase space. The result is still given by (4.21).

It is now convenient to consider the sector partitions of graph C in Fig. 1, postponing to later the sector partitions of Fig. 7 labeled by $s = 2, 3$. The sector numbers can only be assigned as in Fig. 8. Therefore this is again partition (c) of Fig. 4, easily recovered from Eqs. (4.16) and (4.17), provided that in the integral (4.20) the $m \cdot k$ denominator is PV

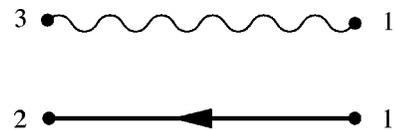


FIG. 8. The sector partition of graph C that occurs when m is spacelike.

prescribed. Again the result is provided by Eq. (4.22). Finally, the only sector partition of the one string reducible graph TL in Fig. 1 (sector numbers 1 to 4 clockwise from right vertex) is again disposed of, thanks to $\langle V_0^\dagger \rangle^{-1}$ instruction in Eq. (4.1).

Going back to the other two sector partitions $s=2,3$ of Fig. 7, they both have a $W_0(p)$ factor (the fermion line in the right) and so take the place of (ζL) of the previous section.

The contribution of these partitions is parametrized, in analogy with Eq. (4.4), by

$$\begin{aligned}
 (zL) = & \frac{\alpha}{\pi} C_F \left[c(\beta(m,p); UV, IR) \right. \\
 & \left. + d(\beta(m,p); UV) \frac{\not{m}\mu}{m \cdot p} \right] W_0(p), \quad (4.27)
 \end{aligned}$$

where, the invariant functions c and d depend on m and p through the variable $\beta(m,p)$ [defined in Eq. (4.7) and now, with $m \in C_0$, satisfying $\beta > 1$] and on the cutoffs (4.5), (4.6): the result of the calculation gives

$$\begin{aligned}
 c(\beta; UV, IR) = & UV \left(\frac{1}{2} + \frac{3}{8} \beta^{-2} + \frac{1}{8} (1 - \beta^{-2}) B(\beta) \right) \\
 & + IR B(\beta) + \frac{5 - \beta^{-2}}{16} Y(\beta) + \frac{2\beta^{-2} - 1}{4} B(\beta) \\
 & - \frac{3}{4} \beta^{-2} - \frac{1}{4} \left(\frac{1}{\xi} - 1 \right) (UV - 2IR + 1) \\
 & + \frac{1}{4\xi} \ln \xi, \quad (4.28)
 \end{aligned}$$

$$\begin{aligned}
 d(\beta; UV) = & UV \left(-\frac{3}{8} \beta^{-2} - \frac{1}{8} (1 - \beta^{-2}) B(\beta) \right) \\
 & + \frac{1}{4} \beta^{-2} - \frac{1}{4} B(\beta) - \frac{1 - \beta^{-2}}{16} Y(\beta), \quad (4.29)
 \end{aligned}$$

in which, we remind the reader, $B(\beta)$ is given by Eq. (4.8) and

$$Y(\beta) = \left[\frac{1}{2\beta} \ln^2 \frac{1+\beta}{2\beta} + \frac{1}{\beta} \text{Li}_2 \left(\frac{1+\beta}{2\beta} \right) \right] + [\beta \rightarrow -\beta]. \quad (4.30)$$

Comparison of the above formulas with the corresponding Eqs. (4.10), (4.11) shows that—contrary to δ —the invariant function d does depend on UV: this means that no choice of the renormalization constant ζ_0 introduced in Eq. (3.1) [and giving rise to counterterms proportional to $W_0(p)$, not to $\not{m}W_0(p)$] can cure the divergence. In addition there also are the coefficients of UV and IR in c that both depend on p through β . As for the IR divergence associated with a space-like string, the result is not new [26].

There is a way out of this problem: choosing $m_0 = 0$ in the rest frame, in which only $p_0 = \mu \neq 0$. In this case in fact

$(zL) \rightarrow \alpha C_F / \pi W_0(p) \times [UV/2 + \text{last two terms of Eq. (4.28)}]$, so that a suitable choice of ζ_0 to order g^2 removes the divergence. Unfortunately, the choice $m_0 = 0$ spoils Lorentz invariance and we will not stick to it.

To summarize: the perturbative theory involving a charged field, dressed with a string in spacelike direction, is nonrenormalizable—at least at finite orders—due to the m string. There survive, in addition, IR divergences carried by both the m and n strings.

C. $m \in C_0, n \in C_0$

The discussion of local graphs as well as of the sector partitions with only one string vertex [Figs. 2, 7 and its analogue giving rise to a partition (bR) and to a contribution (zR) obtained by Eq. (4.27), with a replacement analogue to Eq. (4.12)] presents no novelty with respect to the preceding subsection. The only novel feature is given by graph C of Fig. 1, where the only partition is given by assigning sector numbers from 1 to 4 with clockwise movement, starting from the top right vertex. This is again recovered from Eqs. (4.16) and (4.17), provided the integral (4.20) is now taken with both m and n spacelike, i.e., with both denominators prescribed by PV. The result is once more Eq. (4.22). In this case also, thanks to the (zL) and (zR) contributions, there are UV as well IR divergences due to each string. Choosing both $m_0 = n_0 = 0$ in the rest frame would eliminate the two problems.

However, once more we refrain from breaking Lorentz symmetry, also because there is another way out of this impasse. This is provided exactly by the double limit $m = -n \rightarrow p$. The latter has to be effected first by dragging p in C_0 : this makes the whole two-point function vanish, due to the support of the $\delta(p^2 - \mu^2)$ in the (zL) and (zR) contributions, and to the support of the $\Gamma_2(p)$ two-body phase space in all the other ones. At this point taking the limit is safe and gives zero, in agreement with the naive extrapolation of Eq. (4.25) discussed in the end of Sec. IV A.

V. OUTLOOK OF FOURTH ORDER CALCULATIONS

The calculations of Sec. IV should make it evident that the only two-point function free from both UV and IR problems is that relative to the field (2.36): they provide evidence for the necessity, rather than the possibility, of taking the limit $n \rightarrow p$, whose meaning and implications we have discussed in the final part of Sec. II.

It is also clear that, in order to obtain the result (4.25), commuting the limit $n \rightarrow p$ with the loop integration makes the calculation by far simpler and that, consistently, only the diagrammatic rules of Sec. IV A have to be used in order to get a nonvanishing two-point function. Exactly in this way, we have performed the two-loop calculation of W_2 , Eq. (1.3), in QED. This receives contribution from 12 graphs for a total of 19 nonvanishing partitions, 10 of which involve two-body phase-space, the other 9 involve three-body phase-space. The graphs with only one external fermion line on shell—as the partitions (ζL) or (ζR) of Sec. IV A—have not been included in the counting, because, much as in the case

of Eq. (4.14), they can be renormalized away with a suitable choice of the fourth order contribution ζ_2 to the renormalization constant ζ_{\pm} .

Several graphs exhibit an IR divergences proportional to $\ln \lambda$. In this QED calculation, the photon mass regularization has been adopted, for it is well known not to interfere with either BRST symmetry or unitarity. Indeed, we have a diagrammatic (i.e., without the need of analytic calculations) proof of the decoupling of unphysical degrees of freedom, in the form of gauge-fixing parameter independence

$$\xi \frac{\partial}{\partial \xi} W_2 = 0. \quad (5.1)$$

Furthermore, we have found that the Grammer-Yennie [44] method of control of IR divergences can be extended, in a straightforward way, to the graphs that include the eikonal string vertices. This results in a diagrammatic proof (an analogue to that of Ref. [45]) of a complete cancellation between the IR divergences coming from the three-body cut graphs and the two-body cut graphs.

What is left is the explicit result of the calculation that we report below to give concreteness to what we have said, although in its full form it is not illuminating. For $p^2 < 9\mu^2$ (i.e., omitting graphs involving closed fermion loops, that are irrelevant for the near-mass-shell asymptotics) the two structure functions, defined by Eq. (1.5) with $i=2$, are given by

$$\begin{aligned} a_2 = & \frac{\varrho(-39-82\varrho+37\varrho^2)}{16(1-\varrho)} + \frac{\varrho^3}{2(1-\varrho)} \ln(1-\sqrt{1-\varrho}) \ln(1+\sqrt{1-\varrho}) + \frac{\varrho^2}{2\sqrt{1-\varrho}} \ln \frac{1+\sqrt{1-\varrho}}{1-\sqrt{1-\varrho}} \\ & + \frac{\varrho(3-31\varrho+2\varrho^2-2\varrho^3+2\varrho^4)}{8(1-\varrho)^2} \ln \varrho + \frac{\varrho(-2+5\varrho-5\varrho^2+3\varrho^3-\varrho^4+(-7-2\varrho-\varrho^2+2\varrho^3)\ln \varrho)}{4(1-\varrho)^2} \ln(1-\varrho) \\ & + \frac{\varrho^2(-1+2\varrho)}{8(1-\varrho)} \ln^2 \varrho + \frac{\varrho(-5-6\varrho+3\varrho^2)}{4(1-\varrho)^2} [\text{Li}_2(\varrho) - \text{Li}_2(1)], \end{aligned} \quad (5.2)$$

$$\begin{aligned} b_2 = & \frac{\varrho(5+\varrho)(-9+2\varrho)}{8(1-\varrho)} - \frac{\varrho^2}{2(1-\varrho)} \ln(1-\sqrt{1-\varrho}) \ln(1+\sqrt{1-\varrho}) - \frac{\varrho}{2\sqrt{1-\varrho}} \ln \frac{1+\sqrt{1-\varrho}}{1-\sqrt{1-\varrho}} \\ & + \frac{\varrho(-2-13\varrho+6\varrho^2-5\varrho^3+\varrho^4)}{4(1-\varrho)^2} \ln \varrho - \frac{\varrho^2}{8(1-\varrho)} \ln^2 \varrho + \frac{\varrho(3\varrho-7\varrho^2+5\varrho^3-\varrho^4+(-13+4\varrho+\varrho^2)\ln \varrho)}{4(1-\varrho)^2} \\ & \times \ln(1-\varrho) + \frac{\varrho(-13+2\varrho+3\varrho^2)}{4(1-\varrho)^2} [\text{Li}_2(\varrho) - \text{Li}_2(1)] \end{aligned} \quad (5.3)$$

whose asymptotic form for $\varrho = \mu^2/p^2 \rightarrow 1$ is Eq. (1.7). It should be also noted that in the ultraviolet regime $p^2 \rightarrow +\infty$, i.e., for $\varrho \rightarrow 0$, a_2 and b_2 vanish respectively as $(\ln p^2)/p^2$ and $1/p^2$. So, when dispersed in p^2 , they need no subtraction.

Concerning the QCD counterpart of the above calculation, stated in Eq. (1.8) of Sec. I, apart from the contribution of the non-planar QED graphs (those where the color matrices occur in the sequence $t^a t^b t^a t^b = C_F^2 - \frac{1}{2} C_A C_F$), there are 15 more graphs giving rise to 23 partitions, 8 of them involving two-body phase-space, the other 15 the three-body phase space.

A due remark concerns the IR regularization: giving a mass to the gluon, even according to Ref. [42], preserves BRST symmetry, but only formally preserves unitarity in the limit $\lambda \rightarrow 0$. As a matter of fact we have verified that, in this limit, the RHS of Eq. (5.1) does not vanish. We have therefore abandoned this regularization adopting dimensional regularization for the IR [37,38] and changed dimensional regularization for the UV with non-Lagrangian Pauli-Villars [43]. With these regularizations, Eq. (5.1) indeed holds also in the non-Abelian case.

In writing Eq. (1.8), we have recalculated the Abelian part (proportional to C_F^2) with the new IR and UV regularizations, with the expected result that the cancellation of IR divergences holds also in the new scheme. These details, however, will be part of a forthcoming paper giving the details of the above described two-loop calculation [33].

Concerning instead the non-Abelian part, we can say, referring to the factor $\frac{11}{6}$ appearing in Eq. (1.8), that $\frac{5}{6}$ comes from the sum of all the graphs that include gluon self-energy corrections; a further 1 comes from the $C_A C_F$ part of the non-planar Abelian graphs. For the other graphs (whose sector partitions are, in some cases, IR divergent even as $1/\epsilon^2$, not just as $1/\epsilon$) there is a complete cancellation between the two-body cut and the three-body cut contributions to each of them.

VI. CONCLUSIONS

We have shown how to construct BRST invariant composite fermion fields that carry the global quantum numbers

of the electron and of the quark in QED and QCD. The construction consists in dressing the ordinary Dirac field with a rectilinear string whose space-time direction is characterized by a four-vector that, provisionally, breaks the Lorentz covariance properties of the field. In perturbation theory the string generates new graphs characterized by the occurrence of eikonal vertices. These new vertices require prescriptions (either $\pm i0$ or PV) whose choice is uniquely dictated by the Dirac conjugation properties of the field. Furthermore, after going in momentum representation, the four-vector characterizing the string must be chosen proportional to the four-momentum of the field. This choice: (i) restores Lorentz, (ii) averts some IR as well as some nonrenormalizable UV divergences. The second point indicates that, as a matter of fact, there is little choice. The whole construction survives the check of a fourth order calculation of the two-point function in PT, performed both in QED and QCD.

If these fields are to survive further and more stringent verifications, one can conclude that global charges associated to a Gauss law imply, for the fields carrying such charges, delocalization properties considerably more involved than the single one-dimensional string in three-space, somewhat popular in the literature: since the string is rectilinear in four-momentum space, in coordinate representation the fields rather appear spread out all over Minkowski space, exhibiting a kind of candy-sugar structure.

The construction gives—as an extra bonus—different results for the IR asymptotic dynamics of QED and QCD respectively. In particular, it hints at a mechanism of confinement according to which the quark so constructed seems to behave as a free field at any momentum scale.

The construction presented raises several problems; but the algorithm we have given in this paper also provides the possibility to face them. Among others, a few of them still involve two-point functions: Extending to any order in PT the above results about: (i) IR cancellation in QED and (ii) IR noncancellation and factorization in QCD and the verification that any gauge invariant colored field, first of all the gluon, has the same behavior as the quark.

Other problems pertain instead a study of correlation functions with more than two points: The verification that such fields do indeed carry the expected global charges, e.g., that they satisfy in PT at least the weak commutation relations

$$\langle e(x)[Q, \bar{e}(y)] \rangle = \langle e(x)\bar{e}(y) \rangle,$$

etc., where $Q = \int d^3x: \bar{\psi}\gamma_0\psi:(x)$ is the electric charge (and the analogue in QCD).

A comparison of the present approach with the well established results of QED such as those about inclusive cross section or the electron $g-2$ and, in general, the impact of this construction on the S matrix [although we expect to recover no more than the usual IR divergent S matrix of point (iv)₁ in the Introduction, because also the Steinmann scheme for perturbation theory relies on splitting the Hamiltonian according to $H = H_0 + V$].

For QCD, the proof of the scenario we have hinted at in Sec. I, namely, that amplitudes involving gauge invariant colored fields either vanish or disconnect into the product of free two-point functions relative to colored fields times an amplitude that only involves color singlet fields. These problems are already under our investigation and we will report about them in the near future.

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